Finite Group Schemes over a DVR - 22/10/20

[c.f. Tate, Finite Flat Group Schemes].

Agenda:

- Finite flat gp schemes
- Generic fibre determines the f.f. gp scheme in low ramification
- Quasi-finite étale group schemes.

§1 Finite flat gp schemes

Let S be a locally Noetherian scheme.

A group scheme $/S$ is a group obj in the cat. of schemes $/S$.

A group scheme G/S is finite flat if $G \longrightarrow S$ is finite & flat, i.e. $\mathcal{O}_G$ is locally free of finite rank as an $\mathcal{O}_S$-mod.

This rank is a locally constant function on S, called the order of G/S.

If S is connected, the order $[G:S]$ is a constant number.

Remark:
$$\begin{tikzcd} G_1 \times_S G_2 \arrow[r] \arrow[d] & G_1 \arrow[d] \\ G_2 \arrow[r] & S \end{tikzcd} \qquad [G_1 \times_S G_2 : S] = [G_1 : S][G_2 : S]$$

$$\begin{tikzcd} G \times_S T \arrow[r] \arrow[d] & G \arrow[d] \\ T \arrow[r] & S \end{tikzcd} \qquad [G \times_S T : T] = [G : S]$$

From now on, let $S = \operatorname{Spec} R$ connected
$G = \operatorname{Spec} A$

<u>Examples of f.f gp schemes</u>

(a) The constant gp scheme

For a finite group T,

$$\underline{T} = \operatorname{Spec}\Big(\prod_T R\Big) \quad \text{order} = \# T.$$

(b) Roots of unity

$$\mu_n = \operatorname{Spec}\big(R[t]/(t^n - 1)\big) \quad \text{order } n.$$

(c) $\alpha_p = \operatorname{Spec}\big(R[t]/(t^p)\big)$, $\operatorname{char}(R) = p$
f.f. of order p.

<u>Properties</u>

① (Deligne) "The order kills the group".
If G/S is f.f. $\underbrace{\text{gp}}_{\text{commutative}}$ scheme of order n

$$\begin{tikzcd} G \arrow[rr, "{[n]}"] \arrow[dr] & & G \\ & S \arrow[ur, "e"'] & \end{tikzcd}$$

② We <u>say</u> a f.f. gp scheme G/S is <u>étale</u> if $G \to S$ is étale, i.e. $\Omega^1_{G/S} = 0$.

Every f.f. gp scheme G/S with order invertible on S is étale.

③ $\bar{s}$ geo. pt. of S

$$\left\{\begin{matrix}\text{finite étale gp} \\ \text{schemes}/S\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\text{finite gps with} \\ \text{cts } \pi_1(S, \bar{s})\text{-action}\end{matrix}\right\}$$

④ Cartier duality

For a f.f. comm. gp scheme $G = \operatorname{Spec} A$ over $S = \operatorname{Spec} R$,

$A^* = \operatorname{Hom}_R(A, R)$

$G^\vee = \operatorname{Spec}(A^*)$ finite flat gp scheme.

⑤ Quotients work:

"If $H \subset G$ f.f. gp schemes with H closed in G then G/H exists & f.f. of order $[G/H : S] = [G : S]/[H : S]$.

Γ G gp scheme$/S$

$H \subset G$ f.f. closed subgp scheme $/S$

$a : G \times_S H \longrightarrow G$ "right action of H on G"

restriction of $m : G \times G \to G$

"strictly free" $(id, a) : G \times_S H \longrightarrow G \times_S G$ closed imm.

A theorem of Grothendieck $\Rightarrow$

$\exists$ a scheme G/H over S

& a morphism $G \longrightarrow G/H$ <u>constant on orbits</u>

s.t. $G \longrightarrow X$ const on orbits

$$\begin{tikzcd} G \arrow[r] \arrow[d, squiggly, "\text{finite flat}"'] & X \\ \boxed{G/H} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

"quotient of G by H"

$[G : G/H] = [H : S]$.

G f.f. $\Rightarrow G/H$ f.f.

$\varphi : X \to Y$

$X \circlearrowleft H$

$\varphi(xh) = \varphi(x)$

$\forall x \in X(T)$

$h \in H(T)$

T scheme$/S$.

$$\begin{tikzcd} X \times_S H \arrow[r, "a"] \arrow[d, "pr"'] & X \arrow[d, "\varphi"] \\ X \arrow[r, "\varphi"] & Y \end{tikzcd}$$

$$[G : H] \overset{\text{def}}{=} [G/H : S] = \frac{[G : S]}{[H : S]}$$

G acts on G/H on the left

$$\begin{array}{ccc} G \times G & \xrightarrow{e \times q} & G \times G/H \\ \downarrow m & & \downarrow \text{left action.} \\ G & \xrightarrow{q} & G/H \end{array}$$

If H <u>normal</u> in G,

H acts trivially on G/H

$\rightsquigarrow$ get $G/H \underset{S}{\times} G/H \longrightarrow G/H$

$\rightsquigarrow$ G/H f.f. gp scheme $/S$.

(6) The connected – étale exact sequence over a Henselian local ring e.g. a field / complete DVR.

Let G be a f.f. gp scheme over $S = \operatorname{Spec} R$
$\;= \operatorname{Spec} A$ ... henselian local

(a) R henselian

$\Rightarrow A = \prod_i A_i$ prod. of local rings

S connected $\Rightarrow$ the image of $e : S \longrightarrow G$ lies in one of the $\operatorname{Spec} A_i$'s.

Denote $A_i =: A^\circ$ and $\operatorname{Spec} A^\circ = G^\circ$
"connected component of the identity in G"

Then G° is a flat closed normal subgp scheme.

(b) $G^{\text{ét}} := G/G^\circ$ finite étale.

(c) $0 \longrightarrow G^\circ \longrightarrow G \longrightarrow G^{\text{ét}} \longrightarrow 0$

characterised by the property that

$$\begin{tikzcd} G \arrow[r] \arrow[dr] & H \\ & G^{\text{ét}} \arrow[u] \end{tikzcd}$$

H an étale gp scheme/S

kernel $= G^{\circ}$

(d) The functors
$G \mapsto G^{\circ}$ & $G \mapsto G^{\text{ét}}$
are exact on the cat. of f.f. gp schemes.

Cor: (i) An extension of a $\left\{\begin{array}{l}\text{conn. f.f.}\\ \text{finite étale}\end{array}\right.$ gp sch by a $\left\{\begin{array}{l}\text{conn. f.f.}\\ \text{finite étale}\end{array}\right.$ gp scheme is $\left\{\begin{array}{l}\text{connected}\\ \text{étale}\end{array}\right.$.

[Apply $(-)^{\text{ét}}$ or $(-)^{\circ}$].

(ii) An ext of a connected f.f. gp scheme/S by a finite étale gp scheme is trivial.
e.g. $0 \to \underbrace{\mathbb{Z}/p\mathbb{Z}}_{\text{étale}} \to \mathbb{Z}/p\mathbb{Z} \times \mu_p \to \underbrace{\mu_p}_{\text{conn.}} \to 0$.

§2 Raynaud's prolongation theorem.
Let $K/\mathbb{Q}_p$ finite, k residue field
$e =$ ramification index of $K/\mathbb{Q}_p$.

~~Thm (Raynaud)~~

Def: A prolongation of a finite gp scheme G_0/K is a f.f. gp scheme $G/\mathcal{O}_K$ & an iso $G_K \xrightarrow{\sim} G_0$.

Thm (Raynaud) Suppose $e < p-1$.
Let G_0 be a finite comm. gp scheme/K.
Then any two prolongations of G_0 to $\mathcal{O}_K$ are isomorphic.

Remark: $K = \mathbb{Q}_p(\zeta_p)$, $e = p-1$.

Then two prolongations of $\mathbb{Z}/p\mathbb{Z}$

étale

$\mathbb{Z}/p\mathbb{Z}$ } isomorphic over K

μ_p } non-isomorphic over $\mathcal{O}_K$

conn.

Cor: Let $K/\mathbb{Q}_p$ finite with $e < p-1$.

A/K an ab. variety $/K$

$\mathcal{A}/\mathcal{O}_K$ its Néron model

Then the reduction map

$\mathcal{A}(\mathcal{O}_K)_{tors} \hookrightarrow \mathcal{A}(k)_{tors}$ is injective.

Pf: $G_0 = A(K)_{tors}$ const gp scheme

$G :=$ scheme-theoretic closure of G_0 in $\mathcal{A}$

$\rightsquigarrow$ flat gp scheme $/\mathcal{O}_K$

Néron mapping property $\Rightarrow$ G is finite $/\mathcal{O}_K$.

G_0 prolongs to a const gp scheme

$\xRightarrow{\text{Raynaud}}$ G constant gp scheme

$\Rightarrow G(\mathcal{O}_K) \hookrightarrow G(k)$. $\square$.

§3 $\underbrace{\text{Quasi-finite}}_{\text{fibres are finite}}$ étale gp schemes

Let R be a complete DVR

$K = \mathrm{Frac}(R)$

G quasi-finite étale gp sch $/R$.

$G_K \rightsquigarrow M = G(\bar{K})$ a $\mathrm{Gal}(\bar{K}|K)$-mod

$G_k \rightsquigarrow M_0 = G(\bar{k})$ a $\mathrm{Gal}(\bar{k}|k)$-mod

$M_0 \subseteq M^{I_K}$

<u>Fact</u>: $G \rightsquigarrow (M, M_0)$ equiv of cats.

$\mathrm{Gal}(\bar{K}|K)$-mod $\quad$ $\mathrm{Gal}(\bar{k}|k)$-mod

$M_0 \subseteq M^{I_K}$

- $G \rightsquigarrow (M, M_0)$ $\quad$ H closed $\Leftrightarrow N_0 = M_0 \cap N$

$\cup$ $\quad$ in this case,

$H \rightsquigarrow (N, N_0)$ $\quad$ $G/H \rightsquigarrow (M/N, M_0/N_0)$ q.f. étale.

- $G \rightsquigarrow (M, M_0)$ $\quad$ H max. closed in G which is finite$/R$

$H \rightsquigarrow (M_0, M_0)$ $\Rightarrow$ $H_k = G_k$

<u>Application</u>: E elliptic curve$/K$ with semistable reduction

$\mathcal{E}/R$ minimal Weierstrass model for E

$G = \mathcal{E}_{sm}[\ell^n]$ quasi-finite étale$/R$

$H :=$ max. finite subgp of G

$\Rightarrow H_k = G_k \Rightarrow H(\bar{k})$ contains a pt of order ℓ^n

ℓ^n-torsion in the elliptic curve or torus

$\uparrow \wr$ H étale

$H(K^{ur}) \subset E[\ell^n](K^{ur})$

$\Rightarrow T_\ell(E)^{I_K} \neq 0$

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