ref. Bushnell-Henniart / Bump

Study group Automorphy lifting Theorems;

April 8th!

$$\boxed{\text{representations of } GL_n K \quad (K \text{ local field})}$$

1. motivation
2. locally profinite groups
3. modulus character
4. Hecke Algebras
5. Principal series

1 Motivation: CFT (Hilbert, Takagi, Artin, Hasse, Tate, ...)

- K local NA field, $\mathrm{Art}_K : K^\times \hookrightarrow \mathrm{Gal}(\overline{K}/K)^{ab}$

image W_K^{ab}

20's

- F number field $\mathrm{Art}_F : F^\times \backslash \mathbb{A}_F^\times \twoheadrightarrow \mathrm{Gal}(\overline{F}/F)^{ab}$

kernel identity component

30's-50's
- Hecke L-functions & Artin L-functions

50's
- Harish-Chandra, Bargmann: rep's of semisimple Lie groups

70's $\rightsquigarrow$ Langlands: Automorphic reps & Galois reps

& Godement-Jacquet: Automorphic forms via harmonic analysis

2 Locally profinite groups

def Top group G is profinite if it is the inverse limit of finite discrete groups

examples

- Abs Galois grps, Weil groups, $\mathbb{A}_F^\times$, $GL_n\mathbb{A}^\times$, $GL_n F$
- K NA local field, $GL_n O_K \longrightarrow F^\times$
- F # numberfield

prop Top group G is profinite iff compact Hausdorff & totally disconnected

idea: $(\Leftarrow)$ $G \longrightarrow \varprojlim G/N$ cont^s open

N normal open compact

$H \leq G$ open

$N = \bigcap_{g \in G/H} gHg^{-1}$ $\Big\}$ image dense & closed $\longrightarrow$ surjective

$G \times G \longrightarrow G$ cont^s & compactness

def Top group G is <u>locally profinite</u> if $\exists 1$ has basis of open compact subgroups

examples:
- Algebraic group $\mathbb{G} \longrightarrow \mathbb{G}(K)$ loc. prof.
- $\mathbb{G}(\mathbb{A}_F)$
- Closed subgroups L.p. are L.p. — quotients L.p are L.p.

def (complex) representation $G \longrightarrow GL(V)$ is <u>smooth</u> if any $v \in V$ has open stabilizer (typically V infinite-dim.) $\longleftrightarrow$ $G \times V \to V$ conts where V discrete top.

example $G = \mathbb{Z}_p^\times$ $C^\infty(\mathbb{Z}_p^\times)$ smooth rep its dual is not smooth

<u>def</u> $V^\vee = (V^*)^{\text{smooth}} = \{\lambda \in V^* \mid \lambda \text{ fixed by compact open}\}$

<u>prop</u> (Jacquet) if V smooth + irreducible, and G/K countable for some K open compact, then Schur's lemma holds, i.e. $\mathrm{End}_G V = \mathbb{C}$

<u>proof</u> $v \in V \setminus \{0\}$, $\exists K$ compact open $v \in V^K$

V is countably generated by the G/K-translates of v.

evaluation at v map: $\mathrm{End}_G V \hookrightarrow V$

suppose $\phi \in \mathrm{End}_G V \setminus \mathbb{C}$, then ϕ transcendental$/\mathbb{C}$

$\{(\phi - \alpha)^{-1} \mid \alpha \in \mathbb{C}\}$ linearly indep. $\rightarrow\leftarrow$ $\square$

Prop (Frobenius reciprocity) $\underline{\mathrm{Smo}}_G$ category of smooth reps of G

G locally profinite, $H < G$ closed

Then Restriction $\mathrm{Res}^H_G : \underline{\mathrm{Smo}}_G \to \underline{\mathrm{Smo}}_H$

has right adjoint $\mathrm{Ind}^G_H : \underline{\mathrm{Smo}}_H \to \underline{\mathrm{Smo}}_G$

i.e. $\mathrm{Hom}_G(V, \mathrm{Ind}^G_H W) = \mathrm{Hom}_H(\mathrm{Res}^H_G V, W)$

And $\mathrm{Ind}^G_H(W) = \{ f : G \to W \mid f(hg) = hf(g) \ \forall h \in H \}$ smooth

i.e. $f(gk) = f(g)$

$\forall k \in K$ some compact open K

def G locally profinite, V representation.

V is **Admissable** if smooth and for all $K \subset G$ compact, V^K is finite dimensional

remark (Jacquet) def^n works over any alg closed field of char 0.

Mimics the following: for $G = GL_n \mathbb{R}$, $\mathfrak{g} = \mathfrak{gl}_n$
$K = O_n \mathbb{R}$

$$\left\{\begin{array}{l}\text{Unitary rep}^{n}\text{s} \\ \pi : G \longrightarrow GL(H) \\ H \text{ Hilbert space}\end{array}\right\} \xleftrightarrow{1:1} \left\{(\mathfrak{g}, K)\text{-modules}\right\}$$

"K-finite vectors"

$$\pi \longmapsto \{h \in H \mid \pi(K)h \text{ finite dim}^{e}\}$$

Facts: (Bernstein) smooth & irreducible $\Rightarrow$ Admissable

- $V \in \underline{Adm}_G \iff V^\vee \in \underline{Adm}_G$; moreover V irreducible iff $V^\vee$ is and $V^{\vee\vee} \xrightarrow{\sim} V$ canonically
- if G/H compact, $\mathrm{Ind}_H^G : \underline{Adm}_H \to \underline{Adm}_G$

proof $W \in \underline{Adm}_H$, $V = \mathrm{ind}_H^G W$ for $K \subset G$ compact, $H \backslash G / K$ finite, say represented by $g_1, \dots, g_n$

if $f \in V^K$ determined by $f(g_1), \dots, f(g_n)$

$f(g_i) \in \underbrace{W^{H \cap g_i K g_i^{-1}}}_{\text{finite dim.}}$. Hence V adm $\square$

3. Modulus character

Let G be locally profinite

PROP up to scaling there is a unique left Haar measure $\mu : \{\text{opens } G\} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$

denote $\int_G \cdot \, d\mu : C_c^\infty G \to \mathbb{C}$

compactly supported

for S open, $S \mapsto \mu(Sg)$ Also left Haar measure,

so $\mu(Sg) = \delta(g)\mu(S)$ for

$\delta : G \to \mathbb{C}^\times$ <u>modulus character</u>

def Say G unimodular if δ trivial

examples: • Abelian groups

• Profinite groups

• $GL_n E$ (E local field) + reductive groups

$GL_2 E$

$$GL_2 E = \bigsqcup_{a \le b} GL_2 \mathcal{O}_E \begin{pmatrix} \pi_E^a & 0 \\ 0 & \pi_E^b \end{pmatrix} GL_2 \mathcal{O}_E$$

(Cartan decomp.)

WTS $\delta \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} = 1 \quad g = \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}$

$$U = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{array}{l} c \in M_E \\ a, b, c, d \in \mathcal{O}_E \end{array} \right\}$$ Iwahori subgr.

$$\boxed{gUg^{-1} = (U^{-1})^t} \quad \blacksquare$$

Prop $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset GL_2 E$ is not unimodular

Proof $B = T \ltimes N$ Levi decomposition

$$T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \quad N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

δ_B trivial on N, $Z(B)$.

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \pi_E \end{pmatrix} \qquad g \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g^{-1} = \begin{pmatrix} a & \pi_E b \\ 0 & d \end{pmatrix}$$

$$\delta_B(g) = \mu(B_0 g) / \mu(B_0) = \frac{\mu(g^{-1} B_0 g)}{\mu(B_0)}$$

$B_0 = B \cap GL_2 \mathcal{O}_E$ $\quad \delta_B \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = |d/a|$ $\quad = |\pi_E| = \frac{1}{q}$

How to integrate over quotient

Prop $\theta : H \to \mathbb{C}^\times$ smooth character. Then there exists nonzero G-equivariant

$$c\text{-}\mathrm{Ind}_H^G(\theta) \to \mathbb{C}$$

iff $\underline{\theta = \delta_H^{-1}(\delta_G|_H)}$

[BH §3.4]
Bushnell-Henniart

Write as $f \mapsto \int_{H\backslash G} f(g)\, d\mu_{H\backslash G}$, where f

transforms as $\delta_H^{-1}\delta_G$ under left action H

PROP (DUALITY)

$$(c\mathrm{Ind}_H^G V)^\vee = \mathrm{Ind}_H^G(V^\vee \otimes \delta_H^{-1}\delta_G)$$

$$\begin{tikzcd} c\mathrm{Ind}_H^G V \times \mathrm{Ind}_H^G(V^\vee \otimes \delta_H^{-1}\delta_G) \arrow[r] & c\mathrm{Ind}_H^G \delta_H^{-1}\delta_G \arrow[d, "\int \mu_G"'] \\ & \mathbb{C} \end{tikzcd}$$

check this is perfect

$$I_H^G(V) = \mathrm{Ind}_H^G((\delta_H^{-1}\delta_G)^{1/2} \otimes V)$$

$$I_H^G(V^\vee) = I_H^G(V)^\vee \quad \text{if } G/H \text{ compact}$$

4 Hecke Algebras

fix G locally profinite unimodular, μ Haar measure

$C_c^\infty G$ compactly supported locally constant functions to $\mathbb{C}$.

def $V \in \underline{\mathrm{SMO}}_G$, $F \in C_c^\infty(G)$

$$F * v = \int_G F(g)\, g \cdot v \, d\mu(g)$$

$V = C_c^\infty(G)$ $\rightsquigarrow$ associative ring $\mathcal{H}(G)$ not unital (unless G discrete) compact open $K \subset G$

$$\mathcal{H}(G/K) := e_K \mathcal{H}(G) e_K = C_c^\infty(K \backslash G / K)$$

where $e_K = \frac{1}{\mu(K)} 1_K$ $\qquad e_K * e_K = e_K$

unital ring $\qquad [KgK] = \frac{1}{\mu(K)} 1_{KgK}$

<u>fact</u> (BH 54.3) bijection

$$\left\{ \begin{matrix} \text{irreducible smooth} \\ V \text{ with } V^K \neq 0 \end{matrix} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} \text{Simple} \\ \mathcal{H}(G/K)\text{-modules} \end{matrix} \right\}$$

<u>example</u> K NA local field

$G = GL_2 K$, $\Gamma = GL_2 \mathcal{O}_K$

$\mathcal{H}(G/\Gamma) \cong \mathbb{C}[S, S^{-1}, T]$

where

$$S = \left[\Gamma \begin{pmatrix} \pi_K & 0 \\ 0 & \pi_K \end{pmatrix} \Gamma\right]$$

$$T = \left[\Gamma \begin{pmatrix} \pi_K & 0 \\ 0 & 1 \end{pmatrix} \Gamma\right]$$

$$\left[T * T = \left[K \begin{bmatrix} \pi^2 & 0 \\ 0 & 1 \end{bmatrix} K\right] + (q+1) S\right.$$

5. Principal Series (for GL_2)

Suppose χ_1, χ_2 are smooth characters on K

$$B = \begin{pmatrix} * & * \\ & * \end{pmatrix} \subset GL_2 K = G$$

define $\chi_1 \times \chi_2 := I_B^G\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \chi_1(a)\chi_2(d) \right)$

- $\chi_1^{-1} \times \chi_2^{-1} = (\chi_1 \times \chi_2)^\vee$

$\chi_1 \times \chi_2 = \chi_2 \times \chi_1$

- $\chi_1 / \chi_2 \neq |\cdot|^{\pm}$ iff $\chi_1 \times \chi_2$ irreducible def Steinberg rep^n is kernel of $|\cdot|^{1/2} \times |\cdot|^{-1/2} \to \mathbb{C}$ denoted St (discrete series)

THM
- if $\chi_1/\chi_2 = |\cdot|$, $\chi_1 \times \chi_2$ has codim 1 irred. subrep $St \otimes \chi_2 |\cdot|^{1/2}$
- if $\chi_1/\chi_2 = |\cdot|^{-1}$, quotient by 1-dim subrepr is $St \otimes \chi_2 |\cdot|^{-1/2}$, (so $St^\vee = St$)
- if $\chi_1/\chi_2 \neq |\cdot|^{\pm 1}$, $\chi_1 \times \chi_2$ irreducible

& no further isomorphisms between these.

__def__ Say V irreducible is __principal series__ if it is a subquotient, equivalently a subrepresentation of some $\chi_1 \times \chi_2$

__prop__ Let $N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$

V is principal series iff

N-coinvariants are nonzero

$$V_N \neq \emptyset$$

THM every irreducible rep^{n} V of $GL_2 E$ with $V^K \neq 0$ is either

- $\chi_1 \times \chi_2$ for some unramified smooth characters $\chi_1/\chi_2 \neq |\cdot|^{\pm 1}$

OR

- 1-dim^{e} $\nu \circ \det$ where ν is unramified