Minimal automorphy lifting

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Automorphic Galois representations

Let F be a totally real number field and fix weights (k, η). We recall that for π a regular algebraic cuspidal automorphic representation of $GL_2(\mathbb{A}_F^\infty)$ of weight (k, η), there exists a CM field $L_\pi \subset \mathbb{C}$ and Galois representations

$$r_\lambda(\pi) : G_F \to GL_2(\overline{L_{\pi,\lambda}})$$

for every finite place λ of L_π.

Definition

A semisimple representation $\rho : G_F \to GL_2(\overline{\mathbb{Q}_l})$ is automorphic if $\rho \cong \iota \circ r_\lambda(\pi)$ for some $\iota : \overline{L_{\pi,\lambda}} \cong \bar{\mathbb{Q}}_l$ or $\rho \cong \chi_1 \oplus \chi_2$.
A semisimple representation $\bar{\rho} : G_F \to GL_2(\overline{\mathbb{F}_l})$ is automorphic if $\bar{\rho} \cong (\iota \circ r_\lambda(\pi) \mod l)^{ss}$

Automorphy lifting statements

Conjecture

Suppose $\rho : G_F \to GL_2(\overline{\mathbb{Q}_l})$ is unramified almost everywhere and de Rham, with distinct Hodge-Tate weights for every $i : F \to \overline{\mathbb{Q}_l}$. If $(\rho \mod l)^{ss}$ is automorphic, then ρ is automorphic.

Over the next three talks, we will prove a weaker version of this result. Firstly introduce the following notation. Let L be a finite extension of $\mathbb{Q}_l$, ring of integers $\mathcal{O}$ and uniformizer λ and residue field k.

Definition

Let $\rho, \rho_0 : G_F \to GL_2(L)$ and suppose $\chi := \det \rho = \det \rho_0$ and that $\bar{\rho} := \rho \mod \lambda = \rho_0 \mod \lambda$. For v a finite place of F we say $\rho_v \sim (\rho_0)_v$ if $\rho_v, (\rho_0)_v$ belong to the same irreducible component of $\text{Spec } R^{\square}_{\overline{\rho_v},\chi}[1/l]$. (Recall this was the universal framed deformation ring for the residual representation $\overline{\rho_v}$ of character χ).

Main automorphy lifting theorem

We will prove:

Theorem

Suppose $|F(\zeta_l) : F| \geq 2$ (so necessarily $l \geq 5$). Let $\rho, \rho_0 : G_F \to GL_2(L)$ be unramified outside $S \sqcup T_l$ where S is a finite set of places not dividing l of F and T_l is the set of places dividing l. Suppose that ρ_0 is automorphic, $\rho \mod \lambda \cong \rho_0 \mod \lambda$ is absolutely irreducible. Suppose further that ρ, ρ_0 are crystalline for every $v|l$ and that the Hodge-Tate weights of ρ, ρ_0 are equal, and distinct integers for every embedding $F \to \overline{\mathbb{Q}_l}$. Finally, suppose for every finite $v \in S \sqcup T_l$ that $\rho_v \sim (\rho_0)_v$. Then ρ is automorphic.

(While not stated in the notes, I think we might also need irreducibility of $\bar{\rho}\big|_{G_{F(\zeta_l)}}$ or some other image condition to prove the existence of sets of Taylor-Wiles primes, which we will see in the subsequent lectures).

Finite solvable base change

We will reduce the theorem to proving such a result with additional conditions. Recall firstly the result on finite solvable base change:

Proposition

Let E/F be a finite solvable extension of totally real fields. Then $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}_l})$ is modular if and only if $\rho\big|_{G_E}$ is.

We then have the following fact.

Proposition

Let F a number field and S a finite set of places. Fix for each $v \in S$ a finite Galois extension L_v/F_v. Then there exists a finite solvable extension M/F such that for each place w of M above v, $L_v \cong M_w$ over F_v.

Some reductions

By passing to such a finite solvable extension of totally real fields (the totally real condition given by taking $L_v = F_v$ for every infinite place v) we may make the following assumptions on our representation:

- $|F : \mathbb{Q}|$ is even (e.g. by choosing a quadratic extension at some finite place)
- If ρ or ρ_0 are ramified at some place $v \nmid l$ then $\bar{\rho}|_{G_{F_v}}$ is trivial and $\#k(v) \equiv 1 \mod l$ (for such v set $L'_v = F_v^{\ker \bar{\rho}}$ and then let L_v be an unramified extension of L'_v of degree the order of $\#k(v)$ in $(\mathbb{Z}/l\mathbb{Z})^\times$)
- $\rho(I_v)$ and $\rho_0(I_v)$ are unipotent for all $v \nmid l$ (proof the same as in l-adic monodromy)
- $\chi := \det \rho = \det \rho_0$ (since ρ, ρ_0 are crystalline of equal HT weights, $\det \rho / \det \rho_0$ is unramified at $v|l$ and since $\rho(I_v), \rho_0(I_v)$ is unipotent at $v \nmid l$, $\det \rho / \det \rho_0$ is unramified everywhere so finite order and trivial $\mod l$. So it has l-power order so its kernel defines a finite abelian totally real ($l \neq 2$ so complex conjugation acts trivially) extension unramified away from l).

Deformation conditions

Set $T_l = \{v|l\}$ and $T_r = \{v \nmid l, \infty : \rho$ or ρ_0 ramified at $v\}$. Let $T = T_l \sqcup T_r$. We will impose deformation conditions $\mathcal{D}_v$ for $v \in T$ as follows:

- For $v \in T_r$, $\mathcal{D}_v$ is all lifts (of fixed determinant χ_v)
- For $v \in T_l$, $\mathcal{D}_v$ corresponds to the ideal $\ker(R^{\square}_{\bar{\rho}_v,\chi_v} \to R^{\square}_{\bar{\rho}_v,\chi_v,\mathrm{cr},\mathrm{HT}})$ where $R^{\square}_{\bar{\rho}_v,\chi_v,\mathrm{cr},\{\mathrm{HT}\}}$) is the maximal quotient which is l-torsion free, reduced and whose char 0 points give crystalline representations with HT weights $\{\mathrm{HT}\}$

Our global deformation problem will be $\mathcal{S} = (F, T, \bar{\rho}, \chi, \{\mathcal{D}_v\}_{v\in T})$. Let $R^{\mathrm{univ}} = R^{\mathrm{univ}}_{\mathcal{S}}$ be the universal deformation ring for this deformation problem.

Space of automorphic forms

Let π_0 be the automorphic representation corresponding to ρ_0, unramified outside T. We will take automorphic forms on $D^\times$ where D is the quaternion algebra with $S(D) = \{v|\infty\}$ (we may since $|F : \mathbb{Q}|$ is even). Then our level subgroup

$$U = \prod_{v\nmid\infty} U_v \subset GL_2(\mathbb{A}_F^\infty)$$

will be a compact open subgroup given by

$$U_v = \begin{cases} GL_2(\mathcal{O}_{F_v}) & \text{v} \notin T_r \\ U_v \text{ such that } \pi_{0,v}^{U_v} \neq 0 & v \in T_r \end{cases}$$

Set $\mathcal{S} = \mathcal{S}(U, \mathcal{O}) = \mathcal{S}^U_{k,\eta,\chi|\cdot|_{\mathbb{A}_F^\times}}$ to be our space of automorphic forms.
k, η here are determined by the HT weights of ρ_0.

Hecke algebra

Let $\mathbb{T}_U$ be the Hecke algebra at level U (generated by the images of the unramified Hecke operators T_v, S_v for $v \notin T$ acting on $\mathcal{S}(U, \mathcal{O})$). $\mathbb{T}_U$ is a commutative $\mathcal{O}$-algebra which is finite free over $\mathcal{O}$ and reduced. We recall by strong multiplicity one that we had a bijective correspondence

$$\{\mathbb{T}_U \to \mathbb{C}\} \leftrightarrow \{\pi \subset S_{k,\eta,\chi} \text{ s.t. } \pi^U \neq 0\}.$$

The kernel of the composition

$$\begin{aligned} \mathbb{T}_U &\to \mathcal{O} \to k \\ T_v &\mapsto \operatorname{tr} \rho_0(\mathrm{Frob}_v) \\ S_v &\mapsto q^{-1} \det \rho(\mathrm{Frob}_v) \end{aligned}$$

(arising from the eigenvalue on π_0^U) defines a maximal ideal $\mathfrak{m} = \langle \lambda, \operatorname{tr}(\bar{\rho}(\mathrm{Frob}_v)) - T_v, \det \bar{\rho}(\mathrm{Frob}_v) - q_v S_v \rangle \subset \mathbb{T}_U$. Then $\bar{\rho}_m \cong \bar{\rho}$ and we saw the existence of a lift

$$\rho_{\mathfrak{m}}^{\mathrm{mod}} : G_F \to GL_2(\mathbb{T}_{U,\mathfrak{m}})$$

with $\operatorname{tr} \rho_{\mathfrak{m}}^{\mathrm{mod}}(\mathrm{Frob}_v) = T_v$ and $\det \rho_{\mathfrak{m}}^{\mathrm{mod}}(\mathrm{Frob}_v) = T_v$

The $R \to \mathbb{T}$ map

The representation $\rho_{\mathfrak{m}}^{\text{mod}}$ satisfies the deformation conditions $\{\mathcal{D}_v\}_{v \in T}$ (since all such $\pi \subset S_{k,\eta,\chi}$ s.t. $\pi^U \neq 0$ give rise to Galois representations satisfying the deformation conditions) and is unramified outside T. Hence by the defining property of R^{univ}, we get a map $R^{\text{univ}} \to \mathbb{T}_{U,\mathfrak{m}}$. The map is surjective, as $\operatorname{tr} \rho^{\text{univ}}(\text{Frob}_v) = T_v$ and $\det \rho^{\text{univ}}(\text{Frob}_v) = q_v S_v$.

To prove the theorem, it will suffice to show that for any $\rho : G_F \to GL_2(\mathcal{O})$ of type $\mathcal{S}$ lifting $\bar{\rho}$ that the map $R^{\text{univ}} \to \mathcal{O}$ factors through $\mathbb{T}_{U,\mathfrak{m}}$. Indeed, the map $\mathbb{T}_{U,\mathfrak{m}} \to \mathcal{O} \hookrightarrow \mathbb{C}$ defines an automorphic representation π_ρ, and the Galois representation arising from π_ρ is isomorphic to ρ (as the eigenvalues of Frob_v for $v \notin T$ agree). The map $R^{\text{univ}} \to \mathbb{T}_{U,\mathfrak{m}}$ factors through the reduced quotient $R^{\text{univ,red}} \to \mathbb{T}_{U,\mathfrak{m}}$ and so since $\mathcal{O}$ is reduced it will suffice to show that this map is an isomorphism.

Thanks for listening and feel free to ask any questions.