Mar 11 @ ALT Study group

Global deformation problems : dimension computations

Ref for new input :
[Gal Coh] Serre, Gal. Coh.

Setup and some basic pictures :

- $\mathcal{O}$: r.o.i of a fin. ext'n of $\mathbb{Q}_\ell$ w/ uniformiser λ and residue field k
- $\mathcal{C}$: cat. of cplt noe. local $\mathcal{O}$-alg. (A, m) w/ marked $A/m \simeq k$
- Γ : profinite gp. satisfying finiteness assumption Φ_ℓ / Hyp(ℓ)
- $\bar{\rho} : \Gamma \to GL_n(k)$ abs. irred. rep'n
- $D^{\square}_{\bar{\rho}} : \mathcal{C} \to \text{Sets}$, $A \mapsto \{ \rho : \Gamma \to GL_n(A) \mid \rho \bmod m = \bar{\rho} \}$

  $D_{\bar{\rho}}$: --- --- ------ $/\sim$

- $\underline{\text{Thm}}$: $D^{\square}_{\bar\rho}$, $D_{\bar\rho}$ are representable by $R^{\square}_{\bar\rho}$, $R^{univ}_{\bar\rho} \in \mathcal{C}$

Presentation of $R^{\square}_{\bar\rho}$ and $R^{univ}_{\bar\rho}$:

- Tangent spaces: $\mathrm{Hom}_{\mathcal{C}}(R^{\square}_{\bar\rho}, k[\varepsilon]/(\varepsilon^2)) \simeq Z^1(\Gamma, \mathrm{ad}\bar\rho)$

$\mathrm{Hom}_{\mathcal{C}}(R^{univ}_{\bar\rho}, k[\varepsilon]/(\varepsilon^2)) \simeq H^1(\Gamma, \mathrm{ad}\bar\rho)$ naturally

$\Downarrow$ Nakayama

- $0 \to J^* \hookrightarrow \mathcal{O}[[x_1, x_2, \cdots, x_{d^*}]] \twoheadrightarrow R^*_{\bar\rho} \to 0$

$d^{univ} = h^1(\Gamma, \mathrm{ad}\bar\rho)$

$d^{\square} = h^1(\Gamma, \mathrm{ad}\bar\rho) + n^2 - 1$

- $\mathrm{Hom}_k(J^{\square}/m^{\square}J^{\square}, k) \hookrightarrow H^2(\Gamma, \mathrm{ad}\bar\rho)$ naturally

$\Downarrow$

$\dim R^{\square}_{\bar\rho} \geq n^2 + h^1(\Gamma, \mathrm{ad}\bar\rho) - h^2(\Gamma, \mathrm{ad}\bar\rho)$

$\dim R^{univ}_{\bar\rho} \geq 1 + h^1 - h^2$

Global deformation problems:

- F : # field, S : a finite set of places $\supset \{v \mid l\}$

  $\Gamma := G_{F,S}$, $\quad \Gamma_v := G_{F_v} \subset \Gamma$ $(v \in S)$

- Fix $\triangle$ $\psi : \Gamma \to \mathcal{O}^\times$ lifting $\det \bar\rho$

  $\triangle$ $\forall v \in S$, $\quad \mathcal{D}_v \subseteq D^{\square}_{\bar\rho|_{\Gamma_v}}$ a deformation problem

  $$\Updownarrow$$

  $I(\mathcal{D}_v) \subset R^{\square}_{\bar\rho|_{\Gamma_v}}$ ideal

- For $\mathcal{S} = (F, S, \mathcal{O}, \bar\rho, \psi, \{\mathcal{D}_v\}_{v\in S})$ and $T \subseteq S$, $D^{\square_T}_{\mathcal{S}}$ is the functor

$$\mathcal{C} \to \text{Sets}$$

$$A \mapsto \left\{ (\rho, \{\alpha_v\}_{v\in T}) \;\middle|\; \begin{array}{l} \rho : \Gamma \to GL_n(A) \text{ lifts } \bar\rho, \ \det \rho = \psi \\ \rho|_{\Gamma_v} \in \mathcal{D}_v(A), \ \forall v \in S \\ \alpha_v \in \ker(GL_n(A) \to GL_n(k)), \ \forall v \in T \end{array} \right\} / \sim$$

where $(\rho, \{\alpha_v\}) \sim (\beta\rho\beta^{-1}, \{\beta\alpha_v\})$, $\forall\, \beta \in \ker(GL_n(A) \to GL_n(k))$.

- $D_{\mathcal{S}}^{\square_T}$ is representable by $R_{\mathcal{S}}^{\square_T} \in \mathcal{C}$

- There's a natural map

$$\left(R^{loc}_{\mathcal{S},T}, m^{loc}\right) := \widehat{\bigotimes_{v\in T}} \left(R^{\square}_{\bar\rho|_{\Gamma_v},\psi} / I(D_v)\right) \to \left(R^{\square_T}_{\mathcal{S}}, m\right)$$

- It's natural to study $R_{\mathcal{S}}^{\square_T}$ via its presentation / $R^{loc}_{\mathcal{S},T}$

We'd like some cohomology whose

last time

- $H^1 \cong$ (relative) tangent space

$$\left\{ \begin{tikzcd} R^{loc}_{\mathcal{S},T} \arrow[r] \arrow[d] & k \arrow[d] \\ R^{\square_T}_{\mathcal{S}} \arrow[r, dashed, "e"] & k[\varepsilon]/(\varepsilon^2) \end{tikzcd} \right\} \cong \mathrm{Hom}_k\left(m/(m^2, m^{loc}, \lambda), k\right)$$

- H^2 controls obstructions

· Define

$\triangle$ $\mathcal{C}_0^i = \begin{cases} \mathcal{C}^0(\Gamma, \mathrm{ad}\,\bar\rho) & i=0 \\ \mathcal{C}^i(\Gamma, \mathrm{ad}^0\bar\rho) & i>0 \end{cases}$

$GL_n \xrightarrow{\mathrm{ad}^0} \mathrm{Aut}(sl_n)$

$D_v(k[\varepsilon]/(\varepsilon^2))$

$\mathrm{Hom}(R^{\square}_{\bar\rho|\Gamma_v}/I(D_v), k[\varepsilon]/(\varepsilon^2)) =: \tilde{L}_v$

$\triangle$ $\mathcal{C}^i_{\mathcal{S},T,\mathrm{loc}} = \begin{cases} \bigoplus_{v\in T} \mathcal{C}^0(\Gamma_v, \mathrm{ad}\,\bar\rho) & i=0 \\ \bigoplus_{v\in S} \mathcal{C}^1(\Gamma_v, \mathrm{ad}^0\bar\rho) \Big/ \bigoplus_{v\in S\setminus T} Z^1(\Gamma_v, \mathrm{ad}^0\bar\rho)[I(D_v)] & i=1. \\ \bigoplus_{v\in S} \mathcal{C}^i(\Gamma_v, \mathrm{ad}^0\bar\rho) & i\geq 2 \end{cases}$

Then an extension

$$0 \to \mathcal{C}^{\cdot-1}_{\mathcal{S},T,\mathrm{loc}} \to \mathcal{C}^{\cdot}_{\mathcal{S},T} \to \mathcal{C}^{\cdot}_0 \to 0$$

gives the correct $H^1 =: H^1_{\mathcal{S},T}$

$$\Downarrow$$

$$0 \to J \to R^{\mathrm{loc}}_{\mathcal{S},T}[[x_1, x_2, \cdots, x_{h^1_{\mathcal{S},T}}]] \twoheadrightarrow R^{\square_T}_{\mathcal{S}} \to 0$$

- Moreover similarly $\mathrm{Hom}_k(J/m^{loc}J, k) \hookrightarrow H^2_{\mathfrak{g},T}$

<u>Today's goal</u>: compute $h^{1,2}_{\mathfrak{g},T}$

Techniques: – Euler-Poincaré char.

– Vanishing

– Duality

- Vanishing: Δ for $v<\infty$, $cd(\Gamma_v) = 2$ [Gal Coh, II.5.3]

Δ when $l>2$, $\underset{scd_l \text{ or } scd_l - 1}{\underset{\parallel}{cd_l(\Gamma)}} = 2$ [Gal Coh, II.4.4]

$cd_2(\Gamma) = \begin{cases} \infty & \text{when } S \text{ contains a real place since } cd(\mathbb{Z}/2) = \infty; \\ 2 & \text{otherwise} \end{cases}$

Δ $H^i(G_\mathbb{R}, M) = 0$ for $i>0$ when M is l-primary for $l>2$

- Assume $l>2$, $\Rightarrow H^i_{\mathfrak{g},T} = 0$ for $i>3$.

- Assume $l \nmid n$, $\Rightarrow 0 \to \mathrm{ad}^0\bar{\rho} \to \mathrm{ad}\,\bar{\rho} \to k \to 0$ splits

- Convention for Euler char.: $\chi_* := \sum_i (-1)^{i+1} h^i_*$, $\chi_m := \prod_i (\# H^i)^{(-1)^{i+1}}$

- $\chi_{\mathcal{S},T} = \chi_0 - \chi_{\mathcal{S},T,loc}$

  $\chi_0 = \chi(\Gamma, \mathrm{ad}^0\bar{\rho}) - 1$

  image of $\tilde{L}(D_v)$ in $H^1(\Gamma_v, \mathrm{ad}\,\bar{\rho})$

  $\|$

  $\chi_{\mathcal{S},T,loc} = \sum_{v\in S} \chi(\Gamma_v, \mathrm{ad}^0\bar{\rho}) - \# T + \sum_{v\in S\setminus T} (h^0(\Gamma_v, \mathrm{ad}^0\bar{\rho}) - \dim \mathcal{L}(D_v))$

- [Gal Coh, II.5.7] For $v < \infty$ and finite M, $\chi_m(\Gamma_v, M) = \#(\mathcal{O}_v/\# M)$

  $$\Rightarrow \chi(\Gamma_v, \mathrm{ad}^0\bar{\rho}) = \begin{cases} 0 & v \nmid l \\ (n^2-1)[F_v : \mathbb{Q}_l] & v \mid l \end{cases}$$

- For f.d. M/k, $\chi(\Gamma, M) = \dim_k M \cdot [F:\mathbb{Q}] - \sum_{v\in S, v\mid\infty} h^0(\Gamma_v, M)$

$$\Rightarrow \quad \chi(\Gamma, \mathrm{ad}^0\bar\rho) = \sum_{v\in S} \chi(\Gamma_v, \mathrm{ad}^0\bar\rho),$$

$$\chi_{S,T} = \#T - 1 + \sum_{v\in S\setminus T} (\dim \mathcal{L}(D_v) - h^0(\Gamma_v, \mathrm{ad}^0\bar\rho))$$

- [Gal Coh, II.5.2] Local duality: for $v<\infty$ and finite M,

$$H^i(\Gamma_v, M') \cong H^{2-i}(\Gamma_v, M)^\vee \qquad i=0,1,2,$$

where $M' = \mathrm{Hom}(M, \mathbb{G}_m)$, in particular $(\mathrm{ad}^0\bar\rho)' = \mathrm{ad}^0\bar\rho(1)$

- [GalCoh, II.6.3] Poitou-Tate Thm: for finite M there's an exact seq.

$$0 \to H^0(\Gamma, M) \to \bigoplus_{v\in S} H^0(\Gamma_v, M) \to H^2(\Gamma, M')^\vee$$

$$\to H^1(\Gamma, M) \to \bigoplus_{v\in S} H^1(\Gamma_v, M) \to H^1(\Gamma, M')^\vee$$

$$\to H^2(\Gamma, M) \to \bigoplus_{v\in S} H^2(\Gamma_v, M) \to H^0(\Gamma, M')^\vee \to 0$$

$$H^i(\Gamma, M) \xrightarrow{\sim} \bigoplus_{v\in S} H^i(\Gamma_v, M), \quad i \geq 3$$

· $0 \to \mathcal{C}^{\cdot-1}_{\mathcal{S},T,\mathrm{loc}} \to \mathcal{C}^{\cdot}_{\mathcal{S},T} \to \mathcal{C}^{\cdot}_{0} \to 0$ gives an exact seq.:

$$H^1(\Gamma, \mathrm{ad}^0\bar{\rho}) \to \bigoplus_{v\in S} H^1(\Gamma_v, \mathrm{ad}^0\bar{\rho}) \Big/ \bigoplus_{v\in S\setminus T} \mathcal{L}(D_v) \to H^2_{\mathcal{S},T}$$

$$\to H^2(\Gamma, \mathrm{ad}^0\bar{\rho}) \to \bigoplus_{v\in S} H^2(\Gamma_v, \mathrm{ad}^0\bar{\rho}) \to H^3_{\mathcal{S},T} \to 0$$

Comparing with the Poitou-Tate seq. gives

- $H^3_{\mathcal{S},T} \cong H^0(\Gamma, \mathrm{ad}^0\bar{\rho}(1))^\vee$
- $H^2_{\mathcal{S},T}$ is dual to $\ker\left(H^1(\Gamma, \mathrm{ad}^0\bar{\rho}(1)) \to \bigoplus_{v\in S\setminus T} \dfrac{H^1(\Gamma_v, \mathrm{ad}^0\bar{\rho}(1))}{\mathcal{L}(D_v)^\perp} \right) =: H^1_{\mathcal{S},T}(\Gamma, \mathrm{ad}^0\bar{\rho}(1))$

Besides we have $H^0_{\mathcal{S},T} \cong H^0(\Gamma, \mathrm{ad}\,\bar{\rho})$ has dim 1.

· Summing up, we have $\qquad /\!/\, h^2_{\mathcal{S},T} \qquad \sum_{v\in S_{na}\setminus T} - \sum_{v\mid\infty} h^0$

$$\boxed{h^1_{\mathcal{S},T}} - h^1_{\mathcal{S},T}(\Gamma, \mathrm{ad}^0\bar{\rho}(1)) = \sum_{v\in S\setminus T} \left(\dim \mathcal{L}(D_v) - h^0(\Gamma_v, \mathrm{ad}^0\bar{\rho})\right) - h^0(\Gamma, \mathrm{ad}^0\bar{\rho}(1))$$

· **Prop** $\dim R_{\mathcal{S}}^{\text{univ}} \geqslant 1 + \sum_{v \in S} \left(\dim R^{\square}_{\bar\rho|_{\Gamma_v}, \psi} / I(D_v) - n^2 \right) - h^0(\Gamma, \mathrm{ad}^0 \bar\rho(1))$

$\square_\emptyset$

$\| \quad \dim L(D_v) - h^0(\Gamma_v, \mathrm{ad}^0 \bar\rho)$

· **Prop** For fin. ext'n F'/F, compatible sets S' of places and $\{D_{v'}\}_{v' \in S'}$ and
$\mathcal{S}' = (F', S', \mathcal{O}, \bar\rho|_{G_{F',S'}}, \psi|_{G_{F',S'}}, \{D_{v'}\})$,

$$R_{\mathcal{S}'}^{\text{univ}} \longrightarrow R_{\mathcal{S}}^{\text{univ}} \text{ is finite}$$