Ceresa cycles of curves with extra automorphisms

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- X : smooth projective variety over a field $k \subset \mathbb{C}$.
- For $0 \leq i \leq \dim(X)$ let

$$\mathcal{Z}^i(X) = \text{free abelian group on } \{[Z]\colon Z \subset X \text{ codimension } i\}.$$

- $\mathrm{CH}^i(X) = \mathcal{Z}^i(X)/\text{rational equivalence}$.
- Rational equivalence: generated by

$$[W \times \{0\}] - [W \times \{\infty\}], \quad W \in \mathcal{Z}^i(X \times \mathbb{P}^1)$$

- Set $\mathrm{CH}_i(X) = \mathrm{CH}^{\dim(X)-i}(X)$.
- The abelian groups $\mathrm{CH}^0(X), \mathrm{CH}^1(X), \ldots, \mathrm{CH}^{\dim(X)}(X)$ are fundamental invariants of X.

Examples

- $\mathrm{CH}^0(X) = \mathbb{Z}[X] \simeq \mathbb{Z}$.
- $\mathrm{CH}^1(X) \simeq \mathsf{Pic}(X)$, and $\mathsf{Pic}(X)$ fits into an exact sequence

$$0 \to \mathsf{Pic}^0(X) \to \mathsf{Pic}(X) \to \mathrm{NS}(X) \to 0,$$

  where $\mathrm{NS}(X)$ is finitely generated, and $\mathsf{Pic}^0(X)$ is an abelian variety.
- If $\dim X = 1$ and $k = \mathbb{C}$, have isomorphism

$$\mathsf{Pic}^0(X) \xrightarrow{\mathrm{AJ}} H^0(X, \Omega^1_X)^\vee / H_1(X, \mathbb{Z}) \simeq \mathbb{C}^g/\Lambda$$

- $\mathrm{CH}^i(\mathbb{P}^n) = \mathbb{Z}$ if $0 \leq i \leq n$.

In general, have a cycle class map

$$0 \to \mathrm{CH}^i(X)_{hom} \to \mathrm{CH}^i(X) \xrightarrow{\mathrm{cl}} \mathrm{H}^{2i}(X(\mathbb{C}), \mathbb{Z}).$$

and Abel–Jacobi map

$$\mathrm{AJ}\colon \mathrm{CH}^i(X)_{hom} \to J(H^{2i-1}(X)) = H^{2i-1}(X, \mathbb{C})/\mathrm{Fil}^i + H^{2i-1}(X, \mathbb{Z})$$

1. Image of cl: predicted by the Hodge conjecture.
2. AJ might not be injective/surjective: $\ker(\mathrm{AJ})$ is huge when $h^{2,0} \neq 0$ (Mumford).
3. Beilinson–Bloch conjecture: if k is a number field, then $\mathrm{rank}(\mathrm{CH}^i(X)_{hom}) = \mathrm{ord}_{s=1} L(H^{2i-1}(X)(i), s)$.

We study $\mathrm{CH}^i(X)$ using filtration

$$\mathrm{CH}^i(X)_{alg} \subset \mathrm{CH}^i(X)_{hom} \subset \mathrm{CH}^i(X)$$

$\mathrm{CH}^i(X)_{alg}$ = generated by $[W \times \{t_0\}] - [W \times \{t_1\}]$, where $W \in \mathrm{CH}^i(X \times T)$, (T, t_0, t_1) pointed variety.

Definition

The Griffiths group is $\mathrm{Griff}^i(X) := \mathrm{CH}^i(X)_{hom}/\mathrm{CH}^i(X)_{alg}$.

If $i = 1$, then $\mathrm{Griff}^i(X) = 0$.

Griffiths

$\mathrm{Griff}^2(X) \neq 0$ for a very general quintic 3-fold.

The Ceresa cycle

- C: smooth projective curve of genus $g \geq 2$
- $p \in C(k)$ point
- J: Jacobian variety of C
- $\iota_p \colon C \hookrightarrow J, x \mapsto [x] - [p]$: Abel–Jacobi map
- The *Ceresa cycle based at p* is the 1-cycle

$$\kappa_p(C) = [\iota_p(C)] - (-1)^*[\iota_p(C)] \in \mathrm{CH}_1(J),$$

where $(-1) \colon J \to J$ denotes the inversion map.

Lemma

$\kappa_p(C) \in \mathrm{CH}_1(J)_{hom}$, i.e. $\kappa_p(C)$ is homologically trivial.

Proof.

(-1) acts as the identity on $H_2(J)$. $\square$

$$\kappa_p(C) = [\iota_p(C)] - (-1)^*[\iota_p(C)] \in \mathrm{CH}_1(J)_{hom}.$$

Let $\kappa_{\mathrm{Gr}}(C)$ be image of $\kappa_p(C)$ in $\mathrm{Griff}_1(J)$ (doesn't depend on p).

Motivating question

When is $\kappa_p(C), \kappa_{Gr}(C)$ zero, or torsion?

Why do we care?

- Basic question about geometry of curves
- Evidence for Beilinson–Bloch
- Testing ground for knowledge of algebraic cycles

First observations

Lemma

If C is hyperelliptic and p a Weierstrass point, then $\kappa_p(C) = 0$.

Reason: (-1) restricts to the hyperelliptic involution on $C = \iota_p(C)$.

Lemma

If $\kappa_p(C)$ is torsion in $\mathrm{CH}_1(J)$, then $(2g-2)p = K_C$ in $\mathrm{CH}_1(C) \otimes \mathbb{Q}$.

We therefore get rid of the basepoint and define

Definition (Canonical Ceresa cycle)

$\kappa(C)$: image of $\kappa_e(C)$ in $\mathrm{CH}_1(J) \otimes \mathbb{Q}$, where e is a degree-1 divisor with $(2g-2)e = K_C$.

Fact: $\kappa(C)$ is independent of the choice of e.

Upshot: get canonical elements

$$\kappa(C) \in \mathrm{CH}_1(J) \otimes \mathbb{Q}, \quad \kappa_{\mathrm{Gr}}(C) \in \mathrm{Griff}_1(J).$$

Question

Can we describe

$$V_g^{rat} = \{C \in \mathcal{M}_g : \kappa(C) = 0\}, \tag{1}$$

$$V_g^{alg} = \{C \in \mathcal{M}_g : \kappa_{\mathrm{Gr}}(C) \text{ torsion}\}? \tag{2}$$

Formal: V_g^{rat}, V_g^{alg} are countable unions of closed subvarieties of $\mathcal{M}_g$.

$$\{\text{hyperelliptic locus}\} \subset V_g^{rat} \subset V_g^{alg} \subset \mathcal{M}_g$$

Theorem (Ceresa)

If $g \geq 3$, then a very general C over $\mathbb{C}$ has $\kappa_{\mathrm{Gr}}(C)$ infinite order. In other words, $V_g^{alg} \neq \mathcal{M}_g$.

Theorem (Bloch, Harris)

The Fermat curve has $\kappa_{\mathrm{Gr}}(C)$ of infinite order.

Therefore $\mathrm{Griff}^2(J) \neq 0$ for this curve!

Clemens' question

Does $\kappa_{\mathrm{Gr}}(C) = 0$ imply C is hyperelliptic?

Spirit of the question: *Ceresa only vanishes when there is a good geometric reason.*

Recent results of curves with torsion Ceresa cycle:
Bisogno–Litt-Li–Srinivasan, Beauville–Schoen, Qiu–Zhang, Lilienfeldt–Shnidman...

Chow vanishing criterion

Let H^* denote singular cohomology with $\mathbb{Q}$-coefficients. Let $G = \mathrm{Aut}(C)$.

Theorem (L.-Shnidman)

If $H^3(J)^G = 0$, then $\kappa(C) = 0$ in $\mathrm{CH}_1(J) \otimes \mathbb{Q}$.

This strengthens a criterion of Qiu–Zhang, who required $(H^1(J)^{\otimes 3})^G = 0$. If $V = H^0(C, \Omega^1_C)$, then $H^3(J) = \wedge^3(V + \bar{V})$. Examples:

- $y^3 = x^4 + x$ (Beauville–Schoen, $G = C_9$)
- $y^3 = x^4 + 1$ (Qiu–Zhang, $G =$ order 48)
- $y^3 = x^5 + 1$ (Lilienfeldt–Shnidman, $G = C_{15}$)
- 1-parameter family in genus 4. (Qiu–Zhang)

Proof of Chow vanishing

Theorem (L.-Shnidman)

If $H^3(J)^G = 0$, then $\kappa(C) = 0$ in $\mathrm{CH}_1(J) \otimes \mathbb{Q}$.

Recall the Abel–Jacobi map

$$\mathrm{AJ}\colon \mathrm{CH}^p(X) \to J(H^{2p-1}(X))$$

Theorem (Beauville)

If $H^3(J)^G = 0$, then $\mathrm{AJ}(\kappa(C)) = 0$ in $J(H^{2g-3}(J)) \otimes \mathbb{Q}$.

Proof sketch.

$H^3(J)^G = 0 \Rightarrow H^{2g-3}(J)^G = 0$. By functoriality,
$\mathrm{AJ}(\kappa(C)) \in J(H^{2g-3}(J))^G \otimes \mathbb{Q} = 0$. $\square$

Goal: upgrade this proof to show $\kappa(C) = 0$.

Chow motives

We use the category of Chow motives $\mathrm{Mot}(k)$.

- Every X has a Chow motive $\mathfrak{h}(X) \in \mathrm{Mot}(k)$
- $\mathrm{Hom}(\mathfrak{h}(X), \mathfrak{h}(Y)) = \mathrm{CH}^{\dim(X)}(X \times Y) \otimes \mathbb{Q}$
- If $p \in \mathrm{End}(M)$ idempotent, have decomposition $M = \ker(p) \oplus \ker(1-p)$ in $\mathrm{Mot}(k)$.

Just like X, a motive M has

- Cohomology groups $H^*(M)$
- Chow groups $\mathrm{CH}^*(M)$

Every $\mathfrak{h}(X)$ is expected to have a decomposition

$$\mathfrak{h}(X) = \bigoplus_{i=0}^{2\dim X} \mathfrak{h}^i(X)$$

with $H^*(\mathfrak{h}^i(X)) = H^i(X)$. Known for abelian varieties.

Beauville decomposition for abelian varieties: we have $\mathrm{CH}^p(J)_{\mathbb{Q}} = \bigoplus_s \mathrm{CH}^p_{(s)}(J)$, where

$$\mathrm{CH}^p_{(s)}(J) = \{\alpha \in \mathrm{CH}^p(J)_{\mathbb{Q}} \colon (n)^*\alpha = n^{2p-s}\alpha \,\forall n \in \mathbb{Z}\}$$

Since $[C] \in \mathrm{CH}^{g-1}(J)$, we can write $[C] = [C]_0 + [C]_1 + \cdots + [C]_{g-1}$. So

$$\kappa(C) = [C] - (-1)^*[C] = 2[C]_1 + 2[C]_3 + \cdots$$

Deninger–Murre: have $\mathfrak{h}(J) = \oplus_{i=0}^{2g}\mathfrak{h}^i(J)$ with $\mathrm{CH}^p(\mathfrak{h}^i(J)) = \mathrm{CH}^p_{(2p-i)}(J)$.

Proof of Chow vanishing.

1. S.W. Zhang: $\kappa(C) = 0$ if and only if $[C]_1 = 0$
2. $[C]_1 \in \mathrm{CH}^{g-1}_{(1)}(J)^G = \mathrm{CH}^{g-1}(\mathfrak{h}^{2g-3}(J)^G) \simeq \mathrm{CH}^2(\mathfrak{h}^3(J)^G)$.
3. The motive $M = \mathfrak{h}^3(J)^G$ has $H^*(M) = 0$ by assumption and is finite-dimensional in the sense of Kimura. So $M = 0$ hence $\kappa(C) = 0$.

$\square$

Griffiths vanishing criterion

What about algebraic triviality? Recall

$$\mathrm{CH}_1(J)_{alg} \subset \mathrm{CH}_1(J)_{hom} \subset \mathrm{CH}_1(J)$$

and the element $\kappa_{\mathrm{Gr}}(C) \in \mathrm{Griff}_1(J) = \mathrm{CH}_1(J)_{hom}/\mathrm{CH}_1(J)_{alg}$.

Theorem (L.-Shnidman)

Let $G = \mathrm{Aut}(C)$ and $V = H^0(C, \Omega_C)$. Suppose $(\wedge^3 V)^G = 0$. Assume the Hodge conjecture for abelian varieties. Then $\kappa(C)$ is zero in $\mathrm{Gr}_1(J) \otimes \mathbb{Q}$.

Theorem (L.-Shnidman)

Suppose $H^0(J, \Omega^3_J)^{\mathrm{Aut}(C)} = 0$*. Assume the Hodge conjecture for abelian varieties. Then* $\kappa(C)$ *is zero in* $\mathrm{Gr}_1(J) \otimes \mathbb{Q}$*.*

Proof.

- Assumptions $\Rightarrow H^3(J)^G \simeq H^1(A)(-1)$ for some abelian variety A
- Hodge conjecture for $J \times A \Rightarrow \mathfrak{h}^3(J)^G \simeq \mathfrak{h}^1(A)(-1)$
- Therefore $\kappa(C)$ is element of

$$\mathrm{CH}^{g-1}(\mathfrak{h}^{2g-3}(J)^G) \simeq \mathrm{CH}^2(\mathfrak{h}^3(J)^G) \simeq \mathrm{CH}^1(\mathfrak{h}^1(A)) \simeq \mathrm{Pic}^0(A).$$

- Since every element of $\mathrm{Pic}^0(A)$ is algebraically trivial, done.

$\square$

Examples where this applies?

Picard curves

Consider genus 3 curves of the form

$$C_f \colon y^3 = f(x) = x^4 + ax^2 + bx + c.$$

There is an action of $G = C_3$ on $V = H^0(C, \Omega^1_C)$ with eigenvalues ω, ω, ω^2, where $\omega^3 = 1$. So $\wedge^3 V$ has eigenvalue ω, so $(\wedge^3 V)^G = 0$. Can we verify Hodge conjecture?

Theorem (L.-Shnidman)

$H^3(J_f)^{C_3} \simeq H^1(E_f)(-1)$, where $E_f \colon y^2 = 4x^3 - 27\mathrm{disc}(f)$.

Theorem (Schoen)

Hodge conjecture holds for the 4-fold $J_f \times E_f$.

Conclusion: $\kappa_{\mathrm{Gr}}(C_f)$ torsion for all Picard curves!

$$C_f\colon y^3 = f(x) = x^4 + ax^2 + bx + c.$$

When is $\kappa(C_f) = 0$ in $\mathrm{CH}_1(J_f) \otimes \mathbb{Q}$? A quartic has two invariants:

$$I(f) = a^2 + 12c, \tag{3}$$
$$J(f) = 72ac - 2a^3 - 27b^2, \tag{4}$$

which satisfy the relation $J(f)^2 = 4I(f)^3 - 27 \cdot \mathrm{disc}(f)$. In other words, $P_f = (I(f), J(f))$ lies on the elliptic curve

$$E_f\colon y^2 = 4x^3 - 27\mathrm{disc}(f).$$

Theorem (L.-Shnidman)

$\kappa(C_f) = 0$ if and only if $P_f \in E_f(k)$ is torsion.

Theorem (L.-Shnidman)

$\kappa(C_f) = 0$ if and only if $P_f = (I(f), J(f))$ on $E_f : y^2 = 4x^3 - 27\mathrm{disc}(f)$ is torsion.

Consequences:

- V_3^{rat} contains infinitely many positive-dimensional components, e.g. $y^3 = x^4 - 12x^2 + tx - 12$
- Torsion orders for $\kappa_p(C)$ are unbounded.

Sketch of proof.

1. Prove $H^3(J_f)^G \simeq H^1(E_f)$ using monodromy arguments.
2. Hodge conjecture for $J_f \times E_f$: proved by Schoen.
3. So $\mathfrak{h}^3(J_f)^G \simeq \mathfrak{h}^1(E_f)(-1)$. So $\kappa(C_f)$ "is" an element of $E_f(k)$.
4. Analyze Mordell–Weil group of $E_f \to \{\text{Moduli of Picard curves}\}$: free of rank 1 over $\mathbb{Z}[\omega]$, generated by P_f.

$\square$

Let X_G be the locus of plane quartics with automorphism group G.
Stratification:

$$\begin{tikzcd}[column sep=small]
 & & & & & \{\mathrm{Id}\} \arrow[d, dash] \arrow[dddlll, dash] & & & \dim = 6 \\
 & & & & & C_2 \arrow[d, dash] \arrow[dddlll, dash] \arrow[ddrr, dash] & & & \dim = 4 \\
 & & & & & C_2^2 \arrow[d, dash] & & & \dim = 3 \\
 & & C_3 \arrow[d, dash] \arrow[ddll, dash] & & & D_4 \arrow[dl, dash] \arrow[dr, dash] & & S_3 \arrow[dl, dash] & \dim = 2 \\
 & & C_6 \arrow[d, dash] & & G_{16} \arrow[dll, dash] \arrow[dr, dash] & & S_4 \arrow[dl, dash] \arrow[dr, dash] & & \dim = 1 \\
C_9 & & G_{48} & & & G_{96} & & \mathrm{GL}_3(\mathbb{F}_2) & \dim = 0
\end{tikzcd}$$

We determine when the generic curve with automorphism group G has torsion Ceresa:

Theorem (L.-Shnidman)

- $X_G \subset V_3^{\mathrm{rat}}$ *if and only if* $G = C_9$ *or* G_{48}.
- $X_G \subset V_3^{\mathrm{alg}}$ *if and only if* $G = C_3, C_6, C_9, G_{48}$.

Slogan

Ceresa cycle is controlled by motive $\mathfrak{h}^3(J)^G$.

- If $H^3(J)^G = 0$, then $\mathfrak{h}^3(J)^G = 0$, hence $\kappa(C) = 0$.
- If $H^0(\Omega^3_J)^G = 0$, then $\mathfrak{h}^3(J)^G \simeq \mathfrak{h}^1(A)(-1)$, so $\kappa(C)$ "comes from" divisors.