WEIL-DELIGNE REPS

L1
04/02/21

$L/\mathbb{Q}_\ell$ finite ext.
$\mathcal{O}_L \subseteq L$ ring of integers.

SOME BASICS

RECALL If k a field.

$$\rho : \Gamma \to GL_n(k)$$

Then ρ is semisimple if

$$\rho = \bigoplus \rho_i$$

with ρ_i irreducible.

Given a

$$\rho : \Gamma \to GL_k(V)$$

choose filtration

$$0 \subseteq V_1 \subseteq \cdots \subseteq V_r = V$$

- V_i Γ-invariant
- V_i / V_{i-1} irred.

So the semisimplification ρ^{ss} is ρ acting on $\bigoplus V_i / V_{i-1} =: V^{ss}$.

THM (BRAUER-NESBITT) k a field.
Γ a (top)-group.

$$\rho_1, \rho_2 : \Gamma \to GL_n(k)$$

Suppose either

(1) Characteristic polys of $\rho_1(\gamma)$ and $\rho_2(\gamma)$ are equal $\forall \gamma \in \Gamma$

OR

(2) char $k = 0$ $(> n)$ and $\mathrm{Tr}\, \rho_1(\gamma)$ and $\mathrm{Tr}\, \rho_2(\gamma)$ are equal $\forall \gamma \in \Gamma$

Then $\rho_1^{ss} = \rho_2^{ss}$

CLAIM If Γ compact, and

$$\rho : \Gamma \to GL_n(L)$$

Then there exists a conjugate ρ' of ρ taking values in $GL_n(\mathcal{O}_L)$.

So define reduction mod l of ρ.

$$\bar{\rho} = (\bar{\rho}')^{ss}$$

where

$$\begin{tikzcd} \Gamma \arrow[rr, "\bar{\rho}'"] \arrow[dr, "\rho'"'] & & GL_n(k_\lambda) \\ & GL_n(\mathcal{O}_\lambda) \arrow[ur] & \end{tikzcd}$$

RECALL If K/K' ext of local fields and R top ring.

$$\rho : \mathrm{Gal}(K'/K) \to GL_n(R)$$

is unramified if it factors through $\mathrm{Gal}(K^{ur}/K)$.

Combining Brauer-Nesbitt with Chebotarev we get.

THM F a # field. S fin set of places.

$$\rho_1, \rho_2 : G_F \to GL_n(\mathbb{Q}_\ell)$$

unramified outside S. Suppose

(1) Char poly. $\rho_1(\mathrm{Frob}_v)$ and $\rho_2(\mathrm{Frob}_v)$ equal $\forall v \notin S$ **OR**

(2) $\mathrm{Tr}\, \rho_1(\mathrm{Frob}_v) = \mathrm{Tr}\, \rho_2(\mathrm{Frob}_v)$ equal $\forall v \notin S$

Then $\rho_1^{ss} = \rho_2^{ss}$.

WEIL-DELIGNE REPS

$K/\mathbb{Q}_p$ fin ext.

$\mathcal{O}_K \subseteq K$ ring of ints.

$\pi \in \mathcal{O}_K$ uniformiser

k_K res. field.

$v_K : K^\times \to \mathbb{Z}$ normalised.

PICTURE

$$
\begin{tikzcd}
\bar{K} \arrow[d, dash, "\text{pro } p"] & \\
K^{tame} \arrow[d, dash] & \\
K^{ur} \arrow[d, dash] & \bar{k} \arrow[d, dash] \\
K & k
\end{tikzcd}
$$

G_K

I_K

I_K^{tame} — pro-prime to p

$G_k \cong \hat{\mathbb{Z}}$

$\text{Frob} \mapsto 1 \in \hat{\mathbb{Z}}$

$$0 \to I_K \to G_K \xrightarrow{v_K} G_k \to 0$$

Define the **Weil group**

$$W_K = v_K^{-1}(\mathbb{Z})$$

$W_K \to \mathbb{Z}$ continuous when $\mathbb{Z}$ has the discrete top

THEOREM Let $\ell \neq p$. Let

$$\rho : G_K \to GL_n(L)$$

Then there exists a finite ext K'/K such that $\rho|_{I_{K'}}$ unipotent

all the eigenvalues are 1

CLAIM

$$K^{tame} = \bigcup_{(n,p)=1} K^{ur}(\sqrt[n]{\pi})$$

Proof ANT.

$$\mathrm{Gal}(K^{ur}(\sqrt[n]{\pi})/K^{ur}) \cong \mu_n$$

$$\sigma \longmapsto \sigma(\sqrt[n]{\pi})/\sqrt[n]{\pi}$$

$$\begin{aligned}
\mathrm{Gal}(K^{tame}/K^{ur}) & \\
\| \quad & \\
I_K^{tame} &= \varprojlim_{(n,p)} \mathrm{Gal}(K^{ur}(\sqrt[n]{\pi})/K^{ur}) \\
&\cong \varprojlim_{(n,p)} \mu_n \\
&\cong \varprojlim_{(n,p)} \mathbb{Z}/n\mathbb{Z} \\
&\overset{CRT}{\cong} \prod_{p' \neq p} \mathbb{Z}_{p'}
\end{aligned}$$

(ζ_n)

Define $I_{p'}^{tame}$ to be the inverse image of $\mathbb{Z}_{p'}$ under this.

Define

$$t_{\varsigma,p'} : I_K \to I_K^{tame} \to \mathbb{Z}_{p'}$$

LEMMA Let $\phi \in Gal(K^{tame}/K)$ be lift of Frob. Then conjugating by ϕ is a well defined action of Frob on I_K^{tame}. AND

(1) $\forall \tau \in I_K^{tame}$; $\phi^{-1}\tau\phi = \tau^{\#k}$

(2) $\forall \tau \in I_K$, $\sigma \in W_K$

$$t_{\varsigma,p'}(\sigma^{-1}\tau\sigma) = \#k^{v_K(\sigma)}\, t_{\varsigma,p'}(\tau)$$

Proof (1)

$$\eta = \frac{\tau(\sqrt[n]{\pi})}{\sqrt[n]{\pi}}$$

Suppose π' another uniformi

$$\frac{\tau(\sqrt[n]{\pi'})}{\tau(\sqrt[n]{\pi})} = \frac{\sqrt[n]{\pi'}}{\sqrt[n]{\pi}}$$ since $(p,n)=1$

$$\Rightarrow \frac{\sqrt[n]{\pi}}{\sqrt[n]{\pi'}} \in K^{un}$$

$$\Rightarrow \frac{\tau(\sqrt[n]{\pi'})}{\sqrt[n]{\pi'}} = \eta$$

$$\Rightarrow \frac{\phi^{-1}\tau\phi(\sqrt[n]{\pi})}{\sqrt[n]{\pi}} = \phi^{-1}\left(\frac{\tau\phi(\sqrt[n]{\pi})}{\phi(\sqrt[n]{\pi})}\right)$$

$$= \phi^{-1}(\eta)$$

$$= \eta^{\#k}$$

$$= \frac{\tau^{\# k}(\mathcal{J}_\pi)}{\mathcal{J}_\pi}$$

(ii) Follow your nose

$\boxtimes$

RECALL (TRYING TO PROVE)

$K/\mathbb{Q}_p$ $\quad$ $L/\mathbb{Q}_\ell$

THEOREM Let $\ell \neq p$. Let

$$\rho : G_K \to GL_n(L)$$

Then there exists a finite ext K'/K such that $\rho|_{I_{K'}}$ unipotent

Proof

CLAIM: WLOG $\rho(I_K)$ is pro-ℓ.

Proof

RECALL

$GL_N(\mathcal{O}_L)$ not pro-ℓ BUT

$GL_N^{(1)}(\mathcal{O}_L) = \ker\left(GL_N(\mathcal{O}_L) \to GL_N(k_L)\right)$

is pro-ℓ.

$\Lambda \leq L^n$ G_K-stable

$\bar{\rho} : G_K \to GL_{\mathcal{O}_L}(\Lambda) \to GL_{k_L}(\Lambda/\pi_L\Lambda)$

$U = \ker \bar{\rho}$ open.

Set K again to be

K^U.

Then $\rho(I_K)$ must live in

$GL_N^{(1)}(\Lambda)$

$\square$

CLAIM 2 ρ factors through I_ℓ^{tame} (preimage of $\mathbb{Z}_\ell$ under $I^{tame} \cong \prod_{p' \neq p} \mathbb{Z}_{p'}$)

Proof

$Gal(\bar{K}/K^{tame})$ is inverse limit of p-sylow subgroups of Inertia groups

Gets killed.

Same for all $\mathbb{Z}_{p'}$ in I_K^{tame} when $p' \neq \ell$.

$$\rho : I_K \to I_\ell^{tame} \xrightarrow{\rho^t} GL_n$$

Let $\tau \in I_\ell^{tame}$ be the inverse image of 1 under $I_\ell^{tame} \cong \mathbb{Z}_\ell$.

CLAIM 3 $\rho^t(\tau)$ has eigenvalues which are ℓ^r-th roots of unity.

Proof First part of lemma

$$\phi^{-1}\tau\phi = \tau^{\# k}$$

$$\rho^t(\phi^{-1}\tau\phi) = \rho^t(\tau^{\# k}) = \rho^t(\tau)^{\# k}$$

$\Rightarrow$ $\rho^t(\tau)$ and $\rho^t(\tau)^{\# k}$ have the same eigvalues

$\Rightarrow$ roots of 1.

$\rho^t(\tau), \rho^t(\tau^\ell), \rho^t(\tau^{\ell^2})$ converges to 1

in particular ℓ^r th roots of 1 for some r.

CLAIM 4 There exists $M \geq 1$ such that $\forall \sigma \in I_\ell^{tame}$ $\rho^t(\sigma)^{\ell^M}$ is unipotent.

Proof We know this for τ hence for $\tau^{\mathbb{Z}}$. $\mathbb{Z} \subseteq \mathbb{Z}_\ell$ dense. So by continuity ✓

CLAIM 5 We're really done.

Proof $\rho|_{I_{K'}}$ unipotent.

$$\begin{tikzcd}
I_{K'} \arrow[r, hook] \arrow[rrd, dashed] & I_K \arrow[r] & I_\ell^{tame} \arrow[r, "\rho^t"] \arrow[d, phantom, "\cong"] & {} \\
 & & \mathbb{Z}_\ell & \\
 & & \ell^M \mathbb{Z}_\ell &
\end{tikzcd}$$

Take K'/K such that

$\ell^M \mid e_{K'/K}$. Then you land in $\ell^M \mathbb{Z}_\ell$

$$\begin{tikzcd}
\rho|_{I_{K'}} : I_{K'} \arrow[rr] \arrow[dr] & & GL_n(L) \\
& \ell^M \mathbb{Z}_\ell \arrow[ur] &
\end{tikzcd}$$

So everything in the image is unipotent. $\blacksquare$

COROLLARY There exists a <u>UNIQUE</u> nilpotent $N \in \mathrm{End}(V)$ on a finite extension K'/K such that

(1) $$\rho(\sigma) = \exp\left(t_{J,\ell}(\sigma)\, N\right)$$

for all $\sigma \in I_{K'}$.

$\rho(\sigma) \exp(\ \ ) = 1$ when $\sigma \in I_{K'}$

And N satisfies $\forall \sigma \in W_K$

(2) $$\rho(\sigma) N \rho(\sigma)^{-1} = \# k^{-\nu_K(\sigma)} N$$

Proof. Follow your nose take K' as above $N = \log \rho(I)$.

∎

Finally we are at the topic

DEF A Weil-Deligne representation over a field Ω of char. 0 is a triple (V, ρ, N)

- V is fin dim Ω VS.
- $\rho: W_K \to GL_\Omega(V)$
- $N \in End_\Omega(V)$ nilpotent.

Such that

① ρ cont. wrt discrete top on Ω

($\Leftrightarrow$ ker ρ open in W_K

$\Leftrightarrow \rho(I_K)$ finite)

② $\forall \sigma \in W_K$

$$\rho(\sigma) N \rho(\sigma)^{-1} = \# k^{-v_K(\sigma)} N$$

WHAT'S THE POINT?

Fix ζ as above roots of 1.

ϕ lift of frob. to G_K

Then define functor

$$WD_{\zeta,\phi} : \left\{ \begin{array}{c} G_K\text{-reps of} \\ V / L \end{array} \right\} \to \left\{ \begin{array}{c} \text{WD reps} \\ \text{of } W_K \text{ to } V \end{array} \right\}$$

$$: (V, \rho) \longmapsto (V, r, N)$$

where N nilp. thing before.

$$r(\sigma) = \rho(\sigma) \exp\left(-t_{\zeta,\ell}(\phi^{-v(\sigma)}\sigma)N\right)$$

$$0 \to I_K \to G_K \to G_k \to 0$$

$t_{\zeta,\ell}$ only defined here

LEMMA The functor $WD_{3,\phi}$ is an equivalence of categories

Proof sketch

Faithfulness

The uniqueness of N.

Suppose $f : (V, \rho) \longrightarrow (V', \rho')$

Then the !ness of N

$\Rightarrow f \circ N = N' \circ f$.

in particular $WD_{3,\phi}$ is faithful.