Further properties of deformations of Galois representations

Lukas Kofler

University of Cambridge

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Basic set-up

For now, let k be any field, Γ an abstract group. Let V be a finite-dimensional k-vector space and $\rho : \Gamma \to GL_k(V)$ a representation.

Schur's lemma

If ρ is irreducible, then $\mathrm{End}(\rho)$ is a finite-dimensional division algebra over k.

Proof

$\mathrm{End}(\rho)$ is clearly finite-dimensional. Let $0 \neq \alpha \in \mathrm{End}(\rho)$. $\alpha V \subset V$ is stable under Γ and non-zero, so $\alpha V = V$. Therefore α has an inverse.

Corollary

Let $k = \overline{k}$ and let ρ_1, ρ_2 be irreducible. Then

$$\mathrm{Hom}_{k[\Gamma]}(\rho_1, \rho_2) = \begin{cases} k & \text{if } \rho_1 \cong \rho_2 \\ 0 & \text{otherwise.} \end{cases}$$

Remarks

ρ irreducible does not imply that $\rho \otimes_k \overline{k} : \Gamma \to GL(V \otimes_k \overline{k})$ is irreducible: i.e. $C_4 \to GL_2(\mathbb{R})$, $\sigma \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ becomes reducible over $\mathbb{C}$.

$\mathrm{End}(\rho) = k$ does not imply that ρ is irreducible: take $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \hookrightarrow GL_2(\mathbb{R})$.

Definition

A representation ρ is Schur if $\mathrm{End}_{k[\Gamma]}(\rho) = k$. It is absolutely irreducible if $\forall k'/k$, $\rho \otimes_k k'$ is irreducible.

Lemma

The following are equivalent:

- ρ is absolutely irreducible
- $\rho \otimes_k \overline{k}$ is irreducible
- ρ is Schur and irreducible.

Proof

See [C-R, 29.13].

Set-up

L is a finite extension of $\mathbb{Q}_\ell$ with ring of integers $\mathcal{O}$, uniformiser λ and residue field k. Let Γ be a profinite group. $\widehat{\mathcal{C}_\mathcal{O}}$ is the category of complete Noetherian local $\mathcal{O}$-algebras A with a unique isomorphism $A/\mathfrak{m}_A \cong k$, and $\mathcal{C}_\mathcal{O}$ is the full subcategory of local Artinian $\mathcal{O}$-algebras.

Artinian rings

For a commutative Noetherian ring A, the following are equivalent:

- A is Artinian, i.e. A satisfies the DCC on ideals.
- A has Krull dimension zero.
- Every finitely generated module over A has finite length.
- A is a finite product of commutative Artinian local rings.
- $\mathrm{Spec}(A)$ is finite and discrete.

Quotients and localisations of Artinian rings are Artinian. Artinian local rings are complete. An integral domain is Artinian if and only if it is a field.

Definition

Let $\overline{\rho} : \Gamma \to GL_n(k)$ be a continuous representation which is Schur, i.e. $\mathrm{End}_{k[\Gamma]}(\overline{\rho}) = k$. Define the deformation functor $\mathscr{R}_{\overline{\rho}} : \widehat{\mathcal{C}_{\mathcal{O}}} \to \text{Sets}$ by

$$\mathscr{R}_{\overline{\rho}} : A \mapsto \mathrm{Def}_{\overline{\rho}}(A) = \{\rho : \Gamma \to GL_n(A) \text{ such that } \rho \bmod \mathfrak{m}_A = \overline{\rho}\}/\cong .$$

Lemma

If $\rho_1 \cong \rho_2$, then $a \in GL_n(A)$ such that $a\rho_1 a^{-1} = \rho_2$ can be chosen to lie in $\ker(GL_n(A) \to GL_n(k))$.

Proof

The reduction if a to $GL_n(k)$ must be a scalar since $\overline{\rho}$ is Schur. We can lift this to a scalar $\lambda \in GL_n(A)$. Then replace a by $a\lambda^{-1}$.

Theorem

If $\overline{\rho}$ is Schur, then $\mathscr{R}_{\overline{\rho}}$ is representable by a ring $R^{univ}_{\overline{\rho}} \in \widehat{\mathcal{C}_{\mathcal{O}}}$, the universal deformation ring of $\overline{\rho}$, i.e. $\mathscr{R}_{\overline{\rho}}(A) \cong \mathrm{Hom}_{\widehat{\mathcal{C}_{\mathcal{O}}}}(R^{univ}_{\overline{\rho}}, A)$ for any $A \in \widehat{\mathcal{C}_{\mathcal{O}}}$.

The universal deformation

We can take the inverse image of *id* under the bijection $\mathscr{R}^{univ}_{\overline{\rho}}(R^{univ}_{\overline{\rho}}) \cong \mathrm{Hom}_{\widehat{\mathcal{C}_{\mathcal{O}}}}(R^{univ}_{\overline{\rho}}, R^{univ}_{\overline{\rho}})$.
This gives a universal deformation $\rho^{univ}_{\overline{\rho}} : \Gamma \to GL_n(R^{univ}_{\overline{\rho}})$.
It is universal in the sense that every deformation factors through $\rho^{univ}_{\overline{\rho}}$, i.e. for all $A \in \widehat{\mathcal{C}_{\mathcal{O}}}$ and all $\rho : \Gamma \to GL_n(A)$ lifting $\overline{\rho}$ there is a unique $f_\rho : GL_n(R^{univ}_{\overline{\rho}}) \to GL_n(A)$ such that
$f_\rho \circ \rho^{univ}_{\overline{\rho}} \cong \rho$.
All this applies with $R^{\square}_{\overline{\rho}}$ in place of $R^{univ}_{\overline{\rho}}$, giving the universal framed deformation $\rho^{\square}_{\overline{\rho}}$, except we get a strict equality in the previous line.

Proposition

There is a canonical map $R^{univ}_{\overline{\rho}} \to R^{\square}_{\overline{\rho}}$.

Proof

Taking a deformation ρ to its isomorphism class gives a map $\mathscr{R}^{\square}_{\overline{\rho}}(A) \to \mathscr{R}^{univ}_{\overline{\rho}}(A)$. Apply Yoneda's lemma.
Alternatively, we view $\rho^{\square}_{\overline{\rho}} : \Gamma \to GL_n(R^{\square}_{\overline{\rho}})$ as a deformation of $\overline{\rho}$ to get a unique induced map $R^{univ}_{\overline{\rho}} \to R^{\square}_{\overline{\rho}}$.

Definition

A map $f : A \to B$ in $\widehat{\mathcal{C}_{\mathcal{O}}}$ is *small* if its kernel is a principal ideal annihilated by $\mathfrak{m}_A$.

Schlessinger's criterion

Let $F : \widehat{\mathcal{C}_{\mathcal{O}}} \to \text{Sets}$ be a continuous functor such that $F(k)$ is a singleton.
For $A, B, C \in \widehat{\mathcal{C}_{\mathcal{O}}}$ and maps $\alpha : A \to B$ and $\beta : B \to C$, consider

$$\phi : F(A \times_C B) \to F(A) \times_{F(C)} F(B).$$

F is representable if and only if all the following conditions are satisfied.
H1) If α is small, then ϕ is surjective.
H2) If $A = k[\epsilon]$ and $C = k$, then ϕ is bijective.
H3) $\dim_k F(k[\epsilon]) < \infty$
H4) If $A = B$ and $\alpha = \beta$ is small, then ϕ is bijective.
Note: $F(k[\epsilon])$ has a natural k-vector space structure, see [Ma2].

Proof

See [Sch].

Proof of Theorem (Mazur)

Use Schlessinger's criterion, see [Ma1].

Proof of Theorem (Kisin)

Use formal schemes to make precise the idea that $R_{\overline{\rho}}^{univ} = R_{\overline{\rho}}/PGL_n$. See [Boe].

Proof of Theorem (Faltings)

From the last talk we know that we may assume Γ is topologically finitely generated. Take a set of topological generators $g_1, \ldots, g_r$. Choose lifts $E_i \in M_n(\mathcal{O})$ of the $\overline{\rho}(g_i)$. For $A \in \widehat{\mathcal{C}_{\mathcal{O}}}$ we denote by $M_n^0(A)$ the ring $M_n(A)/(A.I_n)$.
We define a map $i_A : M_n^0(A) \to M_n^r(A)$ by $X \mapsto (XE_i - E_iX)_{i=1}^r$. This is clearly injective. We claim that i_A is a split injection:
Denote the reduction of i_A modulo $\mathfrak{m}_A$ by i_k. If $i_k(X) = 0$ then $X\overline{\rho}(g_i) = \overline{\rho}(g_i)X$. Because $\overline{\rho}$ is Schur, X is a scalar and so i_k is injective. Thus $M_n(k)^r = i_k(M_n^0(k)) \oplus (\bigoplus k\overline{e}_i)$. We can lift these $\overline{e}_i$ to $e_i \in M_n(A)$. Then by Nakayama's lemma, $M_n(A)^r = i_A(M_n^0(A)) \oplus (\bigoplus Ae_i)$, so i_A is indeed split.
Fix a splitting $\pi_{\mathcal{O}}$ of $i_{\mathcal{O}}$. Canonically, $M_n^0(A) \cong M_n^0(\mathcal{O}) \otimes_{\mathcal{O}} A$ and $\pi_A = \pi_{\mathcal{O}} \otimes \mathrm{id}_A$ is a splitting of i_A. Define the map ϕ by sending a deformation ρ to $(\rho(g_i))_{i=1}^r \in M_n(A)^r$.
We say that ρ is *well-placed* if $\pi_A(\phi(\rho)) \in M_n^0(A)$ equals $\pi_{\mathcal{O}}(E_1, \ldots, E_r) \otimes 1$. Now we need to prove a lemma.

Lemma (Faltings)

For every $\rho \in \mathrm{Def}_{\overline{\rho}}(A)$ there is a matrix $M \in \ker(GL_n(A) \to GL_n(k))$ such that $M\rho M^{-1}$ is well-placed, i.e. $\pi_A(\phi(\rho)) = \pi_{\mathcal{O}}(E_1, \ldots, E_r) \otimes 1$. M is unique modulo $1 + \mathfrak{m}_A$.

Proof

Recall that $A = \varprojlim_n A/\mathfrak{m}_A^n$. By completeness it is enough to prove the statement for $A/\mathfrak{m}_A^n$, i.e. for Artinian local $\mathcal{O}$-algebras A. We use induction on the length of A.

The base case $A = k$ is clear. Now assume that $\mathfrak{m}_A^m = 0$ and that ρ is well-placed mod $\mathfrak{m}_A^{m-1}$.

We want to find a unique $H \in M_n^0(\mathfrak{m}_A^{m-1})$ such that $\rho' = (1+H)\rho(1+H)^{-1}$ is well-placed.

Let $H \in M_n^0(\mathfrak{m}_A^{m-1})$ act on ρ by conjugation with $1 + H$, on $M_n(A)^r$ by $X \mapsto X + (1+H)X(1+H)^{-1} = X + HX - XH$ and on $M_n^0(A)$ by $Y \mapsto Y + H$. We can check that this action of $M_n^0(\mathfrak{m}_A^{m-1})$ is compatible with the maps ϕ and π_A.

Therefore $\pi_A(\phi(\rho')) = \pi_A(\phi(\rho)) + H$. We know that this is congruent to $\pi_{\mathcal{O}}(E_1, \ldots, E_r) \otimes 1 \bmod \mathfrak{m}_A^{m-1}$. So choose $H \in M_n^0(\mathfrak{m}_A^{m-1})$ such that the congruence holds modulo $\mathfrak{m}_A^m$.

Theorem

If $\overline{\rho}$ is Schur, then $\mathscr{R}_{\overline{\rho}}$ is representable by a ring $R_{\overline{\rho}}^{univ} \in \widehat{\mathcal{C}_{\mathcal{O}}}$, the universal deformation ring of $\overline{\rho}$, i.e. $\mathscr{R}_{\overline{\rho}}(A) \cong \mathrm{Hom}_{\widehat{\mathcal{C}_{\mathcal{O}}}}(R_{\overline{\rho}}^{univ}, A)$ for any $A \in \widehat{\mathcal{C}_{\mathcal{O}}}$.

Proof of Theorem (Faltings) continued

We apply the lemma to get a well-placed conjugate ρ of $\rho_{\overline{\rho}}^{\square}$. Define R to be the smallest closed sub-$\mathcal{O}$-algebra of $R_{\overline{\rho}}^{\square}$ that contains all entries of $\rho(g)$ for all $g \in \Gamma$. Then ρ lies in $\mathrm{Def}_{\overline{\rho}}(R)$. We claim that R is the universal deformation ring $R_{\overline{\rho}}^{univ}$:
Surjectivity of the map $\mathrm{Hom}_{\widehat{\mathcal{C}_{\mathcal{O}}}}(R, A) \to \mathrm{Def}_{\overline{\rho}}(A)$ follows from the surjectivity of $\mathrm{Hom}_{\widehat{\mathcal{C}_{\mathcal{O}}}}(R_{\overline{\rho}}^{\square}, A) \to \mathrm{Def}_{\overline{\rho}}(A)$.
For injectivity, suppose that for $f_1, f_2 \in \mathrm{Hom}_{\widehat{\mathcal{C}_{\mathcal{O}}}}(R, A)$, the associated well-placed $\rho_1, \rho_2 : G \xrightarrow{\rho} GL_n(R) \xrightarrow{f_1, f_2} GL_n(A)$ lie in the same class in $\mathrm{Def}_{\overline{\rho}}(A)$, i.e. $a\rho_1 a^{-1} = \rho_2$ for some $a \in \ker(GL_n(A) \to GL_n(k))$. By the uniqueness statement in Faltings' lemma, $\rho_1 = \rho_2$. By the definition of R, f_1 equals f_2 and we are done.

Schur's lemma for A-coefficients

Let $A \in \widehat{\mathcal{C}_{\mathcal{O}}}$, let $\rho : \Gamma \to GL_n(A)$ be a continuous representation with $\overline{\rho} := \rho \bmod \mathfrak{m}_A$ absolutely irreducible. Let $a \in GL_n(A)$. If $a\rho a^{-1} = \rho$, then $a \in A^\times$.

Proof

As in the proof of the previous lemma, we can reduce to Artinian local $\mathcal{O}$-algebras A and use induction on the length of A. Base case: $A = k$. We can conclude by Schur's lemma since $\overline{\rho}$ is absolutely irreducible.

Now let A be Artinian local. We have $0 = \mathfrak{m}_A^{e+1} \subsetneq \mathfrak{m}_A^e \subsetneq \ldots \subsetneq A$ with each quotient a k-vector space. Choose a one-dimensional subspace $I \subset \mathfrak{m}_A^e$, i.e. a minimal non-zero ideal I of A.

Let $a \in GL_n(A)$ commute with ρ. By the induction hypothesis,

$a \bmod I \in \mathrm{End}_{A/I}(\rho \bmod I) \cong (A/I)^\times$.

Thus we can write $a = \lambda 1_n + a_0$, where $\lambda \in A^\times$ and $a_0 \in M_n(I)$. For any $\gamma \in \Gamma$ we have

$(\lambda 1_n + a_0)\rho(\gamma) = \rho(\gamma)(\lambda 1_n + a_0)$ and so

$a_0\rho(\gamma) = \rho(\gamma)a_0$ in $M_n(I)$. But $I \cong k$ and A surjects onto k, so

$a_0\overline{\rho}(\gamma) = \overline{\rho}(\gamma)a_0$ in $M_n(k)$. Now we use Schur's lemma again to see that a_0 is a scalar.

Carayol's lemma

Let $A \in \widehat{\mathcal{C}_{\mathcal{O}}}$, $B \subset A$ a closed subring, $B \in \widehat{\mathcal{C}_{\mathcal{O}}}$, and $tr\rho(\Gamma) \subset B$. Then there exists a in $\ker(GL_n(A) \to GL_n(k))$ such that $a\rho(\Gamma)a^{-1}$ lies in $GL_n(B) \subset GL_n(A)$.

Proof

As in the previous proof, we may assume that A and B are local Artinian. The base case $A = k$ is trivial. Take $I \cong k \subset \mathfrak{m}_A$ as before. By the induction hypothesis applied to A/I we may assume that

$$\rho(\Gamma) \bmod I \subset GL_n(B/I \cap B).$$

Since $I \cap B \subset I \cong k$, either $I \cap B = I$ or $I \cap B = 0$. In the first case, $\rho(\gamma) \bmod I \in GL_n(B/I)$ implies that $\rho(\gamma) \in GL_n(B)$, as required.

If $I \cap B = 0$, we can consider the inclusion $B \oplus I\epsilon \hookrightarrow A$ given by $(b, i\epsilon) \mapsto b + i$, where $\epsilon^2 = 0$. It will be enough to find $a \in \ker(GL_n(B \oplus I\epsilon) \to GL_n(k))$ such that $a\rho(\Gamma)a^{-1} \subset GL_n(B)$, so we can assume that $B \oplus I\epsilon = A$.

Now suppose that there exists $\alpha \in M_n(I)$ with

$$(1 + \alpha)\rho(\gamma)(1 + \alpha)^{-1} \bmod \mathfrak{m}_B \in GL_n(B/\mathfrak{m}_B).$$

Then

$$(1 + \alpha)\rho(\gamma)(1 + \alpha)^{-1} \in GL_n(B).$$

Thus we only need to prove the statement in the case $A/\mathfrak{m}_B \cong k \oplus k\epsilon = k[\epsilon]/(\epsilon^2)$ and $B/\mathfrak{m}_B \cong k$. This will be shown in the next talk.

Brauer-Nesbitt for A-coefficients

Let $\overline{\rho} : \Gamma \to GL_n(k)$ be an absolutely irreducible representation, and let $\rho_1, \rho_2 : \Gamma \to GL_n(A)$ be isomorphic to $\overline{\rho}$ mod $\mathfrak{m}_A$. Then $tr\rho_1 = tr\rho_2 \implies \rho_1 \cong \rho_2$.

Corollary

The ring $R^{univ}_{\overline{\rho}}$ is topologically generated over $\mathcal{O}$ by the elements $tr(\rho^{univ}_{\overline{\rho}}(\gamma))$ for γ in a dense subset of Γ.

Proof

Let S be the closure in $R^{univ}_{\overline{\rho}}$ of the subring generated by the $tr(\rho^{univ}_{\overline{\rho}}(\gamma))$. S is an element of $\widehat{\mathcal{C}_{\mathcal{O}}}$ with maximal ideal $\mathfrak{m}_S = \mathfrak{m}_R \cap S$.
By continuity of $\rho^{univ}_{\overline{\rho}}$, $tr(\rho^{univ}_{\overline{\rho}}(\gamma))$ lies in S for every $\gamma \in \Gamma$. Therefore we can apply Carayol's lemma: There exists some a in $\ker(GL_n(R^{univ}_{\overline{\rho}}) \to GL_n(k))$ such that $a\rho^{univ}_{\overline{\rho}}(\gamma)a^{-1} \in GL_n(S)$ for all $\gamma \in \Gamma$.
Thus conjugation by a induces a map $f_a : R^{univ}_{\overline{\rho}} \to S$. Since conjugation preserves traces, this is a retract of the inclusion $i : S \hookrightarrow R^{univ}_{\overline{\rho}}$, i.e. $f_a \circ i = id_S$.
The inclusion i induces a morphism of functors $\mathscr{R}^{univ}_{\overline{\rho}} \to \mathrm{Hom}_{\widehat{\mathcal{C}_{\mathcal{O}}}}(S, -)$. This is surjective because it has a section induced by f_a. It follows from Brauer-Nesbitt for A-coefficients that a deformation is determined by its trace on a dense open subset of Γ. Thus the morphism is also injective. By Yoneda's lemma it follows that the inclusion $S \hookrightarrow R^{univ}_{\overline{\rho}}$ is actually an isomorphism in $\mathcal{C}_{\mathcal{O}}$.

References

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