Automorphic forms for quaternion algebras

Recollections from last time

Last time we talked about the local Langlands correspondence, local base change and the local Jacquet-Langlands correspondence.

Let K be a finite extension of $\mathbb{Q}_p$. The local Langlands correspondence is a bijection

$$\begin{array}{rcl} \text{irreducible smooth} & \overset{\text{rec}}{\longleftrightarrow} & \text{Frobenius-semisimple Weil-Deligne} \\ \mathrm{GL}_n(K)\text{-representations} & & \text{representations of } W_K \end{array}$$

$$\begin{array}{rcl} \text{supercuspidals} & \longleftrightarrow & \text{irreducibles} \\ \text{discrete series} & \longleftrightarrow & \text{indecomposables} \end{array}$$

which we described explicitly in the unramified case using Satake parameters.

If K'/K is a finite extension and $\pi \in \mathrm{Irr}(\mathrm{GL}_n(K))$, then write $\mathrm{BC}(\pi)$ for the image of π under

$$\mathrm{Irr}(\mathrm{GL}_n(K)) \xrightarrow{\mathrm{rec}_K} \mathrm{Rep}(\mathrm{WD}_K)^{\text{F-ss}} \longrightarrow \mathrm{Rep}(\mathrm{WD}_{K'})^{\text{F-ss}} \xrightarrow{\mathrm{rec}_{K'}^{-1}} \mathrm{Irr}(\mathrm{GL}_n(K'))$$

Proposition

Assume K'/K is cyclic, $\mathrm{Gal}(K'/K) = \langle \sigma \rangle$.

1. $\pi' \in \mathrm{Irr}(\mathrm{GL}_n(K'))$ is in the image if and only if $\pi' \simeq \pi' \circ \sigma$ (this has the same underlying space as π', but $\mathrm{GL}_n(K')$ acts through $\pi' \circ \sigma$).
2. $\mathrm{BC}(\pi_1) \simeq \mathrm{BC}(\pi_2) \iff \pi_1 \simeq \pi_2 \otimes \chi \circ \det$ for some smooth $K^\times \longrightarrow \mathbb{C}^\times$ which is trivial on $N_{K'/K}(K'^\times)$.
3. $\mathrm{BC}(\pi)$ is supercuspidal $\iff \pi$ is supercuspidal and $\pi \not\simeq \pi \otimes \chi \circ \det$ for all $\chi \neq 1$ as above.
4. The central character of $\mathrm{BC}(\pi)$ is $\omega_{\mathrm{BC}(\pi)} = \omega_\pi \circ N_{K'/K}$.

Let D be a central division algebra over K of dimension n^2. $D^\times$ is a locally profinite group.

$D \otimes_K \overline{K} \simeq \mathrm{M}_n(\overline{K})$, and we can associate to each regular element $\delta \in D^\times_{\mathrm{reg}}$ (i.e. semisimple with distinct eigenvalues) a regular element $\gamma \in \mathrm{GL}_n(K)_{\mathrm{reg}}$, up to conjugation.

If π is a finite length smooth representation of a locally profinite group G, then there exists a locally constant function $\Theta_\pi \colon G_{\mathrm{reg}} \longrightarrow \mathbb{C}$ such that for all $f \in \mathcal{H}(G)$,

$$\operatorname{tr} \pi(f) = \int_{G_{\mathrm{reg}}} \Theta_\pi(g) f(g) dg.$$

Theorem (Local Jacquet-Langlands correspondence)

There exists a unique bijection

$$JL \colon Irr(D^\times) \xrightarrow{\sim} Irr^2(GL_n(K))$$

such that for $\pi \in Irr(D^\times)$ and δ, γ as above, $\Theta_\pi(\delta) = (-1)^{n-1}\Theta_{JL(\pi)}(\gamma)$.

Goal for today

Today we will look at global analogues for these correspondences in the setting of GL_2 and quaternion algebras.

Some notation

Let F be a number field, S a finite set of places, S_∞ the set of all archimedean places and $S^\infty := S \setminus S_\infty$.

Write $F_S = \prod_{v \in S} F_v, \mathcal{O}_{F,S} = \prod_{v \in S^\infty} \mathcal{O}_{F_v}$ and $\widehat{\mathcal{O}}_F^S = \prod_{v \notin S \cup S_\infty} \mathcal{O}_{F_v}$.

Write also $\mathbb{A}_F^S = \prod'_{v \notin S} F_v$, $\mathbb{A}_F^\infty = \mathbb{A}_F^{S_\infty}$ and $F_\infty = F_{S_\infty}$.

Central simple algebras

Let D be a central simple algebra over F of dimension n^2. If $n = 2$, we say D is a quaternion algebra.

Fact

The set $S(D) = \{v : D \otimes_F F_v \not\simeq \mathsf{M}_n(F_v)\}$ is finite.
If $n = 2$, then $S(D)$ is even, and $D \longmapsto S(D)$ is a bijection between quaternion algebras over F and finite sets of places of F of even cardinality.

Consider the algebraic group $G = D^\times$, defined by $G(R) = (D \otimes_F R)^\times$ for any F-algebra R.

Note that if $v \notin S_\infty$, then $G(F_v)$ is locally profinite, and if $S \supseteq S_\infty$, then so is $G(\mathbb{A}_F^S)$.

Assume $S \supseteq S_\infty$. Let $\{\pi_v\}_{v \notin S}$ be a collection of irreducible (smooth) representations such that π_v is unramified for almost all v, and write S_{ram} for the $v \notin S \cup S(D)$ such that π_v is ramified.

Define

$$\bigotimes_{v \notin S}{}' \pi_v := \varinjlim_{\substack{T \supseteq S_{\text{ram}} \cup (S(D) \setminus S) \text{ finite} \\ T \cap S = \emptyset}} \left(\bigotimes_{v \in T} \pi_v \otimes \bigotimes_{v \notin T \cup S} \pi_v^{G(\mathcal{O}_{F_v})} \right)$$

(recall that if π_v is unramified and irreducible, then $\pi_v^{G(\mathcal{O}_{F_v})}$ is 1-dimensional)

Theorem (Flath)

The association $\{\pi_v\}_v \longmapsto \bigotimes'_{v \notin S} \pi_v$ is a bijection between collections $\{\pi_v\}$ unramified almost everywhere and irreducible smooth representations of $G(\mathbb{A}_F^S)$.

Automorphic forms for quaternion algebras

Assume now that $n = 2$ and F is totally real.

If $v \in S_\infty$, then

$$G(F_v) \simeq \begin{cases} \mathsf{GL}_2(\mathbb{R}) & v \notin S(D) \\ \mathbb{H}^\times & v \in S(D) \end{cases}$$

where $\mathbb{H}$ are the Hamilton quaternions. Thus,

$$U_v = \begin{cases} \mathbb{R}^\times \mathsf{SO}_2(\mathbb{R}) & v \notin S(D) \\ \mathbb{H}^\times & v \in S(D) \end{cases}$$

is a maximal compact-mod-center subgroup of $G(F_v)$.

Set $U_\infty = \prod_{v|\infty} U_v$.

Let $(k_v, \eta_v)_{v|\infty}$ be integers with $k_v \geq 2$ for all v and such that $w = k_v + 2\eta_v - 1$ is independent of v. Consider the representation of U_∞

$$W = \bigotimes_{v \in S_\infty \cap S(D)} \mathsf{Sym}^{k_v - 2}(\mathbb{C}^2) \otimes \det(\mathbb{C}^2)^{\eta_v} \otimes \bigotimes_{v \in S_\infty \setminus S(D)} \mathbb{C}_{k_v, \eta_v}$$

where $\mathbb{C}_{k_v,\eta_v}$ is the 1-dimensional representation of $\mathbb{R}^\times \mathsf{SO}_2(\mathbb{R})$ where $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^\times \mathsf{SO}_2(\mathbb{R})$ acts via $(ci + d)^{k_v} (\det \gamma)^{\eta_v - 1}$.

Definition

The space $\mathcal{S}_{D,k,\eta}$ of *automorphic forms for D of weight (k,η)* is the space of functions $\phi\colon G(F)\backslash G(\mathbb{A}_F) \longrightarrow W$ satisfying

1. ϕ is right-translation invariant by some open compact $U^\infty \subseteq G(\mathbb{A}_F^\infty)$
2. ϕ is equivariant for the action of $U_\infty \subseteq G(F_\infty)$
3. for all $g \in G(\mathbb{A}_F^\infty)$, the function

$$\begin{aligned} (\mathbb{C} \setminus \mathbb{R})^{S_\infty \setminus S(D)} &\longrightarrow W \\ h \cdot (i, ..., i) &\longmapsto h \cdot \phi(gh) \end{aligned}$$

   for $h \in (\mathbb{R}^\times \mathrm{SO}_2(\mathbb{R}))^{S_\infty \setminus S(D)}$ is holomorphic

Definition (continued)

4. if $F = \mathbb{Q}$ and $S(D) = \emptyset$, then for all $g \in G(\mathbb{A}_F^\infty) \simeq \mathsf{GL}_2(\mathbb{A}_F^\infty)$, the function

$$\begin{aligned} \mathbb{C} \setminus \mathbb{R} &\longrightarrow W \\ h \cdot i &\longmapsto \phi(gh)|\operatorname{Im}(hi)|^{k/2} \end{aligned}$$

   is bounded

5. if $S(D) = \emptyset$, then for all $g \in G(\mathbb{A}_F) \simeq \mathsf{GL}_2(\mathbb{A}_F)$,

$$\int_{F/\mathbb{A}_F} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx = 0.$$

$G(\mathbb{A}_F^\infty)$ acts smoothly on $\mathcal{S}_{D,k,\eta}$ by right translation.

Example

If $F = \mathbb{Q}$ and $S(D) = \emptyset$, so $G = \mathsf{GL}_{2/\mathbb{Q}}$, for any $N \geq 1$ set

$$U_1(N) = \{g \in \mathsf{GL}_2(\widehat{\mathbb{Z}}) : g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \mod N\}.$$

Then $\mathcal{S}_{D,k,0}^{U_1(N)}$ is isomorphic to the usual space $\mathcal{S}_k(\Gamma_1(N))$ of modular cusp forms of weight k and level $\Gamma_1(N)$. An automorphic form ϕ corresponds to

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} i \longmapsto (ci + d)^k \phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right).$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^\times \mathsf{SO}_2(\mathbb{R})$.

Fact

$\mathcal{S}_{D,k,\eta}$ is an admissible and semisimple representation of $G(\mathbb{A}_F^\infty)$.

Definition

The irreducible constituents of $\mathcal{S}_{D,k,\eta}$ are called *regular algebraic cuspidal automorphic representations of* $G(\mathbb{A}_F^\infty)$ *of weight* (k,η).

Theorem (Multiplicity one)

Each irreducible $\pi \subseteq \mathcal{S}_{D,k,\eta}$ *appears exactly once as a direct summand. Moreover, if* $\pi \subseteq \mathcal{S}_{D,k,\eta}$ *and* $\pi' \subseteq \mathcal{S}_{D,k',\eta'}$ *satisfy* $\pi_v \simeq \pi'_v$ *for almost all* v*, then* $(k,\eta) = (k',\eta')$ *and* $\pi = \pi'$*.*

If $S(D) \supseteq S_\infty$, we say D is a definite quaternion algebra. In this case, we have:

Fact

$G(F)\backslash G(\mathbb{A}_F^\infty)$ is compact.
For all open compact $U^\infty \subseteq G(\mathbb{A}_F^\infty)$, the space $G(F)\backslash G(\mathbb{A}_F^\infty)/U^\infty$ is finite.

In particular, the spaces $\mathcal{S}_{D,k,\eta}$ are very simple:

$$\mathcal{S}_{D,2,0} = \varinjlim_{U^\infty} \{G(F)\backslash G(\mathbb{A}_F^\infty)/U^\infty \longrightarrow \mathbb{C}\},$$
$$\mathcal{S}_{D,k,\eta} = \mathrm{Hom}_{G(F_\infty)}(W^\vee, \mathcal{S}_{D,2,0}).$$

The global Jacquet-Langlands correspondence

Let D be a quaternion algebra over F, $G = D^\times$.

Theorem (Jacquet-Langlands, Badulescu)

There is a bijection JL from the infinite-dimensional cuspidal automorphic representations of $G(\mathbb{A}_F^\infty)$ of weight (k, η) to the cuspidal automorphic representations of $GL_2(\mathbb{A}_F^\infty)$ of weight (k, η) which are discrete series at all finite places $v \in S(D)$ and such that

$$JL(\pi)_v = \begin{cases} \pi_v & v \notin S(D), \\ JL(\pi_v) & v \in S(D). \end{cases}$$

Theorem (Cyclic base change)

Let E/F be a cyclic extension of prime degree, and fix generators σ, δ of Gal(E/F) and Hom(Gal$(E/F), \mathbb{C}^\times)$ respectively. Let π be a cuspidal automorphic representation of GL$_2(\mathbb{A}_F^\infty)$ of weight (k, η). Then, there exists a cuspidal automorphic representation BC(π) of GL$_2(\mathbb{A}_E^\infty)$ of weight $(\mathit{BC}(k), \mathit{BC}(\eta))$ such that

1. *For all finite places v of E lying above a place w of F, $\mathit{BC}_{E/F}(\pi)_v = \mathit{BC}_{E_v/F_w}(\pi_w)$.*
2. *$\mathit{BC}(k)_v = k_w$ and $\mathit{BC}(\eta)_v = \eta_w$ for v, w as above.*
3. *$\mathit{BC}(\pi) \simeq \mathit{BC}(\pi') \iff \pi \simeq \pi' \otimes (\delta^i \circ \mathit{Art}_F \circ \det)$ for some i.*
4. *A cuspidal automorphic representation π of GL$_2(\mathbb{A}_E^\infty)$ is in the image of BC $\iff \pi = \pi \circ \sigma$.*

The global Langlands correspondence

Let p be prime, F an number field, $\mathcal{G}_F = \mathrm{Gal}(\overline{F}/F)$ and $n \geq 1$. Fix an isomorphism $\iota\colon \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$.

Conjecture (Global Langlands correspondence)

There exists a unique bijection between regular algebraic cuspidal automorhic representations π of $\mathrm{GL}_n(\mathbb{A}_F)$ and irreducible Galois representations $r\colon \mathcal{G}_F \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ which are de Rham at all $v|p$ such that

- For all $v \nmid \infty$ we have $\mathrm{rec}_{F_v}(\pi_v \otimes |\det|^{\frac{1-n}{2}}) \simeq \mathrm{WD}(r|_{\mathcal{G}_{F_v}})^{\text{F-ss}}$.
- For all $v|\infty$, the weight of π at v corresponds to $HT_{\iota^{-1}v}(r|_{\mathcal{G}_{F_v}})$.

In the case where $n = 2$ and F is totally real, we know the following:

Theorem (Carayol, Wiles, Taylor, Blasius-Rogawski, Saito, Skinner)

Let $\pi \in \mathcal{S}_{GL_{2/F},k,\eta}$ be regular algebraic. There exists a CM field $L_\pi \subseteq \mathbb{C}$ and $\{r_\lambda(\pi) : \mathcal{G}_F \longrightarrow GL_2(\overline{L}_{\pi,\lambda})\}_\lambda$ for finite places λ of L_π such that, if we fix an embedding $\overline{L}_\pi \hookrightarrow \mathbb{C}$ and extensions $\iota \colon \overline{L}_{\pi,\lambda} \longrightarrow \mathbb{C}$, then:

1. *For finite places v of F not dividing the residue characteristic of λ, $rec(\pi_v \otimes |\det|^{-1/2}) \simeq WD(r_\lambda(\pi)|_{\mathcal{G}_{F_v}})^{F\text{-}ss}$*
2. *The characteristic polynomial of $r_\lambda(\pi)(Frob_v)$ has coefficients in L_π*
3. *For all v dividing the residue characteristic of λ, $r_\lambda(\pi)|_{\mathcal{G}_{F_v}}$ is de Rham with τ-Hodge-Tate weights $\eta_\tau, \eta_\tau + k_\tau - 1$ for any embedding $\tau \colon F \hookrightarrow \overline{L}_\pi \hookrightarrow \mathbb{C}$ lying over v. If π_v is unramified, then $r_\lambda(\pi)|_{\mathcal{G}_{F_v}}$ is crystalline*
4. *For all $v|\infty$, $\det(r_\lambda(c_v)) = -1$, where c_v is complex conjugation.*

Remark

Condition (1) implies that π_v is unramified $\iff$ $r_\lambda(\pi)|_{\mathcal{G}_{F_v}}$ is unramified. In this case, the characteristic polynomial of $r_\lambda(\pi)(\mathrm{Frob}_v)$ is

$$X^2 - t_v X + (\# k_v) s_v$$

where t_v and s_v are the eigenvalues of $T_v = [\mathrm{GL}_2(\mathcal{O}_v) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_v)]$ and $S_v = [\mathrm{GL}_2(\mathcal{O}_v) \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_v)]$ in $\pi_v^{\mathrm{GL}_2(\mathcal{O}_v)}$.

Definition

A representation $\rho\colon \mathcal{G}_F \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ is modular if it is of the form $\iota \circ r_\lambda(\pi)$ for some π as above and $\iota\colon L_\pi \hookrightarrow \overline{\mathbb{Q}}_\ell$.

Proposition

Let E/F be a finite solvable Galois extension of totally real fields. Then $r\colon \mathcal{G}_F \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ is modular $\iff$ $r|_{\mathcal{G}_E}$ is modular.

Proof

By induction we may assume that E/F is cyclic of prime degree, $\mathrm{Gal}(E/F) = \langle\sigma\rangle$. If r is modular and π is the corresponding automorphic representation, then $r|_{\mathcal{G}_E}$ is modular with corresponding automorphic representation $\mathrm{BC}_{E/F}(\pi)$.

Proof (continued)

Suppose $r|_{\mathcal{G}_E} \simeq \iota \circ r_\lambda(\pi)$ is modular. If v does not divide the residue characteristic of λ and E/F is unramified at v, then $r(\mathrm{Frob}_v)$ and $r(\mathrm{Frob}_{v\sigma})$ have the same characteristic polynomial (as they are conjugate), so by condition (1), $\pi_v \simeq \pi_{\sigma v} \circ \sigma_v = (\pi \circ \sigma)_v$ (where $\sigma_v \colon E_v \longrightarrow E_{v\sigma}$) so by strong multiplicity one we have $\pi \circ \sigma \simeq \pi$. Hence, there exists an automorphic reprensentation π' such that $\pi = \mathrm{BC}_{E/F}(\pi')$.
Let $r' = \iota \circ r_\lambda(\pi')$. If $\widetilde{\sigma}$ is a lift of σ to $\mathcal{G}_F$, then $r'(\widetilde{\sigma})^{-1} r(\widetilde{\sigma}) r(g) = r'(g) r'(\widetilde{\sigma})^{-1} r(\widetilde{\sigma})$ for all $g \in \mathcal{G}_E$, so by Schur's lemma $r'(\widetilde{\sigma})^{-1} r(\widetilde{\sigma})$ must act as a constant.
In particular, if we define $\chi \colon \mathrm{Gal}(E/F) \longrightarrow \overline{\mathbb{Q}}_\ell$, by $\chi(\sigma) = r'(\widetilde{\sigma})^{-1} r(\widetilde{\sigma})$, we have $r = \chi \otimes r'$. By checking the characteristic polynomials of Frobenii, we see that $r = \iota \circ r_\lambda(\pi' \otimes (\chi \circ \mathrm{Art}_F))$. □