EA Global: Enter Arithmetic and Global deformation problems

Vaughan McDonald

March 4, 2021

Outline

Motivation: dimension and modular forms

A refinement from the theorem last time:

Conjecture (Mazur)

If $G = G_{F,S}$ *for* $S \supseteq \{\nu \mid \infty\} \cup \{\nu \mid p\}$ *and* $\bar{\rho} : G_{F,S} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ *cts irred.,*

$$\dim R_{\bar{\rho}} = 1 + h^1(G_{F,S}, \mathrm{ad}\,\bar{\rho}) - h^2(G_{F,S}, \mathrm{ad}\,\bar{\rho}).$$

Remark

This is not true for arbitrary profinite G! Something very arithmetic. When $n = 1$, equivalent to Leopoldt's conjecture.

For $f = \sum a_n q^n \in S_2^{\mathrm{new}}(\Gamma_0(N))$ Carayol's lemma $\implies$, $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ can be lifted to a rep with coeffs in $\mathbb{T}_f$. Want to show something like $R^{\square}_{\bar{\rho}_f} \xrightarrow{\sim} \mathbb{T}_f$. Immediate problem: the ρ_f has special properties! For example, determinant and ramification information, these will cut subschemes smaller than the dimension above.
Idea: to capture modular forms, add conditions to our deformations.

Notation

For this talk:

Notation

- Fix a number field F, a prime p, a finite set of places $S \supseteq \{\nu \mid p\}$. Let F_S be the maximal extension of F unramified $S \cup \{\nu \mid \infty\}$, and $G_{F,S} := \mathrm{Gal}(F_S/F)$.
- Let $\mathcal{O}$ is the ring of integers of a finite extension of $\mathbb{Q}_p$ with maximal ideal λ, and $\mathbb{F} := \mathcal{O}/\lambda$, $\mathcal{C}_{\mathcal{O}}$ will be the category of complete Noetherian (not Artinian!) local $\mathcal{O}$-algebras.
- For a field K, let $\varepsilon_p : G_K \to \mathbb{Z}_p^\times$ be the p-adic cyclotomic character.
- For a representation $\bar{\rho} : \Gamma \to \mathrm{GL}_n(\mathbb{F})$, assume $p \nmid 2n$.
- Denote $\mathbb{F}[\varepsilon] := \mathbb{F}[x]/(x^2)$

Deformation conditions

Definition

A **deformation condition/problem** $\mathcal{D} \subseteq \mathcal{D}_{\bar{\rho}}^{\square}$ is a collection of lifts $\rho : \Gamma \to \mathrm{GL}_n(A)$ for $A \in \mathcal{C}_{\mathcal{O}}$ such that

1. $(\mathbb{F}, \bar{\rho}) \in \mathcal{D}$
2. If $\phi : A \to B \in \mathrm{Hom}_{\mathcal{C}_{\mathcal{O}}}(A, B)$, then $(A, \rho) \in \mathcal{D} \implies (B, \phi \circ \rho) \in \mathcal{D}$
3. If $\phi : A \hookrightarrow B \in \mathrm{Hom}_{\mathcal{C}_{\mathcal{O}}}(A, B)$ is an injection then $(A, \rho) \in \mathcal{D} \iff (B, \phi \circ \rho) \in \mathcal{D}$.
4. If $(A, \rho_A), (B, \rho_B) \in \mathcal{D}$ and there are maps $A, B \to C$ in $\mathcal{C}_{\mathcal{O}}$, then $(A \times_C B, \rho_A \times \rho_B) \in \mathcal{D}$.
5. For an inverse system $(A_i, \rho_i) \in \mathcal{D}$ such that $\varprojlim A_i \in \mathcal{C}_{\mathcal{O}}$ then $(\varprojlim_i A_i, \lim_i \rho_i) \in \mathcal{D}$.
6. $(A, \rho) \in \mathcal{D}, a \in \ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(\mathbb{F})) \implies (A, a\rho a^{-1}) \in \mathcal{D}$.

Remark

We take lifts rather than deformations, so we need not assume $\bar{\rho}$ be Schur.

Deformation conditions are "closed subschemes:"

Lemma

Let $R_{\bar\rho}^{\square} \twoheadrightarrow R$ be a surjection in $\mathcal{C}_{\mathcal{O}}$ such that $(*)$: for any lift $\rho : \Gamma \to \mathrm{GL}_n(A)$ to $A \in \mathcal{C}_{\mathcal{O}}$ and $g \in 1 + M_n(\mathfrak{m}_A)$, the induced map $R^{\square} \to A$ factors through R iff $g\rho g^{-1}$ factors through R. Then the lifts that factors through R form a deformation problem, and every deformation problem $\mathcal{D}$ arises this way

Proof.

Given $\mathcal{D}$, want an $I \subset R_{\bar\rho}^{\square}$. Let $\mathcal{I} = \{I \subseteq R_{\bar\rho}^{\square} : (R_{\bar\rho}^{\square}/I, \rho^{\square} \mod I) \in \mathcal{D}\}$.
Conditions: $(1) \implies \mathcal{I} \neq \emptyset$; $(2),(3) \implies (A,\rho) \in \mathcal{D} \iff \ker(R^{\square} \to A) \in \mathcal{I}$; $+\ (5) \implies \mathcal{I}$ closed under nested intersection; $(4)+(5) \implies \mathcal{I}$ closed under finite intersections. Zorn's lemma produces a minimal $I(\mathcal{D}) \in \mathcal{I}$.
$(6) \implies R := R^{\square}/I(\mathcal{D})$ satisfies $(*)$. First part is easy. $\square$

Remark

Want bijection between def. problems $\mathcal{D}$ and $1 + M_n(\mathfrak{m}_{R_{\bar\rho}^{\square}})$-invariant (radical) ideals $I \subseteq R_{\bar\rho}^{\square}$, but I don't see why $I(\mathcal{D})$ is radical. See [BLGHT, 3.1-3.2].

Deformation conditions: examples

Let $\bar{\rho} : \Gamma \to \mathrm{GL}_n(\mathbb{F})$ be a continuous representation.

Example (Determinant)

Fix a continuous character $\psi : \Gamma \to \mathcal{O}^\times$ such that $\psi \mod \lambda \equiv \det \bar{\rho}$. Then take $D_{\bar{\rho}}^{\square,\psi} : \mathcal{C}_{\mathcal{O}} \to \mathsf{Set}$, $A \mapsto \{(\rho, A) \in D_{\bar{\rho}}^{\square} : \det \rho = \Gamma \xrightarrow{\psi} \mathcal{O}^\times \hookrightarrow A^\times\}$. This is conjugation invariant, and so also defines $D_{\bar{\rho}}^{\psi} \subseteq D_{\bar{\rho}}$.
$D_{\bar{\rho}}^{\square,\psi}$ is a deformation. problem, represented by $R_{\bar{\rho}}^{\square,\psi} = R_{\bar{\rho}}^{\square}/J$, where $J = (\{\det \rho^{\square}(\sigma) - \psi(\sigma) : \sigma \in \Gamma\})$, and same for $D_{\bar{\rho}}^{\psi}$ and $R_{\bar{\rho}}^{\psi}$.
Easy to show $D_{\bar{\rho}}^{\square,\psi}(\mathbb{F}[\varepsilon]) \simeq Z^1(\Gamma, \mathrm{ad}^0 \bar{\rho})$, $D_{\bar{\rho}}^{\psi}(\mathbb{F}[\varepsilon]) \simeq H^1(\Gamma, \mathrm{ad}^0 \bar{\rho})$, (second part uses $p \nmid n$), where $\mathrm{ad}^0 \bar{\rho}$ denotes trace 0.

Theorem

(Elliptic curves): *If $E/\mathbb{Q}$ is an elliptic curve then* $\det \rho_{E,p} \simeq \varepsilon_p$.
(Modular forms): *If $f \in S_2(\Gamma_0(N))$ then* $\det \rho_f \simeq \varepsilon_p$

Deformations: local examples

Example (D^{ord})

For $K/\mathbb{Q}_p$ finite, define $\bar{\rho} : G_K \to \mathrm{GL}_2(\mathbb{F})$ by $g \mapsto \begin{pmatrix} \bar{\chi}_1(g) & * \\ 0 & \bar{\chi}_2(g) \end{pmatrix}$ where $\bar{\rho}(I_K) \neq 1$ and $\bar{\chi}_1 |_{I_K} = 1$.
Fixing cts $\psi : I_K \to \mathcal{O}^\times$, define $D^{\mathrm{ord}} : \mathsf{CNL}_{\mathcal{O}} \to \mathsf{Set}$ as lifts ρ to A strictly equivalent to $\begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$ with $\chi_1 |_{I_K} = 1, \chi_2 |_{I_K} = \psi$.
$\mathcal{D}^{\mathrm{ord}}$ is a deformation problem, and when $\bar{\chi}_1\bar{\chi}_2^{-1} \neq 1, \bar{\varepsilon}_p$ in fact is represented by $R^{\mathrm{ord}} \simeq \mathcal{O}_K[[x_1, \ldots, x_g]]$, $g = 4 + [K : \mathbb{Q}_p]$ (compute with Tate duality).

Theorem

(Elliptic curves): *If an elliptic curve $E/\mathbb{Q}$ has good ordinary reduction at p ($p \nmid a_p$) then $\rho_{E,p} \in \mathcal{D}^{\mathrm{ord}}$.*
(Modular forms): *If $f \in S_2(\Gamma_0(N))$ then $\rho_f |_{G_{\mathbb{Q}_p}} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathcal{O}_{K'}) \in \mathcal{D}^{\mathrm{ord}}$ iff $a_p \in \mathcal{O}_{K'}^\times$.*

Global deformation problems

Definition

A **global deformation problem** is the data $\mathcal{S} = (\bar{\rho}, S, \psi, \mathcal{O}, \{\mathcal{D}_\nu\}_{\nu \in S})$ where $\bar{\rho} : G_{F,S} \to \mathrm{GL}_n(\mathbb{F})$ is Schur, $\psi : G_{F,S} \to \mathcal{O}^\times$ has $\psi \equiv \det \bar{\rho} \mod \lambda$, and $\mathcal{D}_\nu$ is a deformation problem for $\bar{\rho}\,|_{G_\nu}$. We say a lift $\rho : G_{F,S} \to \mathrm{GL}_n(A)$ of $\bar{\rho}$ to A is **of type $\mathcal{S}$** if:

- $\det \rho = \psi$
- $\rho\,|_{G_{F_\nu}} \in D_\nu(A)$ for all $\nu \in S$.

A deformation is **of type $\mathcal{S}$** if any of its lifts are.

Definition

A **T-framed deformation** of type $\mathcal{S}$ to $A \in \mathcal{C}_{\mathcal{O}}$ is a strict equivalence class of tuples $(\rho, \{\beta_\nu\}_{\nu \in T})$ for $\rho : \Gamma \to \mathrm{GL}_n(A)$ a lift of $\bar{\rho}$ of type $\mathcal{S}$ and $\beta_\nu \in \ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(\mathbb{F}))$. Strict equivalence means for $\alpha \in \ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(\mathbb{F}))$, $(\rho, \{\beta_\nu\}_{\nu \in T}) \sim (\alpha\rho\alpha^{-1}, \{\alpha\beta_\nu\}_{\nu \in T})$ (so that $\beta_\nu^{-1}\rho\,|_{G_{F_\nu}}\,\beta_\nu \in D_\nu$ is well-defined).

T-framed deformations

Lemma

The functor $D_{\mathcal{S}}^{\square_T} : \mathcal{C}_{\mathcal{O}} \to \mathsf{Set}$ sending A to T-framed deformations of $\bar{\rho}$ of type $\mathcal{S}$is representable by $R_{\mathcal{S}}^{\square_T} \in \mathcal{C}_{\mathcal{O}}$. In fact, $\mathcal{R}_{\mathcal{S}}^{\square_T} \simeq R_{\mathcal{S}}[[x_1, \ldots, x_{n^2|T|-1}]]$ (non-canonically)

Proof.

We saw fixings determinants representable, so suffices to show local properties + framing representable. Assume for now $\mathcal{T} = \emptyset$. Choosing a lift ρ of the universal $R_{\bar{\rho}}^{\psi}$-deformation and taking $\rho\mid_{G_{F_\nu}}$ gives maps $R^{\square}_{\bar{\rho}|_{G_{F_\nu}}} \to R_{\bar{\rho}}^{\psi}$. By pushout, this gives a map $R_{\mathcal{S}}^{\square} := \widehat{\bigotimes}_{\nu \in S} R^{\square}_{\bar{\rho}|_{G_{F_\nu}}} \to R_{\bar{\rho}}^{\psi}$. Also define $R_{\mathcal{S}}^{\text{loc}} := \widehat{\bigotimes}_{\nu \in S} R_\nu$. Then $D_{\mathcal{S}}$ is represented by $R_{\bar{\rho}}^{\psi} \otimes_{R_{\mathcal{S}}^{\square}} R_{\mathcal{S}}^{\text{loc}}$ (quotient b/c R_ν is quotient of $R^{\square}_{\bar{\rho}|_{G_{F_\nu}}}$). Independent of choice of ρ since the D_ν are deformation problems (condition (6)). To add T-framing, specifying $|T|$ $n \times n$ matrices (subtract 1 for scaling equivalence). See [CHT, 2.2.9]. □

Presentation over local lifting rings

Recall

We showed $\dim R_{\bar\rho} \geq 1 + h^1(G, \mathrm{ad}\,\bar\rho) - h^2(G, \mathrm{ad}\,\bar\rho)$. Can we get a similar theory for $R_{\mathcal{S}}^{\square_T}$?

Note $\beta_\nu^{-1}\rho^{\square_T}|_{G_{F_\nu}}\beta_\nu : G_{F_\nu} \to \mathrm{GL}_n(R_{\mathcal{S}}^{\square_T}) \in D_\nu$ is independent of choice of strict equivalence of T-frame $\implies$ a map $R_\nu \to R_{\mathcal{S}}^T$, which induces a **canonical** map

$$R_{\mathcal{S},T}^{\mathrm{loc}} := \widehat{\bigotimes}_{\nu\in T} R_\nu \to R_{\mathcal{S}}^{\square_T}.$$

Now set $\mathfrak{m} := \mathfrak{m}_{R_{\mathcal{S}}^{\square_T}}$ and $\mathfrak{m}^{\mathrm{loc}} := \mathfrak{m}_{R_{\mathcal{S},T}^{\mathrm{loc}}}$.
Goal: compute the relative tangent of this presentations, i.e. $\dim_{\mathbb{F}} \mathrm{Hom}_{\mathbb{F}}(\mathfrak{m}/(\mathfrak{m}^2, \mathfrak{m}^{\mathrm{loc}}, \lambda), \mathbb{F})$. This will be Galois cohomology subject to local conditions, so more like a Selmer group.

Constructing a complex

We need to deal with 3 adjustments, which we tackle one at a time.

1. Adding framing ($T \neq \emptyset$);
2. Taking relative tangent space (deal with places $\nu \in T$);
3. Adding in general local conditions at $\nu \in S \setminus T$.

(1): Assume $\mathcal{D}_\nu = \mathcal{D}^{\square}_{\rho|_{G_{F_\nu}}}$ are trivial. What is $\mathrm{Hom}_{\mathbb{F}}(\mathfrak{m}/(\mathfrak{m}^2, \lambda), \mathbb{F}) \simeq \mathcal{D}_S^{\square_T}(\mathbb{F}[\varepsilon])$? Just fixing determinants, so:

$$\begin{aligned}(\rho, (\alpha_\nu)_{\nu \in T})/\sim &= (Z^1(G_{F,S}, \mathrm{ad}^0\,\bar{\rho}) \oplus \bigoplus_{\nu \in T}(1 + \varepsilon M_n(\mathbb{F})))/\sim \\ &= \mathrm{coker}\left(\mathrm{ad}\,\bar{\rho} \xrightarrow{\partial \oplus \Delta_T} Z^1(G_{F,S}, \mathrm{ad}^0\,\bar{\rho}) \oplus \bigoplus_{\nu \in T} \mathrm{ad}\,\bar{\rho}\right),\end{aligned}$$

where $\Delta_T(a) = (a, a, \ldots, a)$.

(2): relative tangent space

Claim

For $\{D_\nu\}_{\nu \in S \setminus T}$ *trivial,* $\mathrm{Hom}_{\mathbb{F}}(\mathfrak{m}/(\mathfrak{m}^2, \mathfrak{m}^{\mathrm{loc}}, \lambda), \mathbb{F})$ *is*

$$\ker\left((Z^1(G_{F,S}, \mathrm{ad}^0\,\bar{\rho}) \oplus \bigoplus_{\nu \in T} \mathrm{ad}\,\bar{\rho}) / \operatorname{im}(\partial \oplus \Delta_T) \xrightarrow{\oplus_{\nu \in T} \mathrm{res}_\nu \oplus (-\partial)} \bigoplus_{\nu \in T} Z^1(F_\nu, \mathrm{ad}^0\,\bar{\rho}) \right)$$

Proof.

Note $\mathfrak{m}/(\mathfrak{m}^2, \mathfrak{m}^{\mathrm{loc}}, \lambda) = \operatorname{coker}(\mathfrak{m}^{\mathrm{loc}}/((\mathfrak{m}^{\mathrm{loc}})^2, \lambda) \to \mathfrak{m}/(\mathfrak{m}^2, \lambda))$, so we compute $\ker(\mathcal{D}_{\mathcal{S}}^{\square_T}(\mathbb{F}[\varepsilon]) \to \bigoplus_{\nu \in T} D_\nu(\mathbb{F}[\varepsilon]))$, which sends a lift (independent of class) $(\rho = (1 + \phi\varepsilon)\bar{\rho}, \alpha_\nu = 1 + T_\nu \varepsilon) \mapsto \alpha_\nu^{-1} \rho \alpha_\nu$, where $T_\nu \in M_n(\mathbb{F}), \phi \in Z^1(G, \mathrm{ad}^0\,\bar{\rho})$. Expanding $\alpha_\nu^{-1} \rho \alpha_\nu = \bar{\rho}$,

$$\begin{aligned} \alpha_\nu^{-1} \rho \alpha_\nu &= (1 - T_\nu \varepsilon)(1 + \phi\varepsilon)\bar{\rho}(1 + T_\nu \varepsilon) \\ &= \bar{\rho} + (\phi\bar{\rho} - T_\nu \bar{\rho} + \bar{\rho} T_\nu)\varepsilon = \bar{\rho}, \end{aligned}$$

So $\forall \nu \in T, \phi - T_\nu + \bar{\rho} T_\nu \bar{\rho}^{-1} = \phi - \partial T_\nu = 0$, as desired. □

Constructing a complex (continued)

(3): To add general local $\mathcal{D}_\nu$ at $\nu \in S \setminus T$, note $\widetilde{\mathcal{L}}_\nu := \mathcal{D}_\nu(\mathbb{F}[\varepsilon]) \subseteq Z^1(F_\nu, \mathrm{ad}^0 \bar{\rho}) \subseteq \mathcal{C}^1(F_\nu, \mathrm{ad}^0 \bar{\rho})$ is the full preimage of its image $\mathcal{L}_\nu \subseteq H^1(F_\nu, \mathrm{ad}^0 \bar{\rho})$ (by condition (6)),so $\rho \mid_{G_{F_\nu}} \in \mathcal{D}_\nu(\mathbb{F}[\varepsilon])$ iff

$$\rho \in \ker \left(Z^1(G_{F,S}, \mathrm{ad}^0 \bar{\rho}) \to \bigoplus_{\nu \in S \setminus T} Z^1(F_\nu, \mathrm{ad}^0 \bar{\rho}) / \widetilde{\mathcal{L}}_\nu \right),$$

and this is independent of choice of ρ. Summing up, we have

$$\mathrm{Hom}_{\mathbb{F}}(\mathfrak{m}/(\mathfrak{m}^2, \mathfrak{m}^{\mathrm{loc}}, \lambda), \mathbb{F}) = H^1(K^\bullet),$$

where $K^\bullet$ is the finite complex

$$\mathrm{ad}\, \bar{\rho} \to Z^1(G_{F,S}, \mathrm{ad}^0 \bar{\rho}) \oplus \bigoplus_{\nu \in T} \mathrm{ad}\, \bar{\rho} \to \bigoplus_{\nu \in T} Z^1(F_\nu, \mathrm{ad}^0 \bar{\rho}) \oplus \bigoplus_{\nu \in S \setminus T} Z^1(F_\nu, \mathrm{ad}^0 \bar{\rho}) / \mathcal{L}_\nu.$$

So the tangent space is still an H^1! Motivated by this, we will instead rewrite $K^\bullet$ on the level of inhomogeneous cochains.

Defining the complex

$K^\bullet$ looks like a cone construction, so we make the following definition:

Definition

Let $\mathcal{C}^\bullet_{S,T,\mathrm{loc}}, \mathcal{C}^\bullet_0$ be complexes of $\mathbb{F}$-vector spaces defined by

$$\mathcal{C}^i_{S,T,\mathrm{loc}} = \begin{cases} \bigoplus_{\nu\in T} \mathcal{C}^0(G_\nu, \mathrm{ad}\,\bar\rho) & i=0 \\ \bigoplus_{\nu\in T} \mathcal{C}^1(F_\nu, \mathrm{ad}^0\,\bar\rho) \oplus \bigoplus_{\nu\in S/T} \mathcal{C}^1(F_\nu, \mathrm{ad}^0\,\bar\rho)/\widetilde{\mathcal{L}}_\nu & i=1 \\ \bigoplus_{\nu\in S} \mathcal{C}^i(F_\nu, \mathrm{ad}^0\,\bar\rho) & i>1 \end{cases}$$

and $\mathcal{C}^0_0 = \mathcal{C}^0(G_{F,S}, \mathrm{ad}\,\bar\rho)$ and $\mathcal{C}^i_0 = \mathcal{C}^i(G_{F,S}, \mathrm{ad}^0\,\bar\rho)$ for $i>0$.
Define $\mathcal{C}^\bullet_{\mathcal{S},T} = \mathcal{C}^\bullet_0 \oplus \mathcal{C}^{\bullet-1}_{S,T,\mathrm{loc}}$, with diff. $\partial : (\phi, (\psi_\nu)_{\nu\in S}) \mapsto (\partial\phi, (\phi\,|_{G_{F_\nu}} - \partial\psi_\nu)_\nu)$.
We denote $H^i_{S,T}(G_{F,S}, \mathrm{ad}^0\,\bar\rho) := H^i(\mathcal{C}^\bullet_{S,T})$

Corollary (Immediate from what we have already done)

$\mathrm{Hom}_{\mathbb{F}}(\mathfrak{m}/(\mathfrak{m}^2, \mathfrak{m}^{\mathrm{loc}}, \lambda), \mathbb{F}) \simeq H^1_{\mathcal{S},T}(G_{F,S}, \mathrm{ad}^0\,\bar\rho)$.

How to grok this dimension in terms of the $\mathcal{D}_\nu$ will be discussed next lecture.

References

- Main sources: Patrick Allen's online course on modularity lifting 5-9, Rong's notes, Toby Gee's notes
- Technical things: Clozel-Harris-Taylor Section 2.2 (and its correction in BLGHT)