Geometry of universal local lifting rings and some deformation problems

Notation

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$L/\mathbb{Q}_\ell$ finite extension with ring of integers $\mathcal{O} = \mathcal{O}_L$, uniformizer λ and residue field $\mathbb{F}$.

Γ is a profinite subgroup satisfying condition Φ_ℓ, $\overline{\rho} \colon \Gamma \longrightarrow \mathsf{GL}_n(\mathbb{F})$ is a continuous representation.

K is a finite extension of $\mathbb{Q}_p$ with residue field k.

The reference [BLGGT] is T. Barnet-Lamb, T. Gee, D. Geraghty, and R. Taylor, *Potential automorphy and change of weight*.

Recollections

In the previous weeks we've seen that given a deformation problem $\mathcal{S} = (F, S, \mathcal{O}, \overline{\rho}, \chi, \{\mathcal{D}_v\}_{v\in S})$ and $T \subseteq S$, then $R_{\mathcal{S}}^{\square_T}$ is the quotient of a power series ring in $d = \dim H^1_{\mathcal{S},T}(G_{F,S}, \mathsf{ad}^0\,\overline{\rho})$ variables over

$$R^{\mathsf{loc}}_{\mathcal{S},T} = \widehat{\bigotimes}_{v\in T} \left(R_{\overline{\rho}|_{G_v},\chi} / I(\mathcal{D}_v) \right),$$

and

$$\begin{aligned} d \geq & \#T - \sum_{v|\infty} \dim H^0(G_{F_v}, \mathsf{ad}^0\,\overline{\rho}) \\ & + \sum_{v\in S\setminus T} \left(\dim L(\mathcal{D}_v) - \dim H^0(G_{F_v}, \mathsf{ad}^0\,\overline{\rho}) \right) \\ & - \dim H^0(G_{F,S}, \mathsf{ad}^0\,\overline{\rho}(1)). \end{aligned}$$

Today we will do two things:

1. We'll study the generic fibers of universal lifting (or deformation) rings.
2. We will consider certain deformation problems that we are interested in for applications.

Generic fibers of deformation rings

We will start by looking at $\operatorname{Spec} R_{\overline{\rho}}^{\square}[\frac{1}{\ell}]$.

There is a bijection

$$\begin{array}{rcl} \{\text{maximal ideals of } R_{\overline{\rho}}^{\square}[\frac{1}{\ell}]\} & \longleftrightarrow & \{\phi\colon R_{\overline{\rho}}^{\square} \longrightarrow \mathcal{O}_{L'} \text{ where } L' = L(\phi(R_{\overline{\rho}}^{\square})) \\ & & \text{ is a finite extension of } L\} \\ \mathfrak{m} & \longmapsto & (R_{\overline{\rho}}^{\square} \longrightarrow R_{\overline{\rho}}^{\square}[\frac{1}{\ell}] \longrightarrow R_{\overline{\rho}}^{\square}[\frac{1}{\ell}]/\mathfrak{m}) \\ (\ker\phi)[\frac{1}{\ell}] & \longleftarrow\!\shortmid & \phi \end{array}$$

So closed points of $\operatorname{Spec} R_{\overline{\rho}}^{\square}[\frac{1}{\ell}]$ correspond to lifts of $\overline{\rho}$ to finite extensions of L.

If $x \in \operatorname{Spec} R_{\overline{\rho}}^{\square}[\frac{1}{\ell}]$ is a closed point, write $\phi_x\colon R_{\overline{\rho}}^{\square} \longrightarrow L_x$ for the corresponding map and $\rho_x\colon \Gamma \longrightarrow R_{\overline{\rho}}^{\square} \xrightarrow{\phi_x} \mathsf{GL}_n(\mathcal{O}_{L_x})$ for the corresponding lift.

Lemma.

Let $\mathcal{C}_{L_x}$ be the category of Artinian local L_x-algebras with residue field L_x. Then, the completion of $R_{\overline{\rho}}^{\square}[\frac{1}{\ell}]$ at x,

$$R_{\overline{\rho}}^{\square}\left[\frac{1}{\ell}\right]_x^{\wedge} := \varprojlim_j R_{\overline{\rho}}^{\square}\left[\frac{1}{\ell}\right] / \left(\ker \phi_x \left[\frac{1}{\ell}\right]\right)^j$$

represents the functor

$$\begin{aligned} \mathscr{R}_{\rho_x}^{\square} : \mathcal{C}_{L_x} &\longrightarrow \mathsf{Sets} \\ A &\longmapsto \{\rho\colon \Gamma \longrightarrow \mathsf{GL}_n(A) \mid \rho \mod \mathfrak{m}_A = \rho_x\}. \end{aligned}$$

What does $\mathrm{Spec}\, R_{\overline{\rho}}^{\square}[\frac{1}{\ell}]$ look like?

In the unobstructed case, $R_{\overline{\rho}}^{\square} \simeq \mathcal{O}[[X_1, ..., X_d]]$. We have

$$(\mathrm{Spec}\, \mathcal{O}[[X_1, ..., X_d]])(\overline{\mathbb{F}}_\ell) = \{0\},$$
$$(\mathrm{Spec}\, \mathcal{O}[[X_1, ..., X_d]])(\overline{\mathbb{Q}}_\ell) = \{(x_1, ..., x_d) \in \overline{\mathbb{Q}}_\ell \mid |x_i| < 1\}.$$

Fact.

The closed points of $\mathrm{Spec}\, R_{\overline{\rho}}^{\square}[\frac{1}{\ell}]$ are Zariski dense.

In order to say anything meaningful about $\operatorname{Spec} R^{\square}_{\overline{\rho}}[\frac{1}{\ell}]$, we must specialize to the case when $\Gamma = G_K$ is the absolute Galois group of a finite extension K of $\mathbb{Q}_p$.

We must treat the cases $\ell \neq p$ and $\ell = p$ separately.

The case $\ell \neq p$

Assume $\ell \neq p$.

Lemma.

Each irreducible component of $\operatorname{Spec} R^{\square}_{\overline{\rho}}[\frac{1}{\ell}]$ is generically formally smooth over L of dimension n^2.

Idea of proof.

The first step is to show that the points x such that $H^0(G_K, (\mathrm{ad}\, \rho_x)(1)) = 0$ is Zariski dense [BLGGT, Lemma 1.3.2 (2)]. For such an x, local Tate duality and the Euler-Poincaré characteristic formula show $\dim H^0(G_K, \mathrm{ad}\, \rho_x) = \dim H^1(G_K, \mathrm{ad}\, \rho_x)$ and $H^2(G_K, \mathrm{ad}\, \rho) = 0$. The same argument we did for $\overline{\rho}$ works for liftings of ρ_x to show $R^{\square}_{\overline{\rho}}[\frac{1}{\ell}]^{\wedge}_x \simeq L_x[[y_1, ..., y_{n^2}]]$. □

We can similarly show:

Lemma.

If $\chi\colon \Gamma \longrightarrow \mathcal{O}^\times$ lifts $\det \overline{\rho}$, then each irreducible component of $\operatorname{Spec} R^{\square}_{\overline{\rho},\chi}[\frac{1}{\ell}]$ is generically formally smooth over L of dimension $n^2 - 1$.

Lemma.

If $\overline{\rho}$ is Schur, then each irreducible component $\operatorname{Spec} R^{\square}_{\overline{\rho},\chi}[\frac{1}{\ell}]$ is generically formally smooth over L of dimension 1

Let $\mathcal{C}$ be a non-empty subset of irreducible components of $\operatorname{Spec} R^{\square}_{\overline{\rho}}[\frac{1}{\ell}]$ and χ a lift of $\det \overline{\rho}$.

Definition.

We write $R^{\square}_{\overline{\rho},\chi,\mathcal{C}}$ for the largest reduced and ℓ-torsion free quotient of $R^{\square}_{\overline{\rho}}$ such that $\operatorname{Spec} R^{\square}_{\overline{\rho},\chi,\mathcal{C}}$ is contained in $\mathcal{C}$.

Lemma.

If $I_{\mathcal{C}} = \ker(R^{\square}_{\overline{\rho},\chi} \longrightarrow R^{\square}_{\overline{\rho},\chi,\mathcal{C}})$, then $\mathcal{D}(I_{\mathcal{C}})$ is a deformation problem. Moreover, $R^{\square}_{\overline{\rho},\chi,\mathcal{C}}$ is equidimensional of dimension n^2.

Proof.

The hard part is showing that $\ker(\mathsf{GL}_n(R^{\square}_{\overline{\rho},\chi}) \longrightarrow \mathsf{GL}_n(\mathbb{F}))$ preserves the connected components of $\operatorname{Spec} R^{\mathsf{univ}}_{\overline{\rho}}[\frac{1}{\ell}]$, which is shown in [BLGGT, Lemma 1.2.2]. □

Let W_K be the Weil group of K and $I_K \subseteq W_K$ the interia subgroup. Recall that a Weil-Deligne representation is a triple (V, r, N) where $r\colon W_K \longrightarrow \mathsf{GL}(V)$ is a continuous representation (i.e. has open kernel) satisfying for $\sigma \in W_K$, $r(\sigma)Nr(\sigma)^{-1} = (\#k)^{-v(\sigma)}N$.

Definition

An interial type is an object of the form $(V, r|_{I_K}, N)$ where (V, r, N) is a Weil-Deligne representation. A Weil-Deligne representation has type $\tau = (V, r_0, N)$ if it is isomorphic to a Weil-Deligne representation (V, r, N) with $r_0 = r|_{I_K}$.

If (V, r_0, N) is an interial type, then r_0 has open kernel and r_0 and N commute, but the converse isn't true in general.

Example.

There are four cases for (types of) 2-dimensional Weil-Deligne representations over $\mathbb{C}$:

1. Unramified up to twist: $\tau = (\psi \oplus \psi, 0)$ for some character $\psi \colon I_K \longrightarrow \mathbb{C}^\times$.
2. Steinberg: $\tau = (\psi \oplus \psi, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})$ for ψ as above.
3. Split ramified: $\tau = (\psi_1 \oplus \psi_2, 0)$ for distinct characters $\psi_1, \psi_2 \colon I_K \longrightarrow \mathbb{C}^\times$.
4. Irreducible: $\tau = (r_0, 0)$ where r_0 is the restriction of some irreducible Weil-Deligne representation.

Assume $n = 2$.

Theorem.

After possibly enlarging L:

1. Every irreducible component $\mathcal{C}$ of $\operatorname{Spec} R^{\square}_{\overline{\rho}}[\frac{1}{\ell}]$ has an associated type $\tau_{\mathcal{C}}$ such that:
   - If $\tau_{\mathcal{C}}$ is not Steinberg, then closed points correspond to ρ_x whose Weil-Deligne representation is of type $\tau_{\mathcal{C}}$.
   - If $\tau_{\mathcal{C}}$ is Steinberg, then closed points correspond to (possibly split) extensions
   $$0 \longrightarrow \psi(1) \longrightarrow \rho_x \longrightarrow \psi \longrightarrow 0$$
   for some $\psi \colon G_K \longrightarrow L_x^{\times}$.
2. Each case (1)-(4) occurs in at most one component, except if $\overline{\rho} = \overline{\psi}_1 \oplus \overline{\psi}_2$ for distinct $\overline{\psi}_1, \overline{\psi}_2$ such that $\overline{\psi}_1 \overline{\psi}_2^{-1}$ is unramified, in which case there are two components which are split ramified (case (3)).

Theorem (continued).

3. Two components intersect only when one is unramified and the other is Steinberg and for any x in the intersection, $\rho_x = \psi(1) \oplus \psi$ for some character $\psi\colon G_K \longrightarrow L_x^\times$.
4. Each component is formally smooth

Proof.

See Section 4 of Vincent Pilloni's *The study of 2-dimensional p-adic Galois deformations in the $\ell \neq p$ case*. □

Example: Taylor-Wiles liftings

Proposition.

Assume $\overline{\rho}\colon G_K \longrightarrow \mathsf{GL}_n(\mathbb{F})$ is unramified with distinct eigenvalues, that χ is an unramified lift of $\det \overline{\rho}$ and that $\#k \equiv 1 \mod \ell$. Let ℓ^m be the largest power of ℓ dividing $\#k - 1$. Then

$$R^{\square}_{\overline{\rho},\chi} \simeq \mathcal{O}[[x, y, B, u]]/((1+u)^{\ell^m} - 1).$$

Proof.

For any lift $\rho\colon G_K \longrightarrow \mathsf{GL}_n(A)$, the image of wild inertia $P_K \subseteq G_K$ lands in $\ker(\mathsf{GL}_n(A) \longrightarrow \mathsf{GL}_n(\mathbb{F}))$, so its image is trivial. We can choose a lift of the Frobenius $\varphi \in G_K$ and a topological generator of tame inertia I_K/P_K satisfying $\varphi^{-1}\sigma\varphi = \sigma^{\#k}$. ρ is determined by $\rho(\sigma), \rho(\varphi)$.

Proof (continued).

Let $\overline{\alpha}, \overline{\beta}$ be the eigenvalues of $\overline{\rho}(\varphi)$, and $\alpha, \beta \in \mathcal{O}$ some lifts. An application of Hensel's lemma shows there exist $B, C, u, v, w, z \in \mathfrak{m}_A$ such that

$$\rho(\varphi) = \begin{pmatrix} 1 & y \\ x & 1 \end{pmatrix}^{-1} \begin{pmatrix} \alpha + B & 0 \\ 0 & \beta + C \end{pmatrix} \begin{pmatrix} 1 & y \\ x & 1 \end{pmatrix},$$

$$\rho(\sigma) = \begin{pmatrix} 1 & y \\ x & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 + u & v \\ w & 1 + z \end{pmatrix} \begin{pmatrix} 1 & y \\ x & 1 \end{pmatrix}.$$

From $\varphi^{-1}\sigma\varphi = \sigma^{\#k}$ we deduce $v = w = 0$. If $\det \rho = \chi$ then $C = -\beta + \chi(\psi)/(\alpha + B)$, $1 + z = (1 + u)^{-1}$ and we see $(1 + u)^{\#k} = 1 + u$. This implies $(1 + u)^{\ell^m} = 1$.

Proof (continued).

Thus,

$$\rho(\varphi) = \begin{pmatrix} 1 & y \\ x & 1 \end{pmatrix}^{-1} \begin{pmatrix} \alpha + B & 0 \\ 0 & \chi(\psi)/(\alpha + B) \end{pmatrix} \begin{pmatrix} 1 & y \\ x & 1 \end{pmatrix},$$
$$\rho(\sigma) = \begin{pmatrix} 1 & y \\ x & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 + u & 0 \\ 0 & (1+u)^{-1} \end{pmatrix} \begin{pmatrix} 1 & y \\ x & 1 \end{pmatrix}.$$

These formulas can be used to define $\rho^{\square}$. □

The case $\ell = p$

As is to be expected, the case $\ell = p$ is more complicated. There are many more p-adic representations of G_K than ℓ-adic representations (for $\ell \neq p$) as wild inertia can act in complicated ways. In order to study them, we need to recall the basis of p-adic Hodge theory.

p-adic Hodge theory defines a hierarchy of p-adic representations

$$\text{crystalline} \implies \text{semistable} \implies \text{de Rham} \implies \text{Hodge-Tate}$$

and attaches to a representation in each of these categories some semilinear algebraic data.

We have the following analogy:

abelian var. over K	**ℓ-adic rep.**	**p-adic rep.**	**semilinear data**
		Hodge-Tate	graded vector spaces
		$\Uparrow$	
any red.	all rep.	de Rham	filtered vector spaces
$\Uparrow$	$\Uparrow$	$\Uparrow$	
semistable red.	unipotent	semistable	adm. filtered (φ, N)-modules
$\Uparrow$	$\Uparrow$	$\Uparrow$	
good red.	unramified	crystalline	adm. filtered φ-modules

We say that a representation is potentially $*$ if it becomes $*$ after restricting to $G_{K'}$ for some finite extension K'/K.

For example, potentially de Rham is the same as de Rham, and the p-adic monodromy theorem asserts that potentially semistable is the same as de Rham.

For example, the category of p-adic representations with coefficients in L of G_K which become semistable after restriction to $G_{K'}$ for a finite extension K'/K is equivalent to a certain subcategory of filtered $(\varphi, N, \mathsf{Gal}(K'/K))$-modules over L (for L sufficiently large), defined as follows:

Let K'_0 be the maximal unramified subextension of K'/K, and let $\varphi_0 \colon K'_0 \longrightarrow K'_0$ be the absolute Frobenius. Then a filtered $(\varphi, N, \mathsf{Gal}(K'/K))$ is a tuple $(D, \varphi, N, \mathsf{Fil}^\bullet D_{K'})$ where:

- D is a finite free $K_0 \otimes_{\mathbb{Q}_p} L$-module,
- $\varphi \colon D \longrightarrow D$ is a $\varphi_0 \otimes 1$-semilinear bijection,
- $N \colon D \longrightarrow D$ is a nilpotent operator satisfying $N\varphi = p\varphi N$,
- D has a semilinear action of $\mathsf{Gal}(K'/K)$ commuting with φ and ρ,
- $\mathsf{Fil}^\bullet D_{K'}$ is a decreasing filtration of $D \otimes_{K'_0 \otimes_{\mathbb{Q}_p} L} (K' \otimes_{\mathbb{Q}_p} L)$.

To such an object we can associate a Weil-Deligne representation. Hence, we can associate a Weil-Deligne representation to any potentially semistable representation.

If $\sigma \colon K' \hookrightarrow L$ is an embedding, the filtration $\mathrm{Fil}^\bullet D_{K'}$ induces a filtration on $D \otimes_{K'_0,\sigma} L$. The set HT_σ of indices where this filtration jumps are called the Hodge-Tate weights of $(D, \varphi, N, \mathrm{Fil}^\bullet D_{K'})$.

Everything is the same for (potentially) crystalline representations; this corresponds to the case $N = 0$.

Example.

1. The p-adic cyclotomic character is crystalline and all its Hodge-Tate weights are -1.
2. The p-adic Tate module V of an abelian variety over K has σ-Hodge-Tate weights $\{0, ..., 0, -1, ..., -1\}$. If K has good (resp. semistable, resp. any) reduction, then V is crystalline (resp. semistable, resp. de Rham).
3. If f is an eigenform of weight N, then the associated Galois representation has Hodge-Tate weights $\{0, k-1\}$. It always de Rham, and crystalline if $p \neq N$.

Theorem.

For each $\sigma\colon K' \hookrightarrow L$, let H_σ be a (multi)set of n integers. There exists a unique p-torsion free reduced quotient $R^{\square}_{\overline{\rho},\chi,\{H_\sigma\},K'\text{-st}}$ (resp. $R^{\square}_{\overline{\rho},\chi,\{H_\sigma\}_\sigma,K'\text{-cris}}$) of $R^{\square}_{\overline{\rho},\chi}$ such that a geometric point x of $\operatorname{Spec} R^{\square}_{\overline{\rho},\chi}[\frac{1}{p}]$ lies in the corresponding subscheme if and only if $HT_\sigma(\rho_x) = H_\sigma$ for all σ and $\rho_x|_{G_{K'}}$ is semistable (resp. crystalline).
Moreover, $\operatorname{Spec}(R^{\square}_{\overline{\rho},\chi,\{H_\sigma\},K'\text{-st}})$ is equidimensional of dimension $n^2 + [K : \mathbb{Q}_p]\frac{n(n-1)}{2}$ and its generic fiber is generically smooth.