Néron Models

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Outline

- Models of elliptic curves over a DVR
- Néron models for abelian varieties
- The Néron-Ogg-Shafarevich Criterion

Notation

Let R be a complete discrete valuation ring with $K = \mathrm{Frac}(R)$.
Let v be the valuation, π a uniformiser for R, and $k = R/\pi$.
G_K denotes the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$.

Elliptic Curves

Our goal: extending the K-points of an elliptic curve to R-points in a nice way.

Let E/K be an elliptic curve given by $y^2 = x^3 + ax + b$.
By a change of variables $(x, y) \mapsto (u^2 x, u^3 y)$ for $u \in K^\times$ we can find the *minimal Weierstraß equation*: $a, b \in R$ and $v(\Delta)$ minimal. $v(\Delta) < 12$ is sufficient.

Let $\mathcal{W}$ be the projective R-scheme defined by the minimal Weierstraß equation. It is proper, and so $\mathcal{W}(R) = \mathcal{W}(K) = E(K)$, which is great. But $\mathcal{W}$ might be singular.
Try to fix this by considering the smooth locus $\mathcal{W}_{sm}$. But now this scheme is not proper: we have $\mathcal{W}_{sm}(R) \subsetneq E(K)$.

Intuition for $\mathcal{E}$

The Néron model $\mathcal{E}$ of E has both nice properties: it is a smooth group scheme over R and $\mathcal{E}(R) = E(K)$.

Let $E_0(K)$ be the subgroup of $E(K)$ mapping into $\mathcal{W}_{sm}(k)$ under the reduction map. This map is surjective and $E(K)/E_0(K)$ is finite.

We have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_0(K) & \longrightarrow & E(K) & \longrightarrow & \{finite\} & \longrightarrow & 0 \\ & & \downarrow\!\!\downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{W}_{sm}(k) & \longrightarrow & \mathcal{E}(k) & \longrightarrow & \pi_0(\mathcal{E}(k)) & \longrightarrow & 0 \end{array}$$

The connected component of the identity of $\mathcal{E}$ is $\mathcal{W}_{sm}$.

The minimal regular model

Let C/K be a curve. A *regular model* for C is a proper flat scheme $\mathcal{C}$ over R which is regular and has generic fibre C. We say that $\mathcal{C}$ is *minimal* if for any other regular model $\mathcal{C}'$ there is a map $\mathcal{C}' \to \mathcal{C}$ which extends to the identity in the generic fibre.

Minimal regular models exist and are unique. For an elliptic curve E/K its Néron model $\mathcal{E}$ is the smooth locus of $\mathcal{C}$.

With Tate's algorithm one can compute the special fibre of $\mathcal{C}$ explicitly. It consists of genus 0 curves with possible singularities and intersections. The possibilities were classified by Néron and Kodaira.

Examples

Let $y^2 = x^3 + p$ define the elliptic curve E over $K = \mathbb{Q}_p$. We see that the same equation defines the minimal Weierstraß model $\mathcal{W}$ over $R = \mathbb{Z}_p$. It is clearly smooth everywhere except for the point $P = (0, 0)$ in the special fibre.
We claim that P is regular: take the ring $A = R[x, y]/(y^2 = x^3 + p)$ corresponding to the natural affine chart containing P. Note that A has Krull dimension 2. P corresponds to the maximal ideal $\mathfrak{m} = (x, y, p)$.
$\mathfrak{m}^2$ is generated by x^2, xy, xp, y^2, yp, and p^2. But since $y^2 = x^3 + p$ and $x^3 \in \mathfrak{m}$, we can replace y^2 with p, making px, py, p^2 redundant.
So $\mathfrak{m}^2 = (x^2, xy, p)$ and the quotient $\mathfrak{m}/\mathfrak{m}^2$ has basis x and y, making it a 2-dimensional vector space over $A/\mathfrak{m}$.
So $\mathcal{W}$ is a regular model for E and it is minimal too. So $\mathcal{E} = \mathcal{W}_{sm} = \mathcal{W} \setminus \{P\}$.
The special fibre $\mathcal{E}_k$ is connected and isomorphic to $\mathbb{G}_a$.

Examples

E/K given by $y^2 = x^3 + p^2$. Again this defines a minimal Weierstrass model $\mathcal{W}$ over R and $P = (0,0)$ in the special fibre is the unique singular point. This time, P is not regular: it turns out that $\mathfrak{m}/\mathfrak{m}^2$ is 3-dimensional.
We have to blow up at P to find the minimal regular model. The special fibre of the blow-up is isomorphic to three copies of $\mathbb{P}^1$, all joined at a single point.
We obtain the special fibre of the Néron model by deleting that point, and we see that $\pi_0(\mathcal{E}_k) = \mathbb{Z}/3\mathbb{Z}$.

Results on $\mathcal{C}$

Let E have split multiplicative reduction, $v(j(E)) = -n$. Then $\mathcal{C}_k$ consists of n copies of $\mathbb{P}^1$, each of which intersects two others, forming a circle. Taking the smooth locus we see $\mathcal{E}_k \cong \mathbb{G}_m \times \cdots \times \mathbb{G}_m$; $\pi_0(\mathcal{E}_k) = \mathbb{Z}/n\mathbb{Z}$.

Later we will use the following fact: as long as E doesn't have split multiplicative reduction, $|\pi_0(\mathcal{E}_k)| \leq 4$.

Everything we've done so far also works for R a Dedekind domain.

Motivation for $\mathcal{A}$

How can we define Néron models for abelian varieties?
Weierstraß models use explicit equations.
Minimal regular models are defined only for curves.
It turns out that we can use the functor of points of the Néron model of an elliptic curve:

For any smooth scheme $\mathcal{X}$ over R with $X = \mathcal{X}_K$ we have a bijection $\mathrm{Hom}_R(\mathcal{X}, \mathcal{E}) \to \mathrm{Hom}_K(X, E)$. We use this as the general definition of a Néron model.

Néron models for abelian varieties

Let A/K be a smooth scheme. A Néron model for A is a smooth scheme $\mathcal{A}/R$ such that for all smooth schemes $\mathcal{X}/R$ there is a bijection $\mathrm{Hom}_R(\mathcal{X}, \mathcal{A}) \to \mathrm{Hom}_K(X, A)$.

- This definition actually holds for any smooth A
- For A an abelian variety, $\mathcal{A}$ exists, in the local and in the global case.
- By the Yoneda lemma, $\mathcal{A}$ is unique.
- Taking $\mathcal{X} = R$, we get $\mathcal{A}(R) = A(K)$, just as we wanted.
- The Néron model is compatible with base change to an unramified extension.

The structure of $\mathcal{A}_0$

Theorem (Chevalley): Over a perfect field, every smooth connected group scheme is the extension of an abelian variety by a smooth affine group scheme.

Let A/K be an abelian variety with Néron model $\mathcal{A}/R$, $\mathcal{A}_0$ its special fibre and $\mathcal{A}_0^\circ$ the identity component of $\mathcal{A}_0$.
Then we have an exact sequence $0 \to L \to \mathcal{A}_0^\circ \to B \to 0$, where B is an abelian variety and L is a commutative smooth affine. L contains a maximal torus T such that $L/T = U$ is unipotent.
Over $\overline{K}$, T and U become isomorphic to products of $\mathbb{G}_m$ and $\mathbb{G}_a$, respectively.

Reduction types

We have a canonical filtration of $\mathcal{A}_0$:
$0 = F_0 \subset F_1 \subset F_2 \subset F_3 \subset F_4 = \mathcal{A}_0$ such that $T = F_1/F_0$ is a torus, $U = F_2/F_1$ is unipotent, $B = F_3/F_2$ is an abelian variety and F_4/F_3 is the component group of $\mathcal{A}_0$ which is finite étale by the connected-étale sequence.

- We say that A has *good reduction* if it extends t an abelian scheme over R. This is equivalent to $F_2 = 0$, i.e. equivalent to $\mathcal{A}_0^\circ$ (or $\mathcal{A}_0$) being an abelian variety.
- A has *semi-stable reduction* if $\mathcal{A}_0$ has no unipotent part: $F_2 = F_1$.
- Otherwise A has *bad reduction*.

The Néron-Ogg-Shafarevich Criterion

For ℓ a prime not equal to $\mathrm{char}(k)$, define the ℓ-adic *Tate module* to be $T_\ell(A) = \varprojlim A[\ell^n](\overline{K})$.

Theorem: A has good reduction if and only if $T_\ell(A)$ is unramified, i.e. the inertia subgroup $I_K \subset G_K$ acts trivially on it.

Proof: ($\Rightarrow$) If A has good reduction, $\mathcal{A}[\ell^n]$ is a finite flat group scheme. Since its order is invertible it is étale. It follows that the reduction map $\mathcal{A}[\ell^n](\overline{K}) \to \mathcal{A}_0[\ell^n](\overline{k})$ is an isomorphism, so $T_\ell(A)$ is unramified.

The Néron-Ogg-Shafarevich criterion

($\Rightarrow$) Suppose $T_\ell(A)$ is unramified. Then all ℓ^n-torsion is defined over K^{nr}, so $\mathcal{A}[\ell^n](\overline{K}) = \mathcal{A}[\ell^n](K^{nr})$ has cardinality ℓ^{2ng} where g is the dimension of A. The reduction map $\mathcal{A}[\ell^n](K^{nr}) \to \mathcal{A}(\overline{k})$ is injective, so the cardinality of $\mathcal{A}[\ell^n](\overline{k})$ is at least ℓ^{2ng}. Since $|\mathbb{G}_m[\ell^n]| = \ell^n$ and $|\mathbb{G}_a[\ell^n] = 1$, we see that $\mathcal{A}_0$ has no toric or unipotent part. Therefore $\mathcal{A}_0^\circ$ is an abelian variety and A has good reduction.

The semi-stable reduction theorem

Theorem (Grothendieck): A has semi-stable reduction if and only if the action of I_K on $T_\ell(A)$ is unipotent.
For elliptic curves this was proved last week. For abelian varieties, see SGA 7.1 Theorem 3.5.

Semi-stable reduction theorem: There exists a finite extension K'/K such that $A_{K'}$ has semi-stable reduction.

We will prove this only for finite extensions of $\mathbb{Q}_p$. We need two results:
The wild inertia subgroup $V_K \subseteq G_K$ is a pro-p group.
The tame inertia is topologically generated by τ and F with the single relation $F\tau F^{-1} = \tau^q$, where $q = |k|$.

Proof: Assume that K is a finite extension of $\mathbb{Q}_p$. We only need to show that $I_{K'}$ acts unipotently on $T_\ell(A)$. The proof works for any ℓ-adic representation V of G_K.
Since V_K is pro-p, its image in $GL(V)$ is finite. We can pass to a finite extension of K to get a trivial action of V_K.
τ is the generator of tame inertia and is conjugate to τ^q. If $\alpha_1, ..., \alpha_n$ are the eigenvalues of τ then $\alpha_i^q = \alpha_{\sigma(i)}$ for some $\sigma \in \mathfrak{S}_n$.
Since $\alpha_i^{q^n} = \alpha_i$ for all i, the α_i are roots of unity of order dividing $q^n - 1$. So we can pass to the extension $K' = K(\pi^{1/(q^n-1)})$, which has the effect of replacing τ with τ^{q^n-1}.
Then all eigenvalues of τ become 1, so the action of τ, and thus of all of inertia, becomes unipotent.

Corollaries

If $T_\ell(A)$ is trivial (or unipotent) as an I_K-module for one ℓ then it is for all $\ell \neq \mathrm{char}(k)$.

A has potentially good reduction if and only if I_K acts on $T_\ell(A)$ through a finite quotient.

Isogenous abelian varieties have the same reduction type.

Proof: If A and B are isogenous then $T_\ell(A)[1/\ell]$ and $T_\ell(B)[1/\ell]$ are isomorphic $\mathbb{Q}_\ell$-representations of I_K.