Even further properties of deformations

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Carayol's lemma

Last time we stated and began the proof of Carayol's lemma. Recall:

Lemma (Carayol)

Let $A \in \mathcal{C}_{\mathcal{O}}$ and $\rho : \Gamma \to \mathrm{GL}_n(A)$ a lift of an absolutely irreducible representation $\bar{\rho} : \Gamma \to \mathrm{GL}_n(k)$. If $B \subset A$ is a closed subring of A with $A \in \mathcal{C}_{\mathcal{O}}$ and $\operatorname{tr} \rho(\Gamma) \subset B$ then there exists an $a \in \ker(\mathrm{GL}_n(A) \to GL_n(k))$ such that $a\rho a^{-1}$ has image in $\mathrm{GL}_n(B)$.

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We saw last time that we could reduce the proof of Carayol's lemma to the case of $A = k[\epsilon]/(\epsilon^2)$ and $B = k$.

Continuation of proof of Carayol's lemma

Proof.

Viewing $\rho : k[\Gamma] \to M_n(k[\epsilon]/(\epsilon^2)) = M_n(k) \oplus M_n(k)\epsilon$, we can write

$$\rho(\gamma) = \bar{\rho}(\gamma) + \theta(\gamma)\epsilon$$

which defines a k-linear map $\theta : k[\Gamma] \to M_n(k)$ with the properties

- $\theta(\gamma\delta) = \theta(\gamma)\bar{\rho}(\delta) + \bar{\rho}(\gamma)\theta(\delta)$ (look at coefficient of ϵ)
- $\mathrm{tr}(\theta(\gamma)) = 0$ (assumed $\mathrm{tr}\,\rho$ lies in k)

We claim that θ factors uniquely through $\bar{\rho}$ i.e.

$$\begin{array}{ccc} k[\Gamma] & \xrightarrow{\theta} & M_n(k) \\ & \underset{\bar{\rho}}{\searrow} & \uparrow{\scriptstyle \exists!\theta'} \\ & & M_n(k) \end{array}$$

Continuation of proof of Carayol's lemma

Proof.

To see this, take $\delta \in \ker(\bar{\rho})$ and note that for every $\gamma \in k[\gamma]$ we have $0 = \mathrm{tr}(\theta(\gamma\delta)) = \mathrm{tr}\,\bar{\rho}(\gamma)\theta(\delta)$. Absolute irreducibility of $\bar{\rho}$ implies $\bar{\rho}$ is surjective onto $M_n(k)$ (this follows by Artin-Wedderburn: the image of $\bar{\rho}$ after tensoring to $\bar{k}$ wlog is a semisimple ring containing $M_n(k)$, since k is the only finite division algebra over k and we have a simple module of k-dimension n). Hence $\theta(\delta) = 0$ and we can define $\theta' : M_n(k) \to M_n(k)$ by taking the image under of θ any choice of preimage in $k[\Gamma]$.

Continuation of proof of Carayol's lemma

Proof.

Recall we want to find some $a \in M_n(k)$ such that for every $\gamma \in \Gamma$

$$(1 + a\epsilon)\rho(\gamma)(1 - a\epsilon) \in M_n(k)$$

which, on taking coefficients of ϵ, holds if and only if

$$\theta(\gamma) + a\bar{\rho}(\gamma) - \bar{\rho}(\gamma)a = 0.$$

So we have reduced the problem to the following: if $\theta' : M_n(k) \to M_n(k)$ is k-linear and satisfies for every $\gamma, \delta \in M_n(k)$

- $\theta'(\gamma\delta) = \theta'(\gamma)\delta + \gamma\theta'(\delta)$ (i.e. a k-derivation)
- $\mathrm{tr}(\theta'(\gamma)) = 0$

then we want to find an $a \in M_n(k)$ with $\theta'(\gamma) = \gamma a - a\gamma$.

Continuation of proof of Carayol's lemma

Proof.

Since $\theta'(1_n) = 0$, this is equivalent to showing every derivation of the Lie algebra $\mathfrak{sl}_n$ is inner. We could now conclude by saying $\mathfrak{sl}_n$ is a semisimple Lie algebra and using the fact that every derivation of a semisimple Lie algebra is inner. $\square$

Compatibility with base-change of coefficients

From now on we will consider our usual setup for existence of universal (framed) deformation rings: Γ a profinite group satisfying Φ_l and $\bar{\rho} : \Gamma \to \mathrm{GL}_n(k)$ a continuous representation.

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Recall we took k to be the residue field of a finite extension $L/\mathbb{Q}_l$ with ring of integers $\mathcal{O}$. Consider a finite extension L'/L with ring of integers $\mathcal{O}'$ and residue field k'. Then we can let $\bar{\rho}' = \bar{\rho} \otimes_k k' : \Gamma \to GL_n(k')$ and consider its universal framed deformation ring. The compatibility is as we expect:

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Lemma

There is a canonical isomorphism in $\mathcal{C}_{\mathcal{O}'}$

$$R^{\square}_{\bar{\rho}'} = R^{\square}_{\bar{\rho}} \otimes_{\mathcal{O}} \mathcal{O}'.$$

Proof of base-change

Proof.

Let $\rho' : \Gamma \to \mathrm{GL}_n(A')$ be a lift of $\bar{\rho}'$ and let $A \in \mathcal{C}_{\mathcal{O}}$ be the preimage of k under the map $A' \to k'$.

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Conversely, given such a morphism $R^{\square}_{\bar{\rho}} \otimes_{\mathcal{O}} \mathcal{O}' \to A'$ we obtain a lift of $\bar{\rho}'$ to A' by the composition

$$\Gamma \to \mathrm{GL}_n(R^{\square}_{\bar{\rho}}) \to \mathrm{GL}_n(R^{\square}_{\bar{\rho}} \otimes_{\mathcal{O}} \mathcal{O}') \to \mathrm{GL}_n(A')$$

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Conversely, given such a morphism $R^\square_{\bar{\rho}} \otimes_\mathcal{O} \mathcal{O}' \to A'$ we obtain a lift of $\bar{\rho}'$ to A' by the composition

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and the above constructions are inverse to each other. Thus we get a canonical isomorphism $R^\square_{\bar{\rho}} \otimes_\mathcal{O} \mathcal{O}' \to R^\square_{\bar{\rho}'}$. □

Adjoint representation

We define the adjoint representation of $\bar{\rho}$ to be the composition

$$\begin{aligned}\Gamma \to \mathrm{GL}_n(k) &\xrightarrow{\mathrm{ad}} \mathrm{Aut}_k(M_n(k)) \\ M &\mapsto (N \mapsto MNM^{-1})\end{aligned}$$

and we usually denote the $k[\Gamma]$-module $M_n(k)$ by $\mathrm{ad}\,\bar{\rho}$.

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and we usually denote the $k[\Gamma]$-module $M_n(k)$ by $\mathrm{ad}\,\bar{\rho}$.
The next few lemmas will allow us to understand the universal (framed) deformation ring better through the group cohomologies of this module. Let $R^{\square}$ (resp. R^{univ}) be the universal framed (resp. unframed) deformation ring with maximal ideal $m^{\square}$ (resp. m^{univ}).

Tangent spaces of the framed deformation ring

Lemma

The following are in natural bijection:

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Proof.

(i) $\Leftrightarrow$ (ii): We have $\mathcal{O} + m^\square$ surjects onto $R^\square$, so given $f \in \mathrm{Hom}_k(m^\square/((m^\square)^2, \lambda), k)$, send $a + x \mapsto \bar{a} + f(x)\epsilon$ for $a \in \mathcal{O}$ and $x \in m^\square$. Its well-defined, as $\mathcal{O} \cap m^\square = (\lambda)$ and a morphism in $\mathcal{C}_\mathcal{O}$. Given $g \in \mathrm{Hom}_{\mathcal{C}_\mathcal{O}}(R^\square, k[\epsilon]/(\epsilon^2))$, a morphism of local rings, and so $g(m^\square) \subset \epsilon k[\epsilon] \cong k$. Since $(m^\square)^2$ and λ both map to 0 under g, we get a k-linear map $m^\square/((m^\square)^2, \lambda) \to k$, and these two constructions are inverse.

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Proof.

(ii) $\Leftrightarrow$ (iii): By definition. (iii) $\Leftrightarrow$ (iv): Given a lift

$$\begin{aligned} \rho : \Gamma &\to GL_n(k[\epsilon]) \\ \gamma &\mapsto \bar{\rho}(\gamma) + \theta(\gamma)\epsilon \end{aligned}$$

define a cocycle $\psi_\rho : \gamma \mapsto \theta(\gamma)\bar{\rho}(\gamma)^{-1}$. This has inverse $\psi \mapsto (\gamma \mapsto \bar{\rho}(\gamma) + \psi(\gamma)\bar{\rho}(\gamma)\epsilon)$, and the condition of ρ being a homomorphism is equivalent to ψ_ρ being a 1-cocycle. $\square$

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Proof.

The bijection of (i) and (ii) is as in the previous lemma. For (ii) $\Leftrightarrow$ (iii), we need to show that two liftings are isomorphic if and only if the corresponding cocycles differ by a coboundary. We have for lifts ρ, ρ' that

$$\begin{aligned} &\rho \cong \rho' \\ \Leftrightarrow\ &(1 + a\epsilon)(\bar{\rho}(\gamma) + \theta(\gamma)\epsilon)(1 - a\epsilon) = \bar{\rho}(\gamma) + \theta'(\gamma) \text{ for some } a \in M_n(k) \\ \Leftrightarrow\ &\theta(\gamma) + (a\bar{\rho}(\gamma) - \bar{\rho}(\gamma)a) = \theta'(\gamma) \text{ for some } a \in M_n(k) \\ \Leftrightarrow\ &\psi_\rho(\gamma) + (a - \mathrm{ad}\,\bar{\rho}(\gamma)(a)) = \psi_{\rho'}(\gamma) \text{ for some } a \in M_n(k) \end{aligned}$$

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Proof.

To conclude, we show (ii) $\Leftrightarrow$ (iv). Let ρ be a lift and let V (resp. $\bar{V}$) denote the underlying free $k[\epsilon]$ (resp. k)-module of ρ (resp. $\bar{\rho}$). View V as a free k-module of rank $2n$ and identifying ϵV and $V/\epsilon V$ with $\bar{V}$ one can check the deformation class of ρ gives a well-defined class of extensions of $k[\Gamma]$-modules:

$$0 \to \bar{V} \xrightarrow{\alpha} V \xrightarrow{\beta} \bar{V} \to 0$$

Given such an extension V define a $k[\epsilon]$-module structure by setting multiplication by ϵ to be $\alpha\beta$, thus giving a lift of $\bar{\rho}$. One can check iso extensions give iso lifts. □

Generators for the framed deformation ring

Let $d = \dim_k(Z^1(\Gamma, \mathrm{ad}\,\bar{\rho}) = \dim_k(m^\square/((m^\square)^2, \lambda))$. If we choose

$$\phi : \mathcal{O}[[X]] = \mathcal{O}[[x_1, \dots, x_d]] \to R^\square$$

by insisting $\phi(x_i)$ generate $m^\square/((m^\square)^2, \lambda)$ as a k-vector space, then ϕ is a surjection (by Nakayama say) in $\mathcal{C}_\mathcal{O}$ and an isomorphism on relative cotangent spaces.

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The exact sequence of finite-dimensional k-vector spaces

$$0 \to (\operatorname{ad}\bar{\rho})^\Gamma \to \operatorname{ad}\bar{\rho} \to Z^1(\Gamma, \operatorname{ad}\bar{\rho}) \to H^1(\Gamma, \operatorname{ad}\bar{\rho}) \to 0$$
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gives $d = \dim_k H^1(\Gamma, \mathrm{ad}\,\bar{\rho}) - \dim_k H^0(\Gamma, \mathrm{ad}\,\bar{\rho}) + n^2$. Note also when $\bar{\rho}$ is Schur that $\dim_k(H^0(\Gamma, \mathrm{ad}\,\bar{\rho})) = 1$.

Controlling $\ker \phi$

We had $\phi : \mathcal{O}[[X]] \twoheadrightarrow R^{\square}$. Set $J = \ker \phi$ and $m = (\lambda, x_1, \ldots, x_d)$ the maximal ideal of $\mathcal{O}[[X]]$. The following lemma will give us an interpretation of H^2, which will allow us to further understand our framed deformation ring.

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Lemma

There is a natural injection $\mathrm{Hom}_k(J/mJ, k) \to H^2(\Gamma, \operatorname{ad} \bar{\rho})$.

Summary and consequences

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We have

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i.e. a presentation of $R^{\square}$ as a quotient of a free power series ring over $\mathcal{O}$ with $d = dim_k H^1(\Gamma, \mathrm{ad}\,\bar{\rho}) - \dim_k H^0(\Gamma, \mathrm{ad}\,\bar{\rho}) + n^2$ generators and which we can take to have at most $\dim_k H^2(\Gamma, \mathrm{ad}\,\bar{\rho}) (= \dim_k(J/mJ))$ relations.

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Thus if $H^2(\Gamma, \mathrm{ad}\,\bar{\rho}) = 0$, $R^{\square} = \mathcal{O}[[x_1, \dots, x_d]]$ is a power series ring. In general, we have by Krull's height theorem that

$$\dim R^{\square} \geq d + 1 - \dim_k H^2(\Gamma, \mathrm{ad}\,\bar{\rho}).$$

Proof of lemma

Proof (sketch).

We firstly give the construction of the map

$$\begin{aligned} \mathrm{Hom}_k(J/mJ, k) &\to H^2(\Gamma, \mathrm{ad}\,\bar{\rho}) \\ f &\mapsto [c_f] \end{aligned}$$

Let $\rho^\square : \Gamma \to \mathrm{GL}_n(\mathcal{O}[[X]]/J)$ be the universal lifting and note we have a surjection $\mathrm{GL}_n(\mathcal{O}[[X]]/mJ) \to \mathrm{GL}_n(\mathcal{O}[[X]]/J)$. So let $\tilde{\rho}$ be any choice of set-theoretic lifting of $\rho^\square$ to $\mathrm{GL}_n(\mathcal{O}[[X]]/mJ)$ (not necessarily a homomorphism). Then set

$$c_f(\gamma, \delta) = f(\tilde{\rho}(\gamma\delta)\tilde{\rho}(\delta)^{-1}\tilde{\rho}(\gamma)^{-1} - 1) \in M_n(k)$$

It can be checked that this defines a 2-cocycle (it is helpful to use the isomorphism $(M_n(J/mJ), +) \cong (1 + M_n(J/mJ), \cdot)$ and rewriting the cocycle condition in multiplicative notation).

Proof of lemma

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It can also be checked that the resulting cohomology class $[c_f]$ doesn't depend on the choice of $\tilde{\rho}$, so we have constructed the map. Additionally, if we set $J_f = \ker(J \to J/mJ \xrightarrow{f} k)$, it can be seen that $[c_f] = 0$ if and only if there exists $\tilde{\rho}$ such that $\tilde{\rho} \mod J_f$ is a homomorphism.
So to show injectivity, we suppose $[c_f] = 0$ so there exists such a $\tilde{\rho}$ which is a homomorphism $\mod J_f$. The universal property of $R^\square$ induces a homomorphism $\mathcal{O}[[X]]/J \to \mathcal{O}[[X]]/J_f$ such that the composition

$$\mathcal{O}[[X]]/J \to \mathcal{O}[[X]]/J_f \twoheadrightarrow \mathcal{O}[[X]]/J$$

is the identity (as $\rho^\square$ is mapped to itself under the composition). The second map is an isomorphism on relative cotangent spaces, so the first map is too, and hence the first map is also surjective (by Nakayama). Hence the second map is an injection, so $J_f = J$ and $f = 0$. □

Thanks for listening and feel free to ask any questions.