The local Langlands correspondence

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Outline

- Structure of representations of $GL_n(F)$, for F local
- Structure of Weil-Deligne representations
- Statement of local Langlands correspondence
- Satake transform and unramified local Langlands
- Statement of local base change

Notation

$F/\mathbb{Q}_p$ is a local field with uniformiser ϖ, $|\mathcal{O}_F/(\varpi)| = q$.
We do not assume characters are unitary.
G usually denotes $GL_n(F)$ and H $GL_n(\mathcal{O}_F)$.

Recall that a representation (π, V) of $G = GL_n(F)$ is smooth if for all $v \in V$, $\mathrm{stab}(v)$ is open in G, and admissible if it is smooth and for all open compact $K \geqslant G$, $\dim(V^K) < \infty$.
The irreducible admissible representations of $GL_n(F)$ can be organised as follows:

$$Irr^{sc}(GL_n(F)) \subset Irr^2(GL_n(F)) \subset Irr^{adm}(GL_n(F)),$$

where Irr^{sc} denoted supercuspidal representations and Irr^2 denotes square-integrable (or discrete series representations.
For GL_2, all irr. adm. representations not in Irr^2 are principal series representations.

Fact: Schur's lemma holds for irreducible admissible representations of $G = GL_n(F)$.
Therefore the centre $Z(G)$ of G must act by scalars on V. $Z(G) \cong F^\times$ and so we can define the central character of π by $\omega_\pi : F^\times \to \mathbb{C}^\times$.
In fact any irreducible smooth finite-dimensional representation of G is 1-dimensional and of the form $\chi \circ \det$.
Such representations do not occur as constituents of automorphic representations.

Principal series representation

Let $G = GL_n(F)$, B the Borel subgroup of upper triangular matrices. If $\chi_1, ..., \chi_n$ are characters $F^\times \to \mathbb{C}^\times$, we can construct a character $\chi : B \to \mathbb{C}^\times$ by defining $\chi(b) = \prod \chi_i(b_{ii})$. Define $\delta : B \to \mathbb{C}^\times$ by

$$\delta(b) = \prod_{i<j} |b_{ii}/b_{jj}|.$$

Then define the normalised induced representation of χ from B to G by

$$Ind_B^G(\chi) = \{f : G \to \mathbb{C} | f(bg) = \delta(b)^{1/2}\chi(b)f(g)\}.$$

where the functions f are locally constant, $b \in B$ and $g \in G$. This is a G-representation through $g(f(h)) = f(hg)$.
$Ind_B^G(\chi)$ is admissible. It is irreducible iff $\chi_i \neq \chi_j |.|^{\pm 1}$ for all i, j. Two such representations are isomorphic iff they come from the same characters, possibly after reordering. If the representation is irreducible, it is called a principal series representation.
If it is not irreducible, it contains either a subrepresentation of codimension 1 or a quotient representation by a 1-dimensional subrepresentation which is irreducible and admissible. In either case it is called a special (or Steinberg) representation, denoted Sp.

Discrete series and supercuspidal

Let $(\pi, V) \in Irr^{adm}(GL_n(F))$ with $V^\vee$ its smooth dual. For $v \in V, f \in V^\vee$ define the matrix coefficient $\phi_{v,f} : GL_n(F) \to \mathbb{C}$ by $\phi_{v,f} = f(\pi(g)v)$.

(π, V) is a supercuspidal representation if all matrix coefficients $\phi_{v,f}$ have compact support modulo $F^\times$.

(π, V) is square integrable (or discrete series) if for every $\phi_{v,f}$,

$$\int_{GL_n(F)/F^\times} |\phi_{v,f}(g)\omega_\pi(\det(g))^{-1/n}|^2 d\mu < \infty,$$

where ω_π is the central character of π.
Supercuspidal implies discrete series, but the converse is not true for $n > 1$.

Classification for $GL_2(F)$

An irreducible admissible representation of $GL_2(F)$ is one of the following disjoint types

- 1-dimensional, of the form $\chi \circ \det$,
- principal series $Ind_B^G(\chi_1, \chi_2)$ where $\chi_1\chi_2^{-1} \neq |.|^{\pm 1}$,
- special representation (possibly twisted by a character), $Sp \otimes \chi$,
- supercuspidal.

For GL_n, $n > 2$, things are more complicated.

Bernstein-Zelevinsky classification

Let $\pi \in Irr^{sc}(GL_m(F))$, $m, r \geqslant 1$. Then

$$\pi \times \pi|\det| \times \ldots \times \pi \det^{r-1}$$

has a unique irreducible quotient $Sp_r(m) \in GL_{m^r}(F)$.

For $\pi_i \in Irr^2(GL_{n_i}(F))$ suitably ordered, there exists exactly one irreducible quotient of $\pi_1 \times \ldots \times \pi_r$, called the Langlands quotient and denoted $\pi_1 \boxplus \ldots \boxplus \pi_r$.

$Irr^{adm}(GL_n(F))$ is in bijection with

$$\{(r, \{\pi_i\}_{i=1}^r | r \geqslant 1, \pi_i \in Irr^2(GL_{n_i}(F), \sum n_i = n\},$$

where the bijection is given by sending $(r, \{\pi_i\})$ to $\boxplus_i \pi_i$.

Automorphic representations

Let $f \in S_2(\Gamma_0(N))$ be a newform with character χ. There is a canonical way of associating to f a representation $\pi_{f,p}$ of $GL_2(\mathbb{Q}_p)$. $\pi_{f,p}$ turns out to be admissible and irreducible. Let f correspond to an elliptic curve E by Eichler-Shimura. Then

• if $p \nmid N$ then $\pi_{f,p}$ is a principal series representation coming from two unramified characters χ_1 and χ_2 such that $\chi_1(\varpi)$ and $\chi_2(\varpi)$ are the roots of $X^2 - a_pX + \chi(p)p^{k-1}$. We can see that it is irreducible: if it weren't, we would have $\chi_1(\varpi)/\chi_2(\varpi) = q^{\pm 1}$. But then $a_p = q + 1$, violating the Hasse-Weil bound!

• if p divides N exactly, $\pi_{f,p} = Sp \otimes$ (unram. quadr. char.)

• if $p^2|N$, $\pi_{f,p}$ is ramified principal series iff E acquires good reduction over an abelian extension of $\mathbb{Q}_p$, and supercuspidal or $Sp \otimes$ (ram quadr. char.)

Weil-Deligne representations again

A Weil-Deligne representation is a representation $\rho : W_F \to GL_{\mathbb{C}}(V)$ together with a nilpotent $N \in \mathrm{End}_{\mathbb{C}}(V)$ such that
- $\rho(I_F)$ is finite and
- $\forall \sigma \in W_F, \rho(\sigma)N\rho(\sigma)^{-1} = q^{-v_F(\sigma)}N$.

A WD representation is Frobenius semi-simple if for every lift ϕ of the geometric Frobenius, $\rho(\phi)$ is semi-simple.

$Rep_n^{Fss}(WD_F)$ can be built up from irreducible WD representations as follows, see [Del]:

$Rep_n^{indec}(WD_F)$ is in bijection with $\{(r, \rho) : r \geqslant 1, r|n, \rho \in Rep_{n/r}^{irr}(WD_F)\}$, with map $(r, \rho) \mapsto Sp_r(\rho) = (\rho \oplus \rho|.| \oplus \ldots \oplus \rho|.|^{s-1}, N)$, where $N = (\delta_{i,i+1})_{ij}$.

Furthermore $Rep^{Fss}(WD_F)$ is in bijection with
$\{(n_i, \{\rho_i\})|n_i \geqslant 1, \rho_i \in Rep_{n_i}^{indec}(WD_F), \sum n_i = n\}$, the map given by direct sum.
Notice how this mirrors the Bernstein-Zelevinsky classification.

Local Langlands correspondence (Harris-Taylor, Henniart, Scholze)

Let $F/\mathbb{Q}_p$ be a finite extension. Then there exists exactly one bijection

$$rec_F^n : Irr(GL_n(F)) \to Rep^{Fss}(WD_F)$$

such that rec_F^1 is the Artin map from local class field theory, i.e. sending $\chi : F^\times \to \mathbb{C}^\times$ to $W_F \to W_F^{ab} \xrightarrow{Art_F} K^\times \to \mathbb{C}^\times$, and such that rec_F^n preserves L-factors and ϵ-factors in pairs:

1) $L(s, \pi_1 \times \pi_2) = L(s, rec(\pi_1) \otimes rec(\pi_2))$

2) $\epsilon(s, \pi_1 \times \pi_2) = \epsilon(s, rec(\pi_1) \otimes rec(\pi_2))$

Furthermore,

- rec_F preserves conductors,
- if π corresonds to ρ, then $\pi^\vee$ corresponds to $\rho^\vee$,
- the central character ω_π corresponds to $\det(\rho)$ under rec_F^1,
- $\pi \times (\chi \circ \det)$ corresponds to $\rho \otimes rec_F^1(\chi)$,
- $\pi_1 \boxplus \pi_2$ corresponds to $rec(\pi_1) \oplus rec(\pi_2)$,
- if $\pi \in Irr^{sc}(GL_n(F))$ corresponds to ρ, then $Sp_r(\pi)$ corresponds to $Sp_r(\rho)$ for all $r \geqslant 1$.

Moreover,

$$Irr(GL_n(F)) \to Rep_n^{Fss}(WD_F)$$
$$Irr^2(GL_n(F)) \to Rep_n^{indec}(WD_F)$$
$$Irr^{sc}(GL_n(F)) \to Rep_n^{irr}(WD_F)$$

so by [B-Z] and [Del] it is enough to establish the last correspondence satisfying properties 1) and 2) to get the entire Langlands correspondence.

Conductors, L-factors, e factors

The Hecke algebra

Let G be a locally profinite unimodular group with Haar measure μ and K an open compact subgroup. $\mathcal{H} = \mathcal{H}(G, K)$ is the algebra of compactly supported locally constant K-biinvariant functions $G \to \mathbb{C}$ with convolution product given by

$$(f * g)(x) = \int_G f(xh^{-1})g(h)dh.$$

There is a bijection from $\{$irreducible smooth representations (π, V) of G with $V^K \neq 0\}$ to $\{$simple $\mathcal{H}(G, K)$-modules$\}$ given by $V \mapsto V^K$.
The action of $\mathcal{H}(G, K)$ on V^K is given by

$$\pi(f)(v) = \int_G f(g)\pi(g)vdg.$$

Each $f \in \mathcal{H}(G, K)$ is constant on double cosets KxK. f is compactly supported, so it is a finite linear combination of characteristic functions of double cosets. These characteristic functions give a $\mathbb{Z}$-basis for $\mathcal{H}(G, K)$.

$G = GL_n(F)$. Let B be the Borel subgroup of upper triangular matrices, T the maximal torus of diagonal matrices and N the unipotent radical of B.
There is an isomorphism

$$\mathcal{H}(T(F)/T(\mathcal{O}_F)) \cong \mathbb{C}[x_1^{\pm 1}, ..., x_n^{\pm 1}]$$

given by taking the characteristic function of $T(\mathcal{O}_F) diag(\varpi^{a_1}, ..., \varpi^{a_n}) T(\mathcal{O}_F)$ to $x^{a_1} \cdots x^{a_n}$.
Define $\mathcal{H}^{ur} := \mathcal{H}(GL_n(F), GL_n(\mathcal{O}_F))$.

The Satake transform

The Satake transform is the map

$$S : \mathcal{H}^{ur} \to \mathcal{H}(T(F), T(\mathcal{O}_F)) \cong \mathbb{C}[x_1^{\pm 1}, ..., x_n^{\pm 1}].$$

given by

$$f \mapsto (t \mapsto \delta_B^{1/2}(t) \int_N f(tn) \mu_n).$$

Theorem

The Satake transform S is injective and gives an isomorphism $\mathcal{H}^{ur} \cong \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]^{S_n}$

Let s_i be the i-th symmetric function in $x_1, \ldots, x_n$ and set $T_i = q^{i(n-i)/2} s_i$. Then $\mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]^{S_n} \cong \mathbb{C}[T_1, \ldots, T_{n-1}, T_n^{\pm 1}]$ and the isomorphism with $\mathcal{H}^{ur}$ is given by sending the characteristic function of $GL_n(\mathcal{O}_F) diag(\varpi I_i, I_{n-i}) GL_n(\mathcal{O}_F)$ to T_i.

Corollary

The unramified Hecke algebra $\mathcal{H}^{ur}(GL_n(F))$ is commutative.

Remark

This theorem generalises to reductive groups: $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ can be viewed as the cocharacter variety $\mathbb{X}_*$ of the maximal torus T, and S_n is the Weyl group W of GL_n. In general we have an isomorphism

$$\mathcal{H}(G(F), G(\mathcal{O}_F)) \cong \mathbb{X}_*(T(\mathbb{C}))^W.$$

Definition

$\pi \in Irr(GL_n(F))$ is unramified (or spherical) if $\pi^{GL_n(\mathcal{O}_F)} \neq 0$.
These are important for global automorphic representations: such representations are restricted products with respect to spherical representations.

Let π be an unramified irreducible representation of $GL_n(F)$. $\pi^{GL_n(\mathcal{O}_F)} \neq 0$ is irreducible as an $\mathcal{H}^{ur}$-module. But H^{ur} is commutative, so $\pi^{GL_n(\mathcal{O}_F)} \cong \mathbb{C}$
We obtain a map $S_\pi : \mathcal{H}^{ur} \to \mathbb{C}$ called a Satake parameter. It is defined by the $S_\pi(T_i) = s_i$.
We have shown that giving $\pi \in Irr^{ur}(GL_n(F))$ is the same as giving a point of $(\mathbb{C}^\times)^n/S_n$. We define the Hecke polynomial of π by $\prod(X - s_i)$.

Unramified local Langlands

rec_F^n induces a bijection between $Irr^{ur}(GL_n(F))$ and $Rep_n^{ur}(WD_F)^{Fss}$ making the following diagram commute:

$$\begin{array}{ccc} Irr^{ur}(GL_n(F)) & \xrightarrow{\;rec_F^n\;} & Rep_n^{ur}(WD_F)^{Fss} \\ {\scriptstyle S \mapsto \boxplus_i \chi_{s_i}} \nwarrow & & \nearrow {\scriptstyle S \mapsto \bigoplus_i \sigma_{s_i}} \\ & (\mathbb{C}^\times)^n / S_n & \end{array}$$

where $S = \{s_1, ..., s_n\}$ and χ_{s_i} and σ_{s_i} correspond by class field theory. Furthermore the characteristic polynomial of $\rho(\mathrm{Frob})$ equals the Hecke polynomial of π.

Proof (sketch)

For $\rho \in Rep_n^{ur}(WD_F)^{Fss}$, $\ker(N)$ is a subrepresentation, so we must have $N = 0$. ρ is trivial on I_F, so ρ is just a map $\mathbb{Z} \to GL_n(\mathbb{C})$ such that $\rho(\text{Frob})$ has semi-simple image. Thus ρ is determined by the eigenvalues $s_1, \ldots, s_n$ of the diagonalisation of $\rho(\text{Frob})$.

We clearly have a bijection $(\mathbb{C}^\times)^n / S_n \to Rep_n^{ur}(WD_F)^{Fss}$.

On the other hand, a point $(s_1, \ldots, s_n)$ of $(\mathbb{C}^\times)^n / S_n$ is the same as a set of n unramified characters $\det \circ \chi_i : GL_n(F) \to \mathbb{C}^\times$, where $\chi_i(\varpi) = s_i$.

Again let $G = GL_n(F)$, B the upper triangular Borel subgroup and $K = GL_n(\mathcal{O}_F)$.

Form $\text{Ind}_B^G((\chi_i)) = \{f : G \to \mathbb{C} | f(\begin{pmatrix} a_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & a_n \end{pmatrix} g) = \delta_G(b)^{-1/2} \delta_B(b)^{1/2} \prod \chi_i(a_i) f(g)\}$

The Iwasawa decomposition gives $G = BK$. Define $f^0(bk) = \delta_G(b)^{-1/2} \delta_H(b)^{1/2} \prod \chi_i(a_i)$.

This is well-defined as $B \cap K \subset \ker(f^0)$ and K-invariant.

Conversely if $f \in Ind_B^G((\chi_i))$ satisfies $f(gk) = f(g)$ for all $k \in K$ then $f = cf^0$.

It is left to show that the image of $\mathbb{C} f^0$ in the Langlands quotient $\boxplus \chi_i$ is nontrivial. Then the map to $Rep^{ur}(GL_n(F))$ is well-defined.

Compose with the Satake map and show it gives the identity on $(\mathbb{C}^\times)^n / S_n$. Together with the injectivity of the Satake map this gives a bijection. Now show that the isomorphism is compatible with L-factors and ϵ-factors to see that it must equal rec_F^n.

Let $L'/L/\mathbb{Q}_p$ be finite extensions. If $L'/\mathbb{Q}_p$ is Galois, we can use the ramification filtration to see that $\mathrm{Gal}(L'/\mathbb{Q}_p)$ and hence $\mathrm{Gal}(L'/L)$ is solvable.

Local base change

The following diagram commutes:

$$\begin{array}{ccc} Irr(GL_n(L')) & \xrightarrow{rec_{L'}} & Rep^{Fss}(WD_{L'}) \\ \uparrow & & \uparrow \\ Irr(GL_n(L)) & \xrightarrow{rec_L} & Rep^{Fss}(WD_L) \end{array}$$

The left vertical map is given by base change, the right one by restriction.
For $n = 1$, $\chi : F^\times \to \mathbb{C}^\times$ base changes to $\chi \circ N_{F'/F} : F'^\times \to \mathbb{C}^\times$.

Let D/F be a central division algebra, $[D:F] = n^2$.

We define the reduced norm map $N_D : D \longrightarrow D \otimes_F \overline{F} \cong M_n(\overline{F}) \xrightarrow{\det} \overline{F}$. Its image actually lands in F.

There exists a map

$$D^\times_{reg}/\sim \hookrightarrow GL_n(F)_{reg}/\sim,$$

$\delta \mapsto \gamma$ such that $\mathrm{char}(\delta) = \mathrm{char}(\gamma)$. Regular means distinct eigenvalues, and $\sim$ denotes equivalence under conjugation.
The image of this map is $\{\gamma | \ \mathrm{char}(\gamma) \text{is irreducible over} F\}$. Call these elliptic elements.

Theorem (Jacquet-Langlands, Rogawski, Deligne-Kazhdan-Vigneras)

There is a unique bijection

$$JL : Irr(D^\times) \xrightarrow{\cong} Irr^2(GL_n(F))$$

References

[B1] Buzzard Satake
[Bump]
[B-Z] Bernstein-Zelevinsky
[Del] Deligne Antwerp
[Gr] Gross Satake
[Loe] Loeffler article