Automorphic forms on definite quaternion algebras: Integral theory

Vaughan McDonald

April 29, 2021

Outline

1. Recap and setup for integral theory
2. Integral automorphic forms on definite quaternion algebras
3. Hecke algebras on integral automorphic forms
4. Hecke algebras and Galois representations

Recap of last lecture

Last time we discussed automorphic forms on quaternion algebras $D^\times$, and global aspects of automorphic forms.
These automorphic forms were particularly concrete when $S(D) \supset S_\infty$, i.e. when D/F is **totally definite**. Then $\mathcal{S}_{D,k,\eta}$ is the space of functions $\phi : D^\times \backslash (D \otimes \mathbb{A}_F)^\times \to \bigotimes_{\tau: F \hookrightarrow \mathbb{R}} \mathrm{Sym}^{n_\tau}(\mathbb{C}^2) \otimes (\det)^{m_\tau}$ such that:

- φ is right-invariant under some compact open $U \subset \mathrm{GL}_2(\mathbb{A}_F^\infty)$.
- $\varphi(gh) = h^{-1}\varphi(g)$ for $h \in D_\infty^\times$
- $\varphi(gz) = \chi(z)\varphi(g)$

We also saw for any open subgroup $U \subset G(\mathbb{A}_F^\infty)$, the double quotient $G(F)\backslash G(\mathbb{A}_F^\infty)/U$ is finite.
This totally definite D/F situation is easier, so we focus on it. By Jacquet–Langlands and base change passing to this situation will suffice.

Jacquet–Langlands

Let F be a totally real field and D/F totally definite quaternion algebra.

Definition

The space of **automorphic forms** $\mathcal{A}_0(D^\times\backslash(A\otimes_F \mathbb{A})^\times, \chi)$ for a Hecke character $\chi : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ is the space of functions $\varphi : D^\times\backslash(D$

Theorem

There is a decomposition into irreducible representations of $(D\otimes \mathbb{A}_F)^\times$

$$\mathcal{A}_0(D^\times\backslash(D\otimes_F \mathbb{A}_F)^\times, \chi) \simeq \oplus \pi^D,$$

such that either:

- *(finite dimensional):* $\pi^D = \phi\circ\det$, *where* $\det$ *is the reduced more and* ϕ *is a Hecke character with* $\phi^2 = \chi$
- *(∞-dimensional):* $\pi^D \simeq \pi^\infty \otimes \pi^D_\infty$ *for* π *an aut. rep. of* $\mathrm{GL}_2(\mathbb{A}_F)$, *and* $\pi^D_\infty = \bigotimes_{\tau:F\hookrightarrow\mathbb{R}} \left(\mathrm{Sym}^{n_\tau}(\mathbb{C}^2)\otimes \det^{s_\tau+1/2}\right)^\vee$ *with other conditions*

Today

We will discuss integral theory of automorphic forms on definite quaternion algebras over totally real fields.
We can then use this to form the "$\mathbb{T}$" side of $R = \mathbb{T}$ theorems. That is, we construct a Galois representation valued in a localized Hecke algebra.

Notation

Domain:
Let F be a totally real field, D/F a quaternion algebra, $G = \mathrm{GL}_1(D)$ the associated algebraic group, and $Z \subset G$ its center.
Assumption:
Say D is totally definite, i.e. $S(D) = \{\nu \mid \infty\} =: S_\infty$, so that necessarily $[F : \mathbb{Q}]$ is even. For $\nu \nmid \infty$, $G(F_\nu) \simeq \mathrm{GL}_2(\mathbb{F}_\nu)$. Fix maximal orders $G(\mathcal{O}_{F_\nu}) \simeq \mathrm{GL}_2(\mathcal{O}_{F_\nu})$, and thus an isomorphism $G(\mathbb{A}_F^\infty) \simeq \mathrm{GL}_2(\mathbb{A}_F^\infty)$ The weights $(k, \eta) = ((k_\nu), (\eta_\nu))$ with $w = k_\nu + 2\eta_\nu - 1$ independent of ν. Let $S \subset \{\text{finite } v \nmid \ell\}$ be a finite set of places. For an open compact $U = \prod_{\nu \nmid \infty} U_\nu = U_S U^S \subseteq \mathrm{GL}_2(\mathbb{A}_F^\infty)$, assume $U^S = \prod_{\nu \notin S \cup S_\infty} \mathrm{GL}_2(\mathcal{O}_{F,\nu})$
Coefficients:
Let $L/\mathbb{Q}_\ell$ be a finite extension so that all embeddings $F \hookrightarrow \overline{L}$ are contained in L. $\mathcal{O}$ its ring of integers, $\lambda \subset \mathcal{O}$ a uniformizer, $\mathbb{F} := \mathcal{O}/\lambda$.
Fix an isomorphism $i : \overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$.
Let $\chi : \mathbb{A}_F^\times / F^\times \to \mathbb{C}^\times$ be a central character such that χ unramified outside S and $\chi \mid_{(F_\infty^\times)^0} (z) = z^{1-w}$.

Idelic class characters

Adding characters to the situation is a little fiddly, so just ignore for now. For now, $\chi : \mathbb{A}_F^\times / F^\times \to \mathbb{C}^\times \simeq \overline{\mathbb{Q}}_\ell^\times$ will be a Hecke character, whose infinite component one can twist away as necessary.
More to be added later

Setup

Natural idea: to define integral automorphic forms, take functions valued in $\Lambda = \otimes_{\tau: F \hookrightarrow \overline{L}} \mathrm{Sym}^{n_\tau}(\mathcal{O}^2) \otimes (\det)^{m_\tau}$. The problem is the definition of $S_{k,\chi}(\mathbb{C})$ is that it includes a $D_\infty^\times$-action. We deal with this as follows:

Lemma

$S_{k,\chi}(\mathbb{C}) \simeq \{f : G(\mathbb{A}_F^\infty) \to \otimes_\tau \mathrm{Sym}^{n_\tau}(\mathbb{C}^2) \otimes (\det)^{m_\tau}$:

- *$f(\delta g z) = \tilde{\chi}(z)\delta f(g)$ for $\delta \in D^\times, z \in (\mathbb{A}_F^\infty)^\times$.*
- *f is right-invariant under some open compact subgroup.*$\}$

Proof.

Given $\varphi \in \mathcal{S}_{k,\chi}(\mathbb{C})$, define $f : G(\mathbb{A}_F^\infty) \to \otimes_\tau \mathrm{Sym}^{n_\tau}(\mathbb{C}^2) \otimes (\det)^{m_\tau}$ by $f(g) := g_\infty \varphi(g)$. This is well-defined since if $h \in D^\times$, $f(gh) = g_\infty h \varphi(gh) = g_\infty \varphi(g)$. Conversely, send f to $\varphi(g) := g_\infty^{-1} f(g^\infty)$ □

This definition does not involve the $D_\infty^\times$-action, so we can define $S_{k,\chi}(\overline{\mathbb{Q}}_\ell)$

Definition of integral automorphic forms

We can then undo the unraveling of the ∞-component at ℓ:

Lemma

$S_{k,\chi}(\overline{\mathbb{Q}}_\ell) \simeq \{\varphi : D^\times \backslash G(\mathbb{A}_F^\infty) \to \bigotimes_{\tau : F \hookrightarrow \overline{\mathbb{Q}}_\ell} \mathrm{Sym}^{n_\tau}(\overline{\mathbb{Q}}_\ell^2) \otimes (\det)^{m_\tau} :$
$\varphi(guz) = \chi^{(\ell)}(z) u_\ell^{-1} \varphi(g)$ *for* $z \in (\mathbb{A}_F^\infty)^\times, u \in U$ *some open compact* $\}$.

Proof.

Send $f \in S_{k,\eta}(\overline{\mathbb{Q}}_\ell)$ to $\varphi(g) = g_\ell^{-1} f(g)$. The inverse is $\varphi \mapsto (f \mapsto g_\ell(\varphi(g)))$. □

Definition

For any finitely generated $\mathcal{O}$-module A, the A**-valued automorphic forms** of weight (k, η), level U, and character χ are $\mathcal{S}_{k,\chi}(U, A) :=$

$$\mathcal{S}_{k,\eta,\chi_0,i}(U, A) = \{\phi : G(F) \backslash G(\mathbb{A}_F^\infty) \to \Lambda \otimes_{\mathcal{O}} A : \phi(guz) = \chi(z) u_\ell^{-1} \cdot \phi(g), \}$$

where $g \in G(\mathbb{A}_F^\infty), u \in U, z \in Z(\mathbb{A}_F^\infty)$ and $u_\ell^{-1} \cdot \phi$ comes from the action of $u_\ell \in \mathrm{GL}_2(\mathcal{O}_{F,\ell}) = \prod_{\nu | \ell} \mathrm{GL}_2(\mathcal{O}_{F_\nu})$.

Basic structure

For any $U \subseteq \mathrm{GL}_2(\mathbb{A}_F^\infty)$ such that $U_\ell \subset \mathrm{GL}_2(\mathbb{F}_\ell) \cap M_2(\mathcal{O}_{F,\ell})$. Then U acts on $\mathcal{S}_{k,\chi}(U,A)$ for any A by $(u \cdot \varphi)(g) = u_\ell \varphi(g u_\ell)$.

Lemma

Choose coset representatives $\mathrm{GL}_2(\mathbb{A}_F^\infty) = \sqcup_{i \in I} D^\times g_i U(\mathbb{A}_F^\infty)^\times$. *We then have an isomorphism*

$$\mathcal{S}_{k,\chi}(U,A) \simeq \bigoplus_{i \in I} (\Lambda \otimes_{\mathcal{O}} A)^{(U \cdot (\mathbb{A}_F^\infty)^\times \cap g_i^{-1} D^\times g_i)/F^\times}$$

given by $\varphi \mapsto (\varphi(g_i))$. *(**Warning:** this depends on the choice of the* g_i*!)*

Proof.

The ambiguity of choice $g_i = \delta g_i u z$, i.e. when $uz \in U(\mathbb{A}_F^\infty)^\times \cap g_i^{-1} D^\times g_i$. Since $\varphi(g_i) = \varphi(g_i u z) = \chi(z) u_\ell^{-1} \varphi(g_i)$, so that $\chi(z)^{-1} u_\ell \varphi(g_i) = \varphi(g_i)$. So the map is well-defined and injective (double cosets det'd by one value). Surjective b/c cosets disjoint, and we det'd ambiguity of writing $g_i \in D^\times \backslash \mathrm{GL}_2(\mathbb{A}_F^\infty)/U$. □

Basic structure (continued)

We now discuss some basic structural results. Define $\Delta_{g,U} := (U \cdot (\mathbb{A}_F^\infty)^\times \cap g^{-1} D^\times g)/F^\times$. It is compact and discrete (since $D^\times \subset (D \otimes A)^\times$ is discrete), hence finite.

Definition

Say U is **sufficiently small** for ℓ if $\ell \nmid \#\Delta_{g,U}$ for all g.

Lemma

1. *$\mathcal{S}_{k,\chi}(U, \mathcal{O})$ is a finite free $\mathcal{O}-$module.*
2. *$\mathcal{S}_{k,\chi}(U, \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{C} \simeq S_{k,\chi}^U$ (complex automorphic forms with level U and character χ_0).*
3. *$\mathcal{S}_{k,\chi}(U, A) \simeq \mathcal{S}_{k,\chi}(U, \mathcal{O}) \otimes_{\mathcal{O}} A$*
4. *For $V \subset U$ open, $\mathcal{S}_{k,\chi}(U, A) \hookrightarrow \mathcal{S}_{k,\chi}(V, A)$.*
5. *If $[F(\zeta_\ell) : F] > 2$ then U is sufficiently small for ℓ (true if $\ell > 3$ and F tot. real).*

Proof of lemma

Proof.

1),2) follow from previous lemmas. 4) is clear. To show 3) it suffices by the previous lemma to prove $(\Lambda \otimes_{\mathcal{O}} A)^{\Delta_{g_i,U}} = \Lambda^{\Delta_{g_i,U}} \otimes_{\mathcal{O}} A$. Indeed, the idempotent $\frac{1}{\#\Delta_{g_i,U}}(\sum_{\delta \in \Delta_{g_i,U}} \delta)$ makes $\Lambda^{\Delta_{g_i,U}}$ a direct summand of Λ. But Λ is free, and so $\Lambda^{\Delta_{g_i,U}}$ is projective.

For 5), if $g^{-1}\delta g \in G_i$ then for $\delta \in D^\times$ then $\delta g_i u g_i^{-1} z$ for some $u \in U$ and $z \in Z(\mathbb{A}_F^\infty)$. $\det z = z^2$, so $\delta^2 / \det \delta \in D^\times \cap g_i U g_i^{-1} \det U$ which is compact and discrete, hence finite. So $\delta^2 / \det \delta$ is a root of unity in $D^\times$. But $[F(\zeta_\ell) : F] > 2$, and a quaternion algebra can only contain quadratic extensions of its center. So $\delta^N \in F^\times$ for N prime to ℓ, and the claim follows. □

The Hecke algebra

$\mathrm{GL}_2(\mathbb{A}_F^{\infty,\ell})$ acts on $S_{k,\chi}(A)$, but not $S_{k,\chi}(A)^U$. Instead, we have the usual double coset action: if $UgU = \sqcup g_i U$ and $\varphi \in S_{k,\eta}(A)^U$ then

$$(UgU)\cdot(\varphi) = \sum_i g_i\varphi \in S_{k,\chi}(A)^U.$$

Definition

The **Hecke algebra** $\mathbb{T}_U := \mathbb{T}^S_{k,\chi}(U,A) \subseteq \mathrm{End}_A(S_{k,\chi}(A)^U)$ is the subalgebra generated by $\mathbb{T}_\nu := U\begin{pmatrix}\omega_\nu & \\ & 1\end{pmatrix}U, S_\nu = U\omega_\nu U$ for $\nu \notin S$.

Lemma

$\mathbb{T}_U$ is a commutative $\mathcal{O}$-algebra and finite free as an $\mathcal{O}-$module.

Proof.

First part is easy since $T_\nu's$ are supported at different ν and thus the actions do not interact. It is a finitely generated submodule of $\mathrm{End}_{\mathcal{O}}(S_{k,\chi}(U,\mathcal{O}))$, which is finite free, so $\mathbb{T}_U$ is finite free. □

Hecke eigenspaces

Lemma

We have an isomorphism $T_U \otimes_{\mathcal{O}} \mathbb{C} \simeq \prod_{\pi \subset \mathcal{S}_{k,\chi}, \pi^U \neq 0} \mathbb{C}$, where π runs over RACARs of $G(\mathbb{A}_F^\infty)$ given by $T_\nu, S_\nu \mapsto T_\nu(s_{\pi_\nu}), S_\nu(s_{\pi_\nu})$, where s_{π_ν} is the associated Satake parameter at ν.

Proof.

Using the properties just described, we have a $\mathbb{T}_U$ equivariant isomorphism

$$\mathcal{S}(U, \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{C} \simeq \mathcal{S}^U_{k,\eta,\chi_0} \simeq \bigoplus_{\pi \subset \mathcal{S}_{k,\chi}, \pi^U \neq 0} \pi_S^{U_S} \otimes \left(\bigotimes_{\nu \notin S \cup S_\infty} \pi_\nu^{\mathrm{GL}_2(\mathcal{O}_{F_\nu})} \right).$$

and thus an algebra homomorphism $\mathbb{T}^S_{k,\chi}(U, \mathbb{C}) \to \prod_{\pi \in \mathcal{S}_{k,\chi}} \mathbb{C}$. If it were not surjective, the image would be a proper $\mathbb{C}$-subalgebra, with two coordinates equal. But this would two automorphic representations π, π' have the same T_ν-eigenvalues for almost all ν. Thus $\pi \simeq \pi'$ by strong multiplicity one. □

Hecke-algebra valued Galois representations

Since $\mathbb{T}_U$ is finite free over the complete DVR $\mathcal{O}$, it has only finitely many maximal ideals, and a general commutative algebra fact gives

$$\mathbb{T}_U \simeq \prod_{\mathfrak{m} \subset \operatorname{maxSpec} \mathbb{T}_U} \mathbb{T}_{U,\mathfrak{m}}.$$

We now focus our attention on $\mathbb{T}_{U,\mathfrak{m}}$ for a maximal ideal $\mathfrak{m} \subseteq \mathbb{T}_U$.

Goal

Construct a continuous $\bar{\rho}_{\mathfrak{m}} : G_F \to \mathrm{GL}_2(\mathbb{T}_U/\mathfrak{m})$ such that $\operatorname{tr} \bar{\rho}_{\mathfrak{m}}(\mathrm{Frob}_\nu) = T_\nu, \det \bar{\rho}_{\mathfrak{m}}(\mathrm{Frob}_\nu) = q_\nu S_\nu$. If $\bar{\rho}_{\mathfrak{m}}$ is irreducible (aka **non-Eisenstein**), construct a lift $\rho_{\mathfrak{m}} : G_F \to \mathrm{GL}_2(\mathbb{T}_{U,\mathfrak{m}})$.

Step 1: Use Jacquet–Langlands

By Jacquet–Langlands, an RACAR π of $G(\mathbb{A}_F^\infty)$ corresponds to a RACAR of $\mathrm{GL}_2(\mathbb{A}_F^\infty)^\times$, which have associated ℓ-adic representations $r_\ell(\pi) : G_F \to \mathrm{GL}_2(\bar{L})$ (this is a bit anachronistic). These are unramified outside $S' := S \cup \{\nu \mid \ell\}$ and for$\nu \notin S'$ satisfy

$$\mathrm{tr}(\rho_\pi(\mathrm{Frob}_\nu)) = T_\nu(s_{\pi_\nu}), \quad \det(\rho_\pi(\mathrm{Frob}_\nu)) = q_\nu S_\nu(s_{\pi_\nu}).$$

Grouping together for all $\pi \in \mathcal{S}_{k,\eta,\chi_0}$ and applying the lemma, we get a representation

$$\rho^{\mathrm{mod}} := \prod r_\ell(\pi) : G_F \to \prod_{\pi \subset \mathcal{S}_{k,\chi}} \mathrm{GL}_2(\bar{L}) \simeq \mathrm{GL}_2(\mathbb{T}_U \otimes_{\mathcal{O}} \bar{L}),$$

where $\mathrm{tr}\,\rho^{\mathrm{mod}}(\mathrm{Frob}_\nu) = T_\nu$ and $\det\rho^{\mathrm{mod}}(\mathrm{Frob}_\nu) = S_\nu q_\nu$ (the compatibility follows by the previous lemma).

The residual represention

For our fixed $\mathfrak{m} \subset \mathbb{T}^U$, we can apply going-down to $\operatorname{Spec} \mathbb{T}_U \to \operatorname{Spec} \mathcal{O}_{\bar{L}}$ and $(0) \subsetneq \mathfrak{m} \cap \mathcal{O}_{\bar{L}} \subset \mathcal{O}_{\bar{L}}$ to get a minimal prime $\mathfrak{p} \subsetneq \mathfrak{m} \subset \mathbb{T}_U \otimes \mathcal{O}_{\bar{L}}$. Then we have an injection $\theta : \mathbb{T}_U/\mathfrak{p} \hookrightarrow \overline{L} \simeq \mathbb{C}$. By the previous lemma, this corresponds to a unique π on $\mathrm{GL}_2(\mathbb{A}_F^\infty)$, and thus produces a $r_\ell(\pi)$,
After conjugation, one can assume any $r_\ell(\pi) : G_F \to \mathrm{GL}_2(\bar{L})$ has image in $\mathrm{GL}_2(\bar{\mathcal{O}})$. The $\mod \ell$ reduction then gives $\bar{\rho_\pi} : G_F \to \mathrm{GL}_2(\bar{\mathbb{F}})$. By construction $\mathrm{tr}(\rho_\pi) \in \mathbb{T}_U/\mathfrak{p} \subseteq \mathcal{O}_{\bar{L}}$, so the reduction has $\mathrm{tr}\,\bar{\rho}_\pi \in \mathbb{T}_U/\mathfrak{m} \subset \overline{\mathbb{F}}$. Since $\mathbb{T}_U/\mathfrak{m}$ is finite, the Brauer group is 0 so the image of G_F are the units of a split central simple algebra over $\mathbb{F}$, and thus can be conjugated to give $\bar{\rho}_\mathfrak{m} : G_F \to \mathrm{GL}_2(\mathbb{T}_U/\mathfrak{m})$.
To lift to characteristic 0, need to localize at $\mathfrak{m}$. Applying the previous lemma, this amounts to projecting onto a component of $\mathbb{T}_U$. Doing so gives a representation, $\rho_\mathfrak{m}^{\mathrm{mod}} : G_F \to \mathrm{GL}_2(\mathbb{T}_{U,\mathfrak{m}} \otimes \bar{L}) \simeq \prod_\pi \mathrm{GL}_2(\bar{L})$, where this is π such that $\bar{\rho}_\pi = \bar{\rho}_\mathfrak{m}$.

Lifting to characteristic 0

We can conjugate this to have image in $\prod_\pi \mathrm{GL}_2(\mathcal{O}_{\bar{L}})$. In fact, we can conjugate to lie in subring of elements whose reductions lie in $\mathbb{T}_U/\mathfrak{m}$. This can be summarized as follows:

$$\begin{array}{ccc} G_F & \xrightarrow{\rho_{\mathfrak{m}}^{\mathrm{mod}}} & \mathrm{GL}_2(\mathbb{T}_{U,\mathfrak{m}} \otimes_{\mathcal{O}} \mathcal{O}_{\bar{L}}) = \prod_{\pi : \bar{\rho}_\pi = \bar{\rho}_{\mathfrak{m}}} \mathrm{GL}_2(\mathcal{O}_{\bar{L}}) \\ & \searrow^{\bar{\rho}_{\mathfrak{m}}} & \downarrow \\ & & \mathrm{GL}_2(\mathbb{T}_U/\mathfrak{m}) \end{array}$$

Recall Carayol's lemma:

Lemma (Carayol)

If $(A, \mathfrak{m}_A)$ is a complete Noetherian $\mathcal{O}$-algebra, $\rho : G_F \to \mathrm{GL}_n(A)$ with $\bar{\rho} := \rho \bmod \mathfrak{m}_A$ is absolutely irreducible, and $\mathrm{tr}(\rho(G_F)) \subset B$ for $B \subset A$ a closed subring, then there exists $a \in \ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(\mathbb{F}))$ such that $a\rho a^{-1}$ factors through $\mathrm{GL}_n(B)$.

Thus, if $\bar{\rho}_{\mathfrak{m}}$ is irreducible, we can conjugate to $\rho_{\mathfrak{m}}^{\mathrm{mod}} \to \mathrm{GL}_2(\mathbb{T}_{U,\mathfrak{m}})$.

Ending remarks

A key technical point here was that the algebra $T_{U,\mathfrak{m}}$ is $\mathcal{O}$-flat, so that $T_{U,\mathfrak{m}} \to T_{U,\mathfrak{m}} \otimes \bar{\mathbb{Q}}_\ell$ is an injection.
We showed $\mathbb{T}_{U,\mathfrak{m}}$ is $\mathcal{O}$-flat by showing $\mathbb{T}_U$ is free, because they act faithfully on automorphic forms on a quaternion algebra and those are easy.
What if one tried to do this over $\mathbb{Q}$ with modular forms? Eichler–Shimura isomorphism gives a $\mathbb{T}^S(\Gamma, k)$-equivariant isomorphism

$$M_k(\Gamma, \mathbb{C}) \oplus S_k(\Gamma, \mathbb{C}) \simeq H^1(\Gamma, \operatorname{Sym}^{k-2} \mathbb{C}^2).$$

For $\mathfrak{m} \subset \mathbb{T}^S(\Gamma, k)$ non-Eisenstein corresponding to $g \in S_k(\Gamma, \mathcal{O})$, one shows that $H^1(\Gamma, \operatorname{Sym}^{k-2} \mathcal{O}^2)_\mathfrak{m}$ is finite-free over $\mathcal{O}$, so $\mathbb{T}^S(\Gamma, k)_\mathfrak{m}$ is $\mathcal{O}$-flat.
The easier argument we had is an advantage of base changing to totally real fields and Jacquet–Langlands be an advantage of totally real fields.

References

- Rong's notes, Toby Gee's notes
- Richard Taylor Stanford notes on Automorphy Lifting