May 13 @ ALT Study group

(Picture of) Patching

Goal of the study gp.:

Minimal ALT Thm F: tot. real # field, l: prime # s.t. $[F(\zeta_l):F]>2$

$L/\mathbb{Q}_l$ fin. ext'n containing the images of all $F \hookrightarrow \overline{\mathbb{Q}_l}$, w/ r.o.i $\mathcal{O}$ and uniformiser λ

$\rho, \rho_0 : \mathrm{Gal}_F \to GL_2(\mathcal{O})$ cts rep'ns s.t.

- $\det \rho = \det \rho_0 =: \chi$ ← guaranteed by other assumption + solvable base change.
- $\rho \bmod \lambda = \rho_0 \bmod \lambda =: \bar{\rho}$ is abs. irred.
- $\forall \sigma : F \hookrightarrow L$, $HT_\sigma(\rho) = HT_\sigma(\rho_0) =: HT_\sigma$ consists of 2 distinct integers

technical "minimal"

- $\forall\, v \mid l$, $\rho_v, \rho_{0,v}$ are crystalline
- $\forall\, v \in T = T_l \sqcup T_r$, $\rho_v, \rho_{0,v}$ belong to the same irred component of $\operatorname{Spec} R^{\square}_{\bar\rho_v,\chi}[1/l]$

  where $T_l = \{v \text{ above } l\}$ and $T_r = \{v \nmid l, \infty,\ \rho \text{ or } \rho_0 \text{ ramified at } v\}$

Then ρ_0 is automorphic $\Rightarrow$ ρ is automorphic.

<u>Deformation problem</u> $\mathcal{S} = (F, T, \bar\rho, \chi, \{D_v\}_{v\in T})$

w/ $\begin{cases} v \in T_r, & D_v = \text{all lifts of determinant } \chi_v \\ v \in T_l, & D_v \leftrightarrow \ker(R^{\square}_{\bar\rho_v,\chi_v} \longrightarrow R^{\square}_{\bar\rho_v,\chi_v,\mathrm{cr},\{HT_\tau\}}) \end{cases}$

$R^{\mathrm{univ}} := R^{\mathrm{univ}}_{\mathcal{S}}$

<u>A particular lifting</u> $\mathrm{Gal}_F \longrightarrow \mathrm{GL}_2(\mathbb{T}_{u,m})$

where $\mathbb{T}_U = \mathcal{O}\langle T_v, S_v \rangle \curvearrowright S := S(U, \mathcal{O})$ ← $\mathcal{O}$-coeff. AFs on $D^\times$ of level U
wt ...
central char. ...

$\cup$

$\mathfrak{m} = (\lambda, \mathrm{tr}\,\bar{\rho}(\mathrm{Frob}_v) - T_v, \det \bar{\rho}(\mathrm{Frob}_v) - q_v S_v)$

by construction

$$\begin{tikzcd} {} \arrow[r, squiggly] & R := R^{univ} \arrow[r, two heads] \arrow[dr, "f_\rho"'] & T := \mathbb{T}_{U,\mathfrak{m}} \arrow[d, dashed] \\ & & \mathcal{O} \end{tikzcd}$$

← Statement of Thm

<u>Goal of patching</u> : showing that $R^{red} \xrightarrow{\sim} T$, essentially $\ker(R \to T) \subseteq \sqrt{(0)}$

$\Longleftarrow$

<u>Prop</u> $\mathrm{Supp}_R S = \mathrm{Spec}\, R$

<u>Pf</u> S is f.g. / T and hence R, thus

$$\mathrm{Supp}_R S = \bigcup_{x \in S} V(\mathrm{Ann}_R(x)) = V(\mathrm{Ann}_R(S))$$

$$\mathrm{Supp}_R(S) = R \Rightarrow \ker(R \to T) \subset \mathrm{Ann}_R(S) \subset \sqrt{(0)} \quad \#$$

Depth in the Noetherian local case

Def R: comm. ring, $I \subset R$: ideal, M: R-mod s.t. $IM \subsetneq M$

$$\operatorname{depth}_I(M) := \min\{i : \operatorname{Ext}^i(R/I, M) \neq 0\}.$$

When R is local I is by default the maximal ideal

Def For R, M as above, $x_1 \sim x_n \in R$ is an M-regular sequence if

x_i is not a zero divisor on $M/(x_1, \cdots, x_{i-1})M$, $1 \leq i \leq n$.

In particular if $x_1 \sim x_n \in I$, we have ses

$$0 \to M \xrightarrow{x_1} M \to M/x_1M \to 0$$

$$\Rightarrow \quad 0 \to \operatorname{Ext}^i(R/I, M) \to \operatorname{Ext}^i(R/I, M/x_1M) \to \operatorname{Ext}^{i+1}(R/I, M) \to 0, \quad i \geq 0$$

hence $\quad \operatorname{depth}_I(M/x_1M) = \operatorname{depth}_I(M) - 1$

and $\quad \operatorname{depth}_I(M/(x_1, \cdots, x_n)M) = \operatorname{depth}_I(M) - n.$

When R is noe. and either M is f.g. or R is local; --- $(*)$

$\exists$ non-zero divisor $x \in I$ on $M \Leftarrow\!\!\Rightarrow \operatorname{Hom}(R/I, M) = 0$, therefore

Under $(*)$,

<u>Thm</u> (Rees) $\operatorname{depth}_I(M)$ = length of any maximal M-regular sequence in I.

<u>Def</u> $\dim M :=$ Krull dimension of $\operatorname{supp} M \subseteq \operatorname{Spec} R$

<u>Prop</u> Under $(*)$, $IM \neq M \Rightarrow$

$$\operatorname{depth}_I(M) \leq \dim M.$$

<u>Pf</u> In the local case we have $\forall$ non-zero divisor $x \in \mathfrak{m}$ on M,

$$\dim M/xM = \dim M - 1,$$

hence by induction the prop. reduces to $0 \leq$ a nonneg. integer.

The rest case follows from the local case.

<u>Thm</u> (Auslander – Buchsbaum formula)

R: noe. local ring, M: f.g. R-mod. If $\operatorname{pd}(M) < \infty$, then

$$\operatorname{pd}(M) = \operatorname{depth}(R) - \operatorname{depth}(M)$$

$$\left(\begin{array}{ll} \Rightarrow & \operatorname{depth}(M) = \dim R \Rightarrow M \text{ is free} \\ \Rightarrow & \cdots \end{array}\right)$$

<u>Simplest picture of patching</u>

$$\begin{tikzcd}
J_\infty \arrow[r] & R_\infty \arrow[r, two heads] \arrow[d, phantom, "\circlearrowright"] & R = R^{\text{univ.}} \arrow[d, phantom, "\circlearrowright"] \\
 & S_\infty \arrow[r, two heads] & S = S(U, \mathcal{O})
\end{tikzcd}$$

- J_∞ : power series ring $/ \mathcal{O}$
- $\exists$ ideal $\mathfrak{a}_\infty \triangleleft J_\infty$ s.t. $R = R_\infty / \mathfrak{a}_\infty$ and $S = S_\infty / \mathfrak{a}_\infty$
- $\dim R_\infty = \dim J_\infty =: d$
- S_∞ is free $/ J_\infty$

Prop (i) $\mathrm{Supp}_{R_\infty} S_\infty$ is a union of irred. components of $\mathrm{Spec}\, R_\infty$

(ii) $\mathrm{Supp}_{R_\infty} S_\infty \supset \mathrm{Spec}\, R \Rightarrow \mathrm{Supp}_R S = \mathrm{Spec}\, R \Rightarrow R^{red} \xrightarrow{\sim} \mathbb{T}$.

Pf (i) Let $\mathfrak{p} \in \mathrm{Supp}_{R_\infty} S_\infty$ be minimal, the assertion is that $\mathfrak{p}$ is also minimal in R_∞.

In fact we have

$$d = \dim R_\infty \geq \text{depth } S_\infty \text{ over } R_\infty \geq \text{depth } S_\infty \text{ over } J_\infty = d.$$

$$\dim R_\infty \geq \dim R_\infty/\mathfrak{p} \geq \text{depth } S_\infty \text{ over } R_\infty$$

Thus this equality holds.

(ii) $\mathrm{Supp}_R S = \mathrm{Supp}_{R_\infty} S_\infty / \mathfrak{a}_\infty = \bigcup_{x \in S_\infty} V(\mathrm{Ann}_{R_\infty}(x) + \mathfrak{a}_\infty)$

$$= \bigcup_{x \in S_\infty} \left(V(\mathrm{Ann}_{R_\infty}(x)) \cap \underbrace{V(\mathfrak{a}_\infty)}_{=\mathrm{Spec}\, R}\right)$$

$$= \mathrm{Supp}_{R_\infty} S_\infty \cap \mathrm{Spec}\, R = \mathrm{Spec}\, R.$$

The technical "minimal" assumption

Deformation problem $\mathcal{S}'$: modification of $\mathcal{S}$ as follows:

$\forall\, v \in T_\ell$, shrink D_v to D_v': the irred. component of $R^{\square}_{\bar\rho_v,\chi_v,cr,\{HT_\sigma\}}$ containing $\rho_{0,v}$.

The minimal assumption: ρ is of type $\mathcal{S}'$

$$\Rightarrow \quad \begin{tikzcd} R' \arrow[dr] & R \arrow[l, two heads] \arrow[r] \arrow[d, "f_\rho"] & T \arrow[dl, dashed] \\ & \mathcal{O} & \end{tikzcd}$$

even Spec R is too large

Then previous arguments show that $\operatorname{Spec} R' \subseteq \operatorname{Supp}_{R_\infty} S_\infty$

$$\Rightarrow \operatorname{Supp}_{R'} S_\infty \otimes_{R_\infty} R' = \operatorname{Spec} R'$$

and thus $\ker(R \to T)$ goes into $\sqrt{(0)}$ in R' and 0 in $\mathcal{O}$, and thus f_ρ factors through T.

Enhanced diagram of patching

$$
\begin{tikzcd}
J_\infty \arrow[rr] \arrow[d] & & J[\Delta_Q] \arrow[r] \arrow[d] & \mathcal{O}[\Delta_Q] \arrow[d] & \\
R_\infty \arrow[r] \arrow[d, phantom, "\circlearrowright"] & R_Q \hat{\otimes}_{\mathcal{O}} J \cong & R_Q^{\square} \arrow[r] \arrow[d, phantom, "\circlearrowright"] & R_Q \arrow[r] \arrow[d, phantom, "\circlearrowright"] & R \arrow[d, phantom, "\circlearrowright"] \\
S_\infty \arrow[r] & S_Q \hat{\otimes}_{\mathcal{O}} J \cong & S_Q^{\square} \arrow[r] & S_Q \arrow[r] & S
\end{tikzcd}
$$

- Q: auxiliary set of primes v of F outside T s.t. $\begin{cases} q_v \equiv 1 \bmod \ell \\ \bar{\rho}(\mathrm{Frob}_v) \text{ has distinct evals } \bar{\alpha}_v \neq \bar{\beta}_v \end{cases}$ (T-W primes)
- $J = \mathcal{O}[[X_{v,i,j}]]_{\substack{v \in T \\ 1 \le i,j \le 2}} / (X_{v_0,1,1})$, $v_0 \in T$ fixed

  $J \rhd \mathfrak{a} = \langle X_{v,i,j} \rangle \ldots$
- $\Delta_Q = \prod_{v \in Q} \Delta_v$ finite abelian ℓ-gp.

- $S_Q, S_Q^\square$ f.g. free modules / $\mathcal{O}[\Delta_Q], J[\Delta_Q]$ resp.ly
- Deformation problem for $R_Q, R_Q^\square$:

$$\mathcal{S}_Q = (F, T \sqcup Q, \bar{\rho}, \chi, \{D_v\}_{v \in T \sqcup Q}) : \begin{cases} v \in T, & D_v \text{ as before} \\ v \in Q, & D_v = \text{all lifts} \end{cases}$$

$$\rightsquigarrow R_Q^\square = R_{\mathcal{S}_Q}^{\square_T}, \quad R_Q = R_{\mathcal{S}_Q}^{\text{univ.}}$$

<u>Lemma</u> (i) $\forall\, v \in Q$, $\rho_Q^{\text{univ}}|_{G_{F_v}} \cong \chi_\alpha \oplus \chi_\beta$ where $\chi_\alpha, \chi_\beta : \mathrm{Gal}_{F_v} \to R_Q^\times$ satisfy:

$$\chi_\alpha(\mathrm{Frob}_v) \bmod m_{R_Q} = \bar{\alpha}_v$$
$$\chi_\beta(\mathrm{Frob}_v) \bmod m_{R_Q} = \bar{\beta}_v$$

(ii) χ_α is tamely ramified and $\chi_\alpha|_{I_{F_v}}$ factors through $I_{F_v}^{\text{tame}} \to k(v)^\times \to \Delta_v$

where Δ_v is the max'l l-quotient of $k(v)^\times$

Pf (i) follows from Guillem's talk last term.

(ii) $\bar{\rho}_v$ is unr., so $\chi(I_{F_v}) \subset 1+m_Q$ is pro-l and hence $\chi|_{I_{F_v}}$ factors through the maximal l-quotient of I_{F_v} #

Consequently we get maps

$$\Delta_Q = \prod_{v\in Q} \Delta_v \to R_Q^\times$$

$$\mathcal{O}[\Delta_Q] \to R_Q, \quad J[\Delta_Q] \to R_Q^\square \cong J\hat{\otimes}_{\mathcal{O}} R_Q$$

Lemma (i) $R_Q/I_{\Delta_Q} \cong R$

(ii) $(R_Q^\square)_a \cong R_Q$

Pf (ii) is tautology. (i): $\mathrm{Gal}_{F, Q\cup T} \to GL_2(R_Q) \to GL_2(R_Q/I_{\Delta_Q})$ is unr. at $v\in Q$

$\rightsquigarrow$ map $R \to R_Q/I_{\Delta_Q}$ which is an isom.