Deformations of Galois representations: basics

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Motivation

Let $F\colon (\text{Schemes})^{opp} \to (\text{Sets})$ be a functor represented by some (noetherian) scheme M. (Or more generally a noetherian algebraic stack)
Let x be a 'point' of F, i.e. an element of $F(k)$.

Goal

Study infinitesimal neighbourhood of M around x using F, to obtain information about smoothness, tangent spaces,...

Example

- $F(S) =$ Elliptic curve over S with level structure $\rightsquigarrow X_1(N), X_0(N), \ldots$
- For $g \geq 2$, $F(S) =$ smooth proper $X \to S$ of genus $g \rightsquigarrow \mathcal{M}_g$
- Given a scheme X, consider (fppf sheaffication of) $F(S) = \operatorname{Pic}(X_S)/\operatorname{Pic}(S) \rightsquigarrow \operatorname{Pic}_X$
- $F(S) =$ closed subschemes $Z \subset \mathbb{P}^n_S$, flat over $S \rightsquigarrow \operatorname{Hilb}_{\mathbb{P}^n}$

Motivation

Let $\mathcal{C}$ be the category of local Artinian rings with given isomorphism $A/m_A \simeq k$. Let $F_x \colon \mathcal{C} \to (\text{Sets})$ be given by $F_x(A) =$ elements of $F(\operatorname{Spec} A)$ mapping to x under $F(\operatorname{Spec}(A/m_A)) = F(k)$.

Answer

Formal neighbourhood of x is determined by F_x!

More precisely, if F is represented by M, then F_x is pro-represented by $\widehat{\mathcal{O}}_{M,x}$

Example

If $F = \mathcal{M}_g$ and k algebraically closed, then $\widehat{\mathcal{O}}_{M,x} \simeq k[[t_1, \dots, t_{3g-3}]]$ if char $k = 0$, and $W(k)[[t_1, \dots, t_{3g-3}]]$ otherwise.

Motivation

Imagine we have a moduli space of Galois representations

$$"S \mapsto \{\rho\colon \Gamma \to \mathrm{GL}_n(\mathcal{O}_S)\}"$$

(As written, has very bad properties and barely makes sense, see however the recent work of Emerton-Gee)
We may then similarly fix an $\mathbb{F}_q$-point $\bar{\rho}\colon \Gamma \to \mathrm{GL}_n(\mathbb{F}_q)$ and consider lifts to $\rho\colon \Gamma \to \mathrm{GL}_n(A)$ for Artinian local rings with $A/m_A \simeq \mathbb{F}_q$.

$\rightsquigarrow$ Leads to the theory of deformations of Galois representations, and has the same features as classical deformation theory in algebraic geometry (i.e. cohomological interpretation of tangent spaces, obstructions, deformation problems...)

Condition Φ_p

Lemma

Let Γ be a profinite group. TFAE:

1. *For every open $H \leq \Gamma$, the maximal pro-p quotient of H is topologically finitely generated.*
2. *For every open $H \leq \Gamma$, the $\mathbb{F}_p$-vector space $H^{ab} \otimes_{\mathbb{Z}} \mathbb{F}_p$ is finite-dimensional.*

In both are satisfied, we say Γ satisfies Φ_p.

Proof.

(1) $\Rightarrow$ (2): clear.
(2) $\Rightarrow$ (1): Suppose $H^{ab} \otimes \mathbb{F}_p$ finitely generated. Also assume H is pro-p. Let $S \subset H$ be a lift of the generating set of $H^{ab} \otimes \mathbb{F}_p$. We claim that S topologically generates H. We may suppose that H is finite: now this is exactly Burnside's basis theorem. □

Condition Φ_p

Definition

Recall: Γ satisfies Φ_p if $\dim_{\mathbb{F}_p}(H^{ab} \otimes \mathbb{F}_p) < \infty$ for all open $H \leq \Gamma$

Examples

- Let F be a local field, then G_F satisfies Φ_p.
  Reason: $G_F^{ab} \simeq F^\times/(F^\times)^p$ (LCFT), which is finite. Since every open subgroup of G_F is of the form $G_{F'}$, same argument applies.
- Let F be a global field, S a finite set of primes of F and $G_{F,S} =$ Galois group of maximal extension unramified outside S. Then $G_{F,S}$ satisfies Φ_p.
  Reason: only finitely many Galois extensions of F of degree p unramified outside S.

Notations

Fix a finite field k of characteristic p.

Notation

- Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$ with residue field k.
- $\mathcal{C}_{\mathcal{O}}$ = category of local Artinian $\mathcal{O}$-algebras (A, m_A) with residue field k, with local homomorphisms (Provides unique iso $A/m_A \simeq k$)
- $\widehat{\mathcal{C}_{\mathcal{O}}}$ = category of complete Noetherian local $\mathcal{O}$-algebras with residue field k. (Coincides with pro-objects of $\mathcal{C}_{\mathcal{O}}$ with $\dim_k m_A/m_A^2 < \infty$.)

Definition

A functor $F \colon \widehat{\mathcal{C}_{\mathcal{O}}} \to (\text{Sets})$ is *continuous* if $F(A) = \varprojlim F(A/m_A^n)$ for all $A \in \widehat{\mathcal{C}_{\mathcal{O}}}$, and *representable* if $F(A) = \text{Hom}(R, A)$ for some $R \in \widehat{\mathcal{C}_{\mathcal{O}}}$.

For example, $F(A) = m_A$ is represented by $\mathcal{O}[[X]]$

Deformations

Let Γ be a profinite group satisfying Φ_p. Let $\bar{\rho}\colon \Gamma \to \mathrm{GL}_n(k)$ be a (continuous) representation.

Definition

Define the *framed deformation functor* $D_{\bar{\rho}}^{\square}\colon \widehat{\mathcal{C}_{\mathcal{O}}} \to (\mathrm{Sets})$ by sending $A \in \widehat{\mathcal{C}_{\mathcal{O}}}$ to the set of all $\rho\colon \Gamma \to \mathrm{GL}_n(A)$ such that $\rho \mod m_A = \bar{\rho}$.

Definition

- Say $\rho, \rho' \in D_{\bar{\rho}}^{\square}(A)$ are *strictly equivalent* if there exists $g \in \mathrm{GL}_n(A)$ which conjugates ρ to ρ'. (Such g cqn be chosen to be trivial in $\mathrm{GL}_n(k)$.)
- A deformation is a strict equivalence class of framed deformations.
- The *deformation functor* $D_{\bar{\rho}}\colon \widehat{\mathcal{C}_{\mathcal{O}}} \to (\mathrm{Sets})$ sends $A \in \widehat{\mathcal{C}_{\mathcal{O}}}$ to the set of A-valued deformations.

Representability

$D_{\bar\rho}^{\square}$ and $D_{\bar\rho}$ are continuous functors.

Theorem

Suppose that Γ satisfies Φ_p.

1. *The functor $D_{\bar\rho}^{\square}$ is representable by an element $R_{\bar\rho}^{\square} \in \widehat{\mathcal{C}_{\mathcal{O}}}$.*
2. *If $\bar\rho$ is absolutely irreducible, then $D_{\bar\rho}$ is representable by an element $R_{\bar\rho} \in \widehat{\mathcal{C}_{\mathcal{O}}}$.*

Proof.

We only prove (1) today. Warm-up: suppose that $\Gamma = \widehat{\mathbb{Z}}$.
Let $\widetilde{g} \in \mathrm{GL}_n(\mathcal{O})$ be a lift of $\bar\rho(1) \in \mathrm{GL}_n(k)$. Then an element $\rho \in D_{\bar\rho}^{\square}(A)$ is specified by the element $\rho(1) \in \widetilde{g} + \mathrm{Mat}_n(m_A)$. Therefore $D_{\bar\rho}^{\square}(A) \simeq m_A^{n^2} \simeq \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}[[X_{11}, \dots, X_{nn}]], A)$, so $R_{\bar\rho}^{\square} \simeq \mathcal{O}[[X_{11}, \dots, X_{nn}]]$. □

Proof of general case

Lemma

To prove that $D_{\bar{\rho}}^{\square}$ is representable, we may assume that Γ is topologically finitely generated.

Proof:
Let $H = \ker(\bar{\rho}\colon \Gamma \to \mathrm{GL}_n(k))$, an open subgroup of Γ.
Let $H_0 = \ker(H \to \text{pro-p quotient of } H) \trianglelefteq \Gamma$ (normal since H_0 is a characteristic subgroup of H)

Step 1: $I + \mathrm{Mat}_n(m_A)$ is pro-p

Proof: We may suppose A is Artinian, so $m_A^N = 0$ for some $N \geq 1$. If $B \in \mathrm{Mat}_n(m_A)$, then $(I+B)^p = I + pB + B^2(\dots)$. Since $B^2 \in \mathrm{Mat}_n(m_A^2)$ and $p \in m_A$, see that $(I+B)^p \in I + \mathrm{Mat}_n(m_A^2)$. So $(I+B)^{p^N} = I$.

Proof of general case

Step 2: every $\rho \in D_{\bar{\rho}}^{\square}(A)$ factors through Γ/H_0

Proof: $\rho|_H \colon \Gamma \to I + \mathrm{Mat}_n(m_A)$ has pro-p image. Use Step 1.

Step 3: Γ/H_0 is topologically finitely generated

Proof: By property Φ_p, H/H_0 is topologically finitely generated. Since it is a finite index normal subgroup of Γ/H_0, the latter is also finitely generated.

By replacing Γ by Γ/H_0, the lemma is proved.

Proof of general case

Let $\bar{\rho}\colon \Gamma \to \mathrm{GL}_n(k)$ and assume Γ topologically finitely generated. Let $F_d = \langle e_1, \ldots, e_d \rangle$ be the free group on d generators, and choose

$$\begin{array}{ccc} F_d & \xrightarrow{\phi_0} & \Gamma \\ \downarrow & & \downarrow \\ \widehat{F_d} & \xrightarrow{\phi} & \Gamma \end{array}$$

where ϕ surjective. Then $\ker \phi_0$ is dense in $\ker \phi$ (exercise). $\ker \phi_0$ generated by words $r(e_1, \ldots, e_d)$.
Since $\mathrm{GL}_n(A)$ is profinite, giving a homomorphism $\rho\colon \Gamma \to \mathrm{GL}_n(A)$ is the same as giving elements $g_1, \ldots, g_d \in \mathrm{GL}_n(A)$ such that $r(g_1, \ldots, g_d) = 1$ for all $r \in \ker \phi_0$.

Proof of general case

Let $\tilde{g}_1, \ldots, \tilde{g}_d$ denote lifts of $\bar{\rho}(\phi(e_i)) \in \mathrm{GL}_n(k)$ to $\mathrm{GL}_n(\mathcal{O})$.

Conclusion

Giving an element $\rho \in D_{\bar{\rho}}^{\square}(A) \Leftrightarrow$ giving $g_1, \ldots, g_d \in \mathrm{GL}_n(A)$ with the property that:

1. $r(g_1, \ldots, g_d) = 1$ for all $r \in \ker \phi_0$.
2. $g_i \in \tilde{g}_i + \mathrm{Mat}_n(m_A)$ for $i = 1, \ldots, d$.

End of proof: Each g_k defines an element $(X_{ij}^k)_{1 \leq i,j \leq n}$ of $m_A^{n^2}$ by (2). Conditions (1) cut out an ideal of $\mathcal{O}[[\{X_{ij}^k\}]]$: the corresponding quotient will represent $D_{\bar{\rho}}^{\square}$.

□

References

- Motivation for deformation theory: Chapter 6 of 'FGA explained'
- Galois deformation theory: Mazur's paper on 'Deforming Galois representations'
- Notes linked to in study group.