

# Finite Group Schemes over a DVR - 22/10/20

[c.f. Tate, Finite Flat Group Schemes].

## Agenda:

- Finite flat gp schemes
- Generic fibre determines the f.f. gp scheme in low ramification
- Quasi-finite étale group schemes.

## §1 Finite flat gp schemes

Let  $S$  be a locally Noetherian scheme.

A group scheme  $G/S$  is a group obj in the cat. of schemes  $/S$ .

A group scheme  $G/S$  is finite flat if  $G \rightarrow S$  is finite & flat,

i.e.  $\mathcal{O}_G$  is locally free of finite rank as an  $\mathcal{O}_S$ -mod.

This rank is a locally constant function on  $S$ , called the order of  $G/S$ .

If  $S$  is connected, the order  $[G:S]$  is a constant number.

Remark:

$$\begin{array}{ccc} G_1 \times_S G_2 & \rightarrow & G_1 \\ \downarrow & & \downarrow \\ G_2 & \rightarrow & S \end{array} \quad [G_1 \times_S G_2 : S] = [G_1 : S][G_2 : S]$$
$$\begin{array}{ccc} G \times_S T & \rightarrow & G \\ \downarrow & & \downarrow \\ T & \rightarrow & S \end{array} \quad [G \times_S T : T] = [G : S]$$

From now on, let  $S = \text{Spec } R$  connected  
 $G = \text{Spec } A$

### Examples of f.f. gp schemes

(a) The constant gp scheme

For a finite group  $T$ ,

$$\underline{T} = \text{Spec} \left( \prod_T R \right) \quad \text{order} = \#T.$$

(b) Roots of unity

$$\mu_n = \text{Spec} (R[t]/(t^n - 1)) \quad \text{order } n.$$

(c)  $\alpha_p = \text{Spec} (R[t]/(t^p - t))$ ,  $\text{char}(R) = p$   
 f.f. of order  $p$ .

### Properties

① (Deligne) "The order kills the group"

If  $G/S$  is f.f. gp scheme of order  $n$   
 commutative

$$\begin{array}{ccc} G & \xrightarrow{[n]} & G \\ & \searrow & \nearrow e \\ & S & \end{array}$$

② We say a f.f. gp scheme  $G/S$  is étale

if  $G \rightarrow S$  is étale, i.e.  $\Omega'_{G/S} = 0$ .

Every f.f. gp scheme  $G/S$  with order invertible on  $S$  is étale.

③  $\exists$  geo. pt. of  $S$

$$\left\{ \begin{array}{l} \text{finite étale gp} \\ \text{schemes } / S \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{l} \text{finite gps with} \\ \text{cts } \pi_1(S, \bar{s})\text{-action} \end{array} \right\}$$

#### ④ Cartier duality

For a f.f. comm. gp scheme  $G = \text{Spec } A$  over  $S = \text{Spec } R$ ,

$$A^* = \text{Hom}_R(A, R)$$

$G^v = \text{Spec}(A^*)$  finite flat gp scheme.

#### ⑤ Quotients work:

"If  $H \subset G$  f.f. gp schemes with  $H$  closed in  $G$  then  $G/H$  exists & f.f. of order  $[G/H : S] = [G : S] / [H : S]$ ."

$G$  gp scheme /  $S$

$H \subset G$  f.f. closed subgroup scheme

$a: G \times_S H \rightarrow G$  "right action of  $H$  on  $G$ "  
restriction of  $m: G \times G \rightarrow G$

"strictly free"  $(\text{id}, a): G \times_S H \rightarrow G \times_S G$  closed imm.

A theorem of Grothendieck  $\Rightarrow$

$\exists$  a scheme  $G/H$  over  $S$

& a morphism  $G \rightarrow G/H$  constant on orbits

s.t.  $G \rightarrow X$  const on orbits

$$\varphi: X \rightarrow Y$$

$$X \supset H$$

$$\varphi(xh) = \varphi(x)$$

$$\forall x \in X(T)$$

$$h \in H(T)$$

$T$  scheme /  $S$ .

$$X \times_S H \xrightarrow{a} X$$

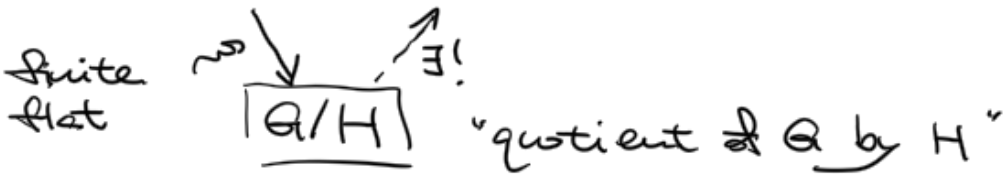
$$\text{pr} \downarrow$$

$$\downarrow \varphi$$

$$X \xrightarrow{\varphi} Y$$

$$[G : G/H] = [H : S].$$

$G$  f.f.  $\Rightarrow G/H$  f.f.





characterised by the property that

$$G \rightarrow H \leftarrow \text{étale gp scheme / } S$$

$$\begin{array}{ccc} & & \nearrow \\ G & \xrightarrow{\text{ét}} & H \\ \downarrow & & \uparrow \\ G^{\circ} & & \end{array} \quad \text{kernel} = G^{\circ}$$

(d) The functors

$$G \mapsto G^{\circ} \quad \& \quad G \mapsto G^{\text{ét}}$$

are exact on the cat. of f.f. gp schemes.

Cor: (i) An extension of a  $\begin{cases} \text{conn. f.f.} \\ \text{finite étale gp sch} \end{cases}$  by a  $\begin{cases} \text{conn. f.f.} \\ \text{finite étale gp scheme} \end{cases}$  is  $\begin{cases} \text{connected} \\ \text{étale} \end{cases}$ .

[Apply  $(-)^{\text{ét}}$  or  $(-)^{\circ}$ ].

(ii) An ext of a connected f.f. gp scheme /  $S$  by a finite étale gp scheme is trivial.

$$\text{e.g. } 0 \rightarrow \frac{\mathbb{Z}/p\mathbb{Z}}{\text{étale}} \rightarrow \frac{\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}}{\text{conn.}} \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow 0$$

§2 Raynaud's prolongation theorem.

Let  $K/\mathbb{Q}_p$  finite,  $k$  residue field

$e =$  ramification index of  $K/\mathbb{Q}_p$ .

~~Theorem~~

Def: A prolongation of a finite gp scheme  $G_0/K$  is a f.f. gp scheme  $G_1/\mathbb{O}_K$

& an iso  $G_1 \times_{\mathbb{O}_K} k \xrightarrow{\sim} G_0$ .

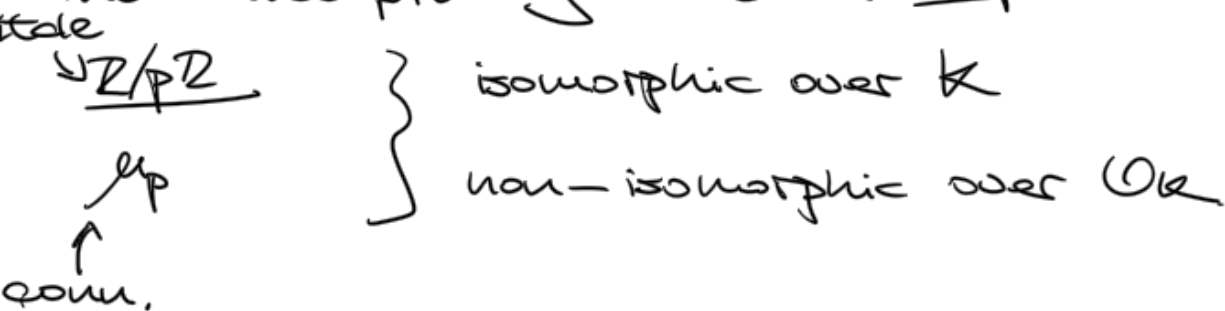
Thm (Raynaud) Suppose  $e < p-1$ .

Let  $G_0$  be a finite conn. gp scheme /  $k$ .

Then any two prolongations of  $G_0$  to  $\mathbb{O}_K$  are isomorphic.

Remark:  $K = \mathbb{Q}_p(\zeta_p)$ ,  $e = p-1$ .

Then two prolongations of  $\mathbb{Z}/p\mathbb{Z}$



Cor: Let  $K/\mathbb{Q}_p$  finite with  $e < p-1$ .

$A/K$  an ab. variety /  $K$

$A/\mathcal{O}_K$  its Néron model

Then the reduction map

$A(\mathcal{O}_K)_{\text{tors}} \hookrightarrow A(K)_{\text{tors}}$  is injective.

PF:  $\mathcal{G}_0 = A(K)_{\text{tors}}$  const gp scheme

$\mathcal{G} :=$  scheme-theoretic closure of  $\mathcal{G}_0$  in  $A$   
 $\mathcal{G}$  flat gp scheme /  $\mathcal{O}_K$ .

Néron mapping property  $\Rightarrow \mathcal{G}_i$  is finite /  $\mathcal{O}_K$ .

$\mathcal{G}_0$  prolongs to a const gp scheme

Raynaud  $\mathcal{G}_i$  constant gp scheme

$\Rightarrow \mathcal{G}(\mathcal{O}_K) \hookrightarrow \mathcal{G}(K)$ .

□

§ 3 Quasi-finite étale gp schemes  
 fibres are finite

Let  $R$  be a complete DVR

$K = \text{Frac}(R)$

$\mathcal{G}$  quasi-finite étale gp sch /  $R$

$\mathcal{G}_K \rightsquigarrow M = \mathcal{G}(R)$  a  $\text{Gal}(K|K)$ -mod

$\mathcal{G}_K \rightsquigarrow M_0 = \mathcal{G}(K)$  a  $\text{Gal}(K|K)$ -mod

