Admissible group schemes

Motivation

Next week, we will prove

Theorem. Let A be an abelian variety over $\mathbb{Q}$ and let $N \neq p$ be primes with $N > 2$. Assume

(1) A has good reduction away from N.

(2) The Jordan-Hölder factors of $A[p](\overline{\mathbb{Q}})$ are all isomorphic to either $\mathbb{Z}/p\mathbb{Z}$ or μ_p.

(3) A has completely toric reduction at N.

Then, $A(\mathbb{Q})$ has rank 0.

Idea of proof.

Recall

$$A(\mathbb{Q})/p^nA(\mathbb{Q}) \hookrightarrow H^1(G_{\mathbb{Q},Np}, A[p^n]) \cong H^1_{\text{ét}}(\operatorname{Spec}\mathbb{Z}[1/Np], \mathcal{A}[p^n]),$$

where $\mathcal{A}$ is the Néron model of A. To prove that $A(\mathbb{Q})$ has rank 0, it's enough to show the right hand side is bounded independent of n.

But this is false!

Instead, $A(\mathbb{Q})/p^nA(\mathbb{Q}) \hookrightarrow H^1_{\text{fppf}}(\operatorname{Spec}\mathbb{Z}, \mathcal{A}[p^n])$.

fppf cohomology

Definition. An *fppf cover* of a scheme X is a family of morphisms $\{f_i \colon X_i \longrightarrow X\}_i$ where each f_i is flat and locally of finite presentation, and such that $X = \bigcup_i f_i(X_i)$.

Let X_{fppf} be the category whose objects are flat and locally of finite presentation maps to X and morphisms are commutative diagrams

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ & \searrow \quad \swarrow & \\ & X. & \end{array}$$

X_{fppf} is a site, so we can define *fppf cohomology* of an fppf sheaf as usual.

We can consider a finite flat group scheme G over S as a sheaf over S_{fppf}.

Then, $H^0_{\mathrm{fppf}}(S, G) = G(S)$.

$H^1_{\mathrm{fppf}}(S, G)$ also has a geometric interpretation, in terms of fppf torsors.

A *torsor* for G is a scheme T/S equipped with an action of G such that for any S'/S, the action of $G(S')$ on $T(S')$ is simply transitive.

T is called an *fppf torsor* (resp. étale torsor) if there exists an fppf (resp. étale) cover $S' \longrightarrow S$ such that $T(S') \neq \emptyset$.

$H^1_{\mathrm{fppf}}(S, G)$ is in bijection with the set of isomorphism classes of fppf torsors.

Remark. G étale $\implies H^1_{\text{fppf}}(S, G) = H^1_{\text{ét}}(S, G)$.

Indeed, any étale torsor is an fppf torsor, and if T/S is an fppf torsor for G, there exists an fppf cover $S' \longrightarrow S$ such that $T_{S'} = G_{S'}$, in particular $T_{S'}$ is étale. This implies T is étale.

Admissible group schemes

The hypotheses of the theorem are

(1) A has good reduction away from N.
$\mathcal{A}[p^n]$ is *pre-admissibe*.

(2) The Jordan-Hölder factors of $A[p](\overline{\mathbb{Q}})$ are all isomorphic to either $\mathbb{Z}/p\mathbb{Z}$ or μ_p.
$\mathcal{A}[p^n]$ is *admissibe*.

(3) A has completely toric reduction at N.
This will allow us to compute certain invariants δ, α.

Definition.

(1) A group scheme G over $\mathbb{Z}[1/N]$ is *pre-admissible* if it is flat, commutative, killed by a power of p and finite.

(2) A group scheme G over $\mathbb{Z}$ is *pre-admissible* if it is flat, commutative, killed by a power of p, quasi-finite, finite over $\mathbb{Z}[1/N]$, separated, of finite presentation.

Example. If A is an abelian variety over $\mathbb{Q}$ with good reduction away from N and $\mathcal{A}$ is its Néron model, then $\mathcal{A}[p^n]$ is pre-admissible over $\mathbb{Z}$.

Definition.

(1) A pre-admissible group over $\mathbb{Z}[1/N]$ is *admissible* if there exists an increasing filtration

$$0 = F^0 G \subseteq F^1 G \subseteq \cdots \subseteq F^n G = G$$

by closed subgroups such that $F^{n+1}G/F^n G$ is isomorphic to either $\mathbb{Z}/p\mathbb{Z}$ or μ_p. $F^\bullet G$ is called an *admissible filtration*.

(2) A pre-admissible group over $\mathbb{Z}$ is *admissible* if its restriction to $\mathbb{Z}[1/N]$ is admissible.

Definition. A $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module V is *admissible* if there exists a filtration $F^\bullet V$ by submodules whose successive quotients are $\mathbb{Z}/p\mathbb{Z}$ or μ_p.

Proposition. Let G be a pre-admissible group over $\mathbb{Z}[1/N]$. Then G is admissible if and only if $G(\overline{\mathbb{Q}})$ is admissible.

Proof. Let V be the first piece of an admissible filtration of $G(\overline{\mathbb{Q}})$, and let H be the closure of V in G.

$$\begin{aligned} H \text{ finite étale over } \mathbb{Z}[1/Np] &\implies H_{\mathbb{Z}[1/Np]} \cong \mathbb{Z}/p\mathbb{Z} \text{ or } \mu_p \\ &\implies H_{\mathbb{Q}_p} \cong \mathbb{Z}/p\mathbb{Z} \text{ or } \mu_p. \end{aligned}$$

It follows from Raynaud's theorem when $p > 2$ (or a theorem of Fontaine when $p = 2$) that, over $\mathbb{Z}_p$, H is isomorphic to either $\mathbb{Z}/p\mathbb{Z}$ or μ_p.

In fact, a global version of Raynaud's (or Fontaine's) theorem shows that H is isomorphic to either μ_p or $\mathbb{Z}/p\mathbb{Z}$ over $\mathbb{Z}[1/N]$. Do the same for G/H and use induction. □

Example. Let A be an abelian variety over $\mathbb{Q}$ with good reduction away from N and $\mathcal{A}$ is its Néron model, and assume the Jordan-Hölder factors of $A[p](\overline{\mathbb{Q}})$ are all isomorphic to either $\mathbb{Z}/p\mathbb{Z}$ or μ_p. Then $\mathcal{A}[p^n]$ is admissible over $\mathbb{Z}$.

Indeed, our assumption and the proposition show that $\mathcal{A}[p]$ is admissible and $\mathcal{A}[p^n]$ is an iterated extension of $\mathcal{A}[p]$'s.

Definition. Let G be an admissible group over $\mathbb{Z}$. Define the invariants

$$\begin{aligned}
\ell(G) &= \log_p(\#G_{\mathbb{Q}}) = \text{ length of an admissible filtration on } G,\\
\delta(G) &= \log_p(\#G_{\mathbb{Q}}) - \log_p(\#G_{\mathbb{F}_N}),\\
\alpha(G) &= \log_p(\#G(\overline{\mathbb{F}_p}))\\
&= \text{number of } \mathbb{Z}/p\mathbb{Z}\text{'s appearing in an admissible filtration of } G,\\
h^0(G) &= \log_p(\#H^0_{\mathrm{fppf}}(\operatorname{Spec}\mathbb{Z}, G)) = \log_p(\#G(\mathbb{Z})),\\
h^1(G) &= \log_p(\#H^1_{\mathrm{fppf}}(\operatorname{Spec}\mathbb{Z}, G)).
\end{aligned}$$

Elementary admissible groups

Definition. An admissible group scheme G is *elementary* if $\ell(G) = 1$.

Over $\mathbb{Z}[1/N]$ the only elementary admissible groups are $\mathbb{Z}/p\mathbb{Z}$ and μ_p.

What about over $\mathbb{Z}$?

Proposition. Let H be a pre-admissible group over $\mathbb{Z}[1/N]$. Then, there is a bijection between extensions of H to $\mathbb{Z}$ and sub-Gal$(\overline{\mathbb{Q}}_N/\mathbb{Q}_N)$-modules V of $H(\overline{\mathbb{Q}})$ whose elements are fixed by the inertia subgroup at N.

Corollary. If $H(\overline{\mathbb{Q}}_N)$ is unramified and one-dimensional (over $\mathbb{F}_p$), then there are only two extensions to $\mathbb{Z}$, H and $H^\flat$.

More concretely, $H^\flat$ can be seen as $j_! H_{\mathbb{Z}[1/N]}$, where $j\colon \operatorname{Spec}\mathbb{Z}[1/N] \longrightarrow \operatorname{Spec}\mathbb{Z}$.

Hence, over $\mathbb{Z}$ there are four elementary admissible groups: $\mathbb{Z}/p\mathbb{Z}, (\mathbb{Z}/p\mathbb{Z})^\flat, \mu_p$ and $\mu_p^\flat$.

Proposition. The invariants of the elementary admissible groups are as follows:

	$\mathbb{Z}/p\mathbb{Z}$	$(\mathbb{Z}/p\mathbb{Z})^\flat$	μ_p	$\mu_p^\flat$
δ	0	1	0	1
α	1	1	0	0
h^0	1	0	$\begin{cases} 1 & \text{if } p = 2, \\ 0 & \text{if } p > 2, \end{cases}$	0
h^1	0	0	$\begin{cases} 1 & \text{if } p = 2, \\ 0 & \text{if } p > 2, \end{cases}$	$\begin{cases} 1 & \text{if } p = 2, N \equiv 1 \mod 4, \\ 1 & \text{if } p > 2, N \equiv 1 \mod p, \\ 0 & \text{otherwise}, \end{cases}$

Proof. The first three lines are easy, so let's look at the last one. Write $S = \operatorname{Spec} \mathbb{Z}$.

$$\mathbb{Z}/p\mathbb{Z} \text{ étale over } S \implies H^1_{\text{fppf}}(S, \mathbb{Z}/p\mathbb{Z}) = H_1^{\text{ét}}(S, \mathbb{Z}/p\mathbb{Z}) \\ = \operatorname{Hom}(\pi^1_{\text{ét}}(S), \mathbb{Z}/p\mathbb{Z}) \\ = \operatorname{Hom}(1, \mathbb{Z}/p\mathbb{Z})$$

So $h^1(\mathbb{Z}/p\mathbb{Z}) = 0$.

There is a short exact sequence

$$0 \longrightarrow (\mathbb{Z}/p\mathbb{Z})^\flat \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow i_*((\mathbb{Z}/p\mathbb{Z})_{\mathbb{F}_N}) \longrightarrow 0$$

where $i\colon \operatorname{Spec}\mathbb{F}_N \hookrightarrow \operatorname{Spec}\mathbb{Z}$.

We get a long exact sequence

$$\begin{aligned} &H^0_{\text{fppf}}(S, (\mathbb{Z}/p\mathbb{Z})^\flat)) \longrightarrow H^0_{\text{fppf}}(S, \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^0_{\text{fppf}}(S, i_*((\mathbb{Z}/p\mathbb{Z})_{\mathbb{F}_N})) \\ \longrightarrow &H^1_{\text{fppf}}(S, (\mathbb{Z}/p\mathbb{Z})^\flat)) \longrightarrow H^1_{\text{fppf}}(S, \mathbb{Z}/p\mathbb{Z}) \end{aligned}$$

i.e.

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow H^1_{\text{fppf}}(S, (\mathbb{Z}/p\mathbb{Z})^\flat)) \longrightarrow 0.$$

So $h^1((\mathbb{Z}/p\mathbb{Z})^\flat) = 0$.

We have an exact sequence of fppf sheaves

$$0 \longrightarrow \mu_p \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_m \longrightarrow 0,$$

so we have a long exact sequence

$$0 \longrightarrow H^0_{\mathrm{fppf}}(S, \mu_p) \longrightarrow H^0_{\mathrm{fppf}}(S, \mathbb{G}_m) \longrightarrow H^0_{\mathrm{fppf}}(S, \mathbb{G}_m)$$

$$\longrightarrow H^1_{\mathrm{fppf}}(S, \mu_p) \longrightarrow H^1_{\mathrm{fppf}}(S, \mathbb{G}_m)[p] \longrightarrow 0$$

i.e.

$$0 \longrightarrow \mu_p(S) \longrightarrow \{\pm 1\} \xrightarrow{(-)^p} \{\pm 1\}$$

$$\longrightarrow H^1_{\mathrm{fppf}}(S, \mu_p) \longrightarrow H^1_{\mathrm{fppf}}(S, \mathbb{G}_m)[p] \longrightarrow 0$$

Fact. $H^1_{\mathrm{fppf}}(S, \mathbb{G}_m) = H^1_{\mathrm{Zar}}(S, \mathbb{G}_m) = \mathrm{Pic}(S) = \mathrm{Cl}(\mathbb{Z}) = 0.$

So $h^1(\mu_p) = \begin{cases} 1 & \text{if } p = 2, \\ 0 & \text{if } p > 2. \end{cases}$

There is a short exact sequence

$$0 \longrightarrow \mu_p^\flat \longrightarrow \mu_p \longrightarrow G := i_*((\mu_p)_{\mathbb{F}_N}) \longrightarrow 0$$

so we have a long exact sequence

$$\begin{aligned} & H^0_{\mathrm{fppf}}(S, \mu_p^\flat)) \longrightarrow H^0_{\mathrm{fppf}}(S, \mu_p) \longrightarrow H^0_{\mathrm{fppf}}(S, G) \\ \longrightarrow & H^1_{\mathrm{fppf}}(S, \mu_p^\flat)) \longrightarrow H^1_{\mathrm{fppf}}(S, \mu_p) \longrightarrow H^1_{\mathrm{fppf}}(S, G) \end{aligned}$$

Assume $p > 2$. Then $H^0_{\mathrm{fppf}}(S, \mu_p) = H^1_{\mathrm{fppf}}(S, \mu_p) = 0$, so $h^1(\mu_p^\flat) = \log_p(\#\mu_p(\mathbb{F}_N)) = \begin{cases} 1 & p \mid N-1, \\ 0 & \text{otherwise.} \end{cases}$

For $p = 2$, the map $H^0_{\mathrm{fppf}}(S, \mu_p) \longrightarrow H^0_{\mathrm{fppf}}(S, G)$ is an isomorphism, so

$$H^1_{\mathrm{fppf}}(S, \mu_p^\flat) = \ker(H^1_{\mathrm{fppf}}(S, \mu_p) \longrightarrow H^1_{\mathrm{fppf}}(S, G)).$$

$H^1_{\mathrm{fppf}}(S, \mu_p)$ has order 2, and from Kummer theory we see that the non-trivial element corresponds to the torsor $f : \operatorname{Spec} \mathbb{Z}[\sqrt{-1}] \longrightarrow S$.

This goes to 0 $\iff$ $\operatorname{Spec}(\mathbb{F}_N[\sqrt{-1}])$ has an $\mathbb{F}_N$-point $\iff$ $-1 \in (\mathbb{F}_N^\times)^2 \iff N \equiv 1 \mod 4$.

So $h^1(\mu_p^\flat) = \begin{cases} 1 & N \equiv 1 \mod 4, \\ 0 & \text{otherwise.} \end{cases}$

Corollary. Let G be admissible over $\mathbb{Z}$. Then,

$$h^1(G) - h^0(G) \leq \delta(G) - \alpha(G)$$

Proof. This holds for the elementary admissible groups, and given a short exact sequence

$$0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 0$$

of admissible groups, we have

$$\begin{gathered}\delta(G_2) - \alpha(G_2) = (\delta(G_1) - \alpha(G_1)) + (\delta(G_3) - \alpha(G_3)) \\ h^1(G_2) - h^0(G_2) \leq (h^1(G_1) - h^1(G_1)) + (h^0(G_3) - h^0(G_3)),\end{gathered}$$

so the result follows from induction on the length of G. $\square$