STUDY GROUP ON THE WORK OF CHARLES, SHANKAR, TANG ET. AL. ON ARITHMETIC INTERSECTIONS ON SHIMURA VARIETIES–MICHAELMAS 2021

The aim of the seminar is to understand the circle of ideas surrounding the work of Charles, Shankar–Tang, and others on applications of Arakelov intersection theory on Shimura varieties. The idea originated in the work of Charles [Cha18] who used it to prove that for any two elliptic curves E and E' over a number field K, there exist infinitely many primes of K such that the reductions of E and E' are geometrically isogenous.

Shankar and Tang further developed the idea in some more complicated geometric cases, the culmination of their program being the proofs of some new cases of the Hecke–Orbit conjecture, a problem which has seen a flurry of activity in recent years.

We will begin with the first paper of Shankar–Tang [ST20], which works in a more interesting geometric setting than that of Charles, and which also uses some more serious properties of Shimura varieties. The plan is then to spend Lent term on the sequels to this paper.

1. List of talks

1.1. **Talk 1.** Introduction. Jef.

1.2. **Talk 2.** Hilbert modular surfaces. §2.1 of [ST20]. Definition of Hilbert modular surface over $\mathbb{Z}$, description of its complex points, existence of compactification. Definition of Hirzebruch–Zagier divisors, special endomorphisms.

1.3. **Talk 3.** Arakelov Theory. Definition of Arakelov intersection product. Define Faltings height of an abelian variety. Prove Lemma 5.1.5 assuming the existence of a Ψ from Lemma 5.1.1.

1.4. **Talks 4+5.** Archimedean computation. Define Hecke orbit $T_{\mathfrak{p}}([\mathcal{A}])$. Explain the equidistribution theorem of Clozel–Oh–Ullmo. Deduce Theorem 3.2.1 from Proposition 3.1.2. Prove Proposition 3.1.2

1.5. **Talk 6+7.** Non-archimedean computation. Review of Grothendieck–Messing theory and the Serre–Tate theorem. Geometry of numbers argument and proof of main non-archimedean estimate (Theorem 4.3.4).

1.6. **Talk 8.** Proof of Theorem 1 (Can be a longer talk if needed). Review of Borcherds products. Prove Lemma 5.1.1. Put everything together to prove Theorem 1.

References

[Cha18] F. Charles, *Exceptional isogenies between reductions of pairs of elliptic curves*, Duke Math. J. **167** (2018), no. 11, 2039–2072.

[ST20] A. N. Shankar and Y. Tang, *Exceptional splitting of reductions of abelian surfaces*, Duke Math. J. **169** (2020), no. 3, 397–434.