The étale homotopy type of a scheme

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In this note we define the étale homotopy type of a locally noetherian scheme X, assuming familiarity with the basic language of simplicial sets. It roughly follows the notes of Schlank-Skorogobatov [SS10] but contains more details and examples. Comments and corrections should be sent to `jcsl5@cam.ac.uk`.

Notation and conventions: If $\mathcal{C}$ denotes a category we denote $s\mathcal{C}$ be the category of simplicial objects in $\mathcal{C}$. If $\mathcal{C}$ has a terminal object then $\mathcal{C}_0$ denotes the corresponding pointed category. Write Top for the category of CW-complexes. We will usually denote a simplicial set by a boldface letter like $\mathbf{X}, \mathbf{Y}$ etcetera. We often omit basepoints in our notation of homotopy groups and silently assume the object (simplicial set, scheme, topological space) is pointed. All schemes considered are locally noetherian.

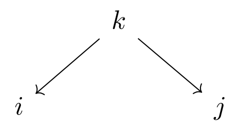
Contents

1 Categorical background

1.1 Pro-categories

Definition 1.1. *A small category $\mathcal{I}$ is* cofiltering *if*

1. *For all objects i, j of $\mathcal{I}$, there is a diagram*



2. *Every pair of arrows $f, g \colon i \to j$ is equalized by an arrow from some k i.e. there exists an arrow $h \colon k \to i$ such that $fh = gh$.*

If a cofiltering category has at most one arrow between any two objects we call it a codirected system or more commonly an inverse system.

Definition 1.2. *Let $\mathcal{C}$ be a category. We define the* pro-category *$\operatorname{pro}\mathcal{C}$ of $\mathcal{C}$ as follows:*

- *An object in $\operatorname{pro}\mathcal{C}$ is a functor $F \colon \mathcal{I} \to \mathcal{C}$.*
- *Morphisms are given by*

$$\operatorname{Hom}_{\operatorname{pro}\mathcal{C}}(\{C_i\}_{i\in\mathcal{I}}, \{D_j\}_{j\in\mathcal{J}}) = \varprojlim_{\mathcal{J}} \varinjlim_{\mathcal{I}} \operatorname{Hom}_{\mathcal{C}}(C_i, D_j).$$

We call an object of $\operatorname{pro}\mathcal{C}$ a pro-object of $\mathcal{C}$.

There is a fully faithful functor $\mathcal{C} \to \operatorname{pro}\mathcal{C}$ and one should view $\operatorname{pro}\mathcal{C}$ as an enlargement of $\mathcal{C}$ by formally adjoining all cofiltered limits. A few remarks are in order:

Remark 1.3 (Alternative definition of the pro-category)**.** *Every functor $F \colon \mathcal{C} \to \mathrm{Set}$ is a colimit of representable functors (this is formal and straightforward, see Ashwin's notes). We say F is* pro-representable *if it is a filtered colimit of representable functors, i.e. the colimit is indexed by a category $\mathcal{I}^{opp}$ where $\mathcal{I}$ is cofiltered. The functor*

$$\operatorname{pro}\mathcal{C}^{opp} \to [\mathcal{C}, \mathrm{Set}], \{X_i\} \mapsto \operatorname{Hom}_{\operatorname{pro}\mathcal{C}}(\{X_i\}, -)$$

is fully faithful and its essential image consists of the set of pro-representable functors. This shows that an alternative definition of $\operatorname{pro}\mathcal{C}$ would be the opposite category of the full subcategory of $[\mathcal{C}, \mathrm{Set}]$ consisting of the pro-representable functors.

Remark 1.4 (Cofiltered vs. inverse limits)**.** *The fact that cofiltering categories are more general than the (more familiar) inverse systems might be uncomfortable but there is essentially no difference: every object in $\operatorname{pro}\mathcal{C}$ is isomorphic to a pro-object indexed by an inverse system, see* [AR94, Theorem 1.5].

A pro-object of a category can contain more information than its limit if it exists:

Example 1.5. *Let* FinGrps *be the category of finite groups. Then every pro-object in this category has a limit in the larger category of (all) groups* Grps*, but the functor* $\varprojlim\colon \operatorname{pro}\mathrm{FinGrps} \to \mathrm{Grps}$ *is not full. In fact, every element in the image of this functor canonically has the structure of a topological group, and the corresponding functor* $\operatorname{pro}\mathrm{FinGrps} \to \mathrm{TopGrps}$ *is fully faithful; its essential image consists of those topological groups which are Hausdorff, compact and totally disconnected. It is therefore justified to call such a topological group a profinite group!*

1.2 Simplicial enrichment

Let $\mathcal{C}$ be a category which is enriched in sSet. This means that $\mathcal{C}$ consists of a class of objects and for each pair of objects X, Y we are given a simplicial set $\mathbf{Hom}(X, Y)$ and for any three objects X, Y, Z we are given a morphism of simplicial sets $\mathbf{Hom}(X, Y) \times \mathbf{Hom}(Y, Z) \to \mathbf{Hom}(X, Z)$ ('composition') satisfying the same axioms as the Hom-sets of an ordinary category. The Hom set $\mathrm{Hom}(X, Y) = \mathbf{Hom}(X, Y)_0$ gives $\mathcal{C}$ the structure of an ordinary category.

In any simplicially enriched category, we can talk about homotopy.

Definition 1.6 (Homotopies)**.** *Let $\mathcal{C}$ be a category enriched in* sSet*. Let $f, g \in \mathrm{Hom}(X, Y) = \mathbf{Hom}(X, Y)_0$. We say f and g are strictly homotopic if there exists an element of $\mathbf{Hom}(X, Y)_1$ whose restrictions along the faces 0 and 1 is f and g respectively. We say f and g are homotopic if they are equivalent in the equivalence relation generated by strict homotopies, i.e. if they can be connected by a chain of strict homotopies.*

The equivalence relation of homotopy is stable under pre- and post-composing morphisms (exercise), so the following is well-defined:

Definition 1.7 (The homotopy category)**.** *Let $\mathcal{C}$ be a category enriched in* sSet*. The homotopy category of $\mathcal{C}$, denoted $\mathrm{Ho}(\mathcal{C})$, is the category with*

- *Objects the same as $\mathcal{C}$.*
- *Morphisms are elements of $\mathrm{Hom}(X, Y)$ up to homotopy equivalence.*

Any functor $F\colon \mathcal{C} \to \mathcal{D}$ between simplicially enriched categories (i.e. a map between the objects of $\mathcal{C}$ and $\mathcal{D}$ together with simplicial maps $\mathbf{Hom}(X, Y) \to \mathbf{Hom}(FX, FY)$ satisfying the usual axioms of a functor) induces a functor $\mathrm{Ho}(F)\colon \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ between the homotopy categories.

We will only consider simplicially enriched categories in the context of the following example:

Example 1.8. *Let $\mathcal{C}$ be a category with finite coproducts. If X is in $\mathcal{C}$ and A is a finite set we define $X \otimes A$ as the coproduct of X with itself indexed by A. Similarly if $\mathbf{X}$ is a simplicial object in $\mathcal{C}$ and $\mathbf{A}$ a simplicial set such that $\mathbf{A}_n$ is finite for each $n \geq 0$, we may define the simplicial object $\mathbf{X} \otimes \mathbf{A}$ of $\mathcal{C}$. This allows us to define a simplicial enrichment of $\mathcal{C}$ with simplicial Hom-sets given by*

$$\mathbf{Hom}(\mathbf{X}, \mathbf{Y})_n = \mathrm{Hom}_{s\mathcal{C}}(\mathbf{X} \otimes \Delta^n, \mathbf{Y}).$$

(Here Δ^n denotes the standard n-simplex.) The previous definitions then apply to $s\mathcal{C}$ and the category $\mathrm{Ho}(s\mathcal{C})$ is defined. The same holds for any full subcategory of $s\mathcal{C}$.

Example 1.9. *Let $\mathrm{Ho}^{Kan}(\mathrm{sSet})$ be the homotopy category of the full subcategory of* sSet *consisting of Kan simplicial sets. Then $\mathrm{Ho}^{Kan}(\mathrm{sSet})$ is equivalent to $\mathrm{Ho}(\mathrm{Top})$, the homotopy category of CW-complexes.*

2 Definition of the étale homotopy type

2.1 Motivation

Let X be a noetherian scheme and $x\colon \operatorname{Spec} k \to X$ a geometric point. The étale fundamental group $\pi_1(X,x)$ is a profinite group classifying covering spaces over X: the category of finite étale covers over X is equivalent to the category of discrete $\pi_1(X,x)$-sets. Moreover if X is a connected normal finite-type $\mathbb{C}$-scheme, every finite covering space of $X(\mathbb{C})$ is the analytification of a finite étale cover of X, yielding a comparision theorem over $\mathbb{C}$.

We would like to have an equally satisfactory algebro-geometric definition for the higher homotopy groups a scheme X. Even better would be a functorial association of a homotopy equivalence class of a 'space' attached to any scheme which has similar applications as for ordinary topological spaces. This leads to the notion of the étale homotopy type.

The starting observation is the following.

Lemma 2.1. *Let $\mathcal{U} = \sqcup_{i\in I} U_i$ be an open cover of a topological space X such that every connected component of every finite intersection of U_i's is contractible (a 'good covering'). Let $\pi_0(\mathcal{U})_\bullet$ be the siset with*

$$\pi_0(\mathcal{U})_n := \pi_0(\mathcal{U} \times_X \cdots \times_X \mathcal{U}) \quad ((n+1) \text{ times}).$$

(The connected component of the Čech nerve of $\mathcal{U}$.) Then the geometrical realization of $\pi_0(\mathcal{U})_\bullet$ is homotopy equivalent to X.

If follows that for any locally contractible topological space (like a manifold, a finite CW-complex or the analytification of a finite type $\mathbb{C}$-scheme), its homotopy type (i.e. its homotopy equivalence class) can be represented as the connected component of the Čech nerve of a good covering. Moreover every covering of such a space can be refined to a good covering.

In the world of schemes, connected components, Čech nerves and coverings have satisfactory analogues, but the notion of a good cover does not. The idea instead is to take the formal limit over all coverings simultaneously, hoping that finer coverings will yield better and better approximations. This is very similar to defining the universal cover of a connected scheme X by the pro-scheme $\{X_i\}$ where $X_i \to X$ runs over the (cofiltered!) category of connected finite étale covers of X, and we know that this works well. We will try to imitate the latter construction but we will need more background to implement in a good way.

2.2 Hypercoverings

Let X_{et} denote the small étale site of a scheme X, with objects étale X-schemes and coverings jointly surjective families. Let $\operatorname{Cov}(X)$ denote the full subcategory of X_{et} consisting of coverings, i.e. surjective étale morphisms $Y \to X$ (not necessarily finite). We would like to implement the same strategy as the definition of the universal cover. However, ordinary coverings of X are not fine enough for the purposes of homotopy, a phenomenon already visible in cohomology. Indeed, taking cohomology of the simplicial sets $\pi_0(\mathcal{U}_\bullet)$ (valued in a finite abelian group A) indexed by coverings essentially gives us the system of Čech cohomology groups $\{H^*(\mathcal{U}, A)\}$ and the colimit of this system gives $\check{H}(X,A)$, which is not always the same as $H_{et}(X,A)$. (It will be if X is quasi-projective over an affine base by a theorem of Artin, but this class doesn't even include all smooth proper varieties over $\mathbb{C}$!)

This problem can be rectified by considering hypercoverings instead.

Definition 2.2. *A hypercovering of X is a simplicial object $\mathcal{U}_\bullet$ of X_{et} such that*

1. *$\mathcal{U}_0 \to X$ is a covering.*
2. *For every $n \geq 0$, $\mathcal{U}_{n+1} \to (cosk_n \mathcal{U}_\bullet)_{n+1}$ is a covering.*

Write $\mathrm{HC}(X)$ *for the full subcategory of $s(X_{et})$ of hypercoverings.*

The Čech nerve construction realizes every covering as a hypercovering, but the latter allows extra freedom at every level. This extra freedom has the following essential consequences, due to Verdier. If $\mathcal{U}_\bullet$ is any hypercovering and $\mathscr{F}$ is a sheaf on X_{et} we write $H^*(\mathcal{U}_\bullet, \mathscr{F})$ for the cohomology of the cosimplicial abelian group $\mathscr{F}(\mathcal{U}_\bullet)$. The following is [AM69, Theorem 8.16]:

Theorem 2.3. *Let $\mathscr{F}$ be an abelian sheaf on X_{et}. Then there is a canonical isomorphism*

$$H^n(X, \mathscr{F}) = \operatorname{colim} H^n(\mathcal{U}_\bullet, \mathscr{F}),$$

where the colimit is taken over the category $\mathrm{HC}(X)^{opp}$.

Note that if $\mathscr{F}$ is constant then $H^n(\mathcal{U}_\bullet, \mathscr{F}) = H^n(\pi_0(\mathcal{U}_\bullet), \mathscr{F})$, so the above theorem says that the cohomology of X valued in constant sheaves can be computed as the colimit of the cohomology of the connected component of the Čech nerve over all hypercoverings.

For the next important result, recall that $\mathrm{HC}(X)$ is simplicially enriched so its homotopy category is well-defined. We have ([AM69, Corollary 8.13]):

Theorem 2.4. *The homotopy category* $\mathrm{Ho}(\mathrm{HC}(X))$ *is cofiltering.*

The categories $\mathrm{HC}(X)$ and $\mathrm{Cov}(X)$ are not cofiltered, as the following example shows:

Example 2.5. *Consider the objects $\{X \xrightarrow{\mathrm{Id}} X\}$ and $\{X \sqcup X \xrightarrow{\mathrm{Id} \sqcup \mathrm{Id}} X\}$ of $\mathrm{Cov}(X)$. Let f and g denote the inclusion of X into the first and second factor of $X \sqcup X$. Then no (hyper)covering of X equalizes f and g.*

2.3 The definition

We are now ready to define the étale homotopy type. Let

$$\pi_0 \colon X_{et} \to \mathrm{Set}$$

be the connected component functor. Since every étale X-scheme is the coproduct of its connected components (X is locally noetherian), this functor is well-defined. Then π_0 extends to a functor $s(X_{et}) \to \mathrm{sSet}$ compatible with the simplicial enrichments on both sides. Restricting to $\mathrm{HC}(X)$ and passing to the homotopy category defines a functor

$$\mathrm{Ho}(\mathrm{HC}(X)) \to \mathrm{Ho}(\mathrm{sSet}).$$

Taking the topological realization defines a functor

$$\mathbf{Et}(X) \colon \mathrm{Ho}(\mathrm{HC}(X)) \to \mathrm{Ho}(\mathrm{Top}). \tag{1}$$

(Here Top denotes the category of CW complexes.) Since $\mathrm{Ho}(\mathrm{HC}(X))$ is cofiltered, $\mathbf{Et}(X)$ defines a pro-object of Ho(Top). We call this pro-object the étale homotopy type of X. If we fix a geometric point of X

and consider pointed hypercoverings of X, we can upgrade $\mathbf{Et}(X)$ to a pro-object of the pointed homotopy category of pointed CW complexes $\mathrm{Ho}(\mathrm{Top}_0)$.

The étale homotopy type defines a functor

$$\mathbf{Et}\colon \mathrm{NoethSchemes} \to \mathrm{pro}\,\mathrm{Ho}(\mathrm{Top}).$$

We may consider truncated versions using the Postnikov tower. Briefly, define $\mathbf{Et}^\sharp(X)$ as the geometric realization of the pro-object $\{\mathrm{cosk}_k \pi_0(\mathcal{U}_\bullet)\}$ indexed by the cofiltering category $\mathrm{Ho}(\mathrm{HC}(X)) \times \mathbb{N}$. (Here $i \to j$ if $i \geq j$ in $\mathbb{N}$.) Moreover define $\mathbf{Et}^\sharp_n(X)$ as the same pro-object but only indexed by $\mathrm{Ho}(\mathrm{HC}(X)) \times \{1, \ldots, n\}$.

3 Properties

3.1 Homotopy, homology and cohomology

Let $\{\mathbf{X}_i\}$ be a pro-simplicial set. Then any functor defined on simplicial sets extends to a functor between the pro-categories. In particular, we write (for any abelian group A)

$$\pi_n(\{\mathbf{X}_i\}) := \{\pi_n(\mathbf{X}_i)\},$$
$$H_n(\{\mathbf{X}_i\}, A) := \{H_n(\mathbf{X}_i, A)\},$$

which are pro-groups, abelian except in the case of π_1. (In the definition of π_n the $\mathbf{X}_i$ have to be pointed.) We could define $H^n(\{\mathbf{X}_i\}, A)$ as $\{H^n(\mathbf{X}_i, A)\}$ which is and ind-abelian group (a pro-object in the opposite category). However, we prefer to take the colimit and define

$$H^n(\{\mathbf{X}_i\}, A) := \varinjlim H^n(\mathbf{X}_i, A).$$

Since $\varinjlim$ is exact, this will cause no harm. The functors π_n, H_n, H^n factor through the homotopy category. The same definitions also work for pro-CW complexes.

The above shows that for any locally noetherian scheme X we may define the pro-groups $H_n(\mathbf{Et}(X), A)$, $\pi_n(\mathbf{Et}(X), A)$ (after picking a base-point) and the abelian groups $H^n(\mathbf{Et}(X), A)$.

Theorem 2.3 (and the remark after it) immediately imply the following:

Proposition 3.1. *Let X be a locally noetherian scheme and A an abelian group. Then we have a canonical isomorphism*

$$H^n_{et}(X, A) \simeq H^n(\mathbf{Et}(X), A).$$

The homology and homotopy groups do give genuinely new invariants of a scheme X.

We say a few words about the difference between weak equivalences and isomorphisms in $\mathrm{pro}\,\mathrm{Ho}(\mathrm{Top})$ (or its pointed version $\mathrm{pro}\,\mathrm{Ho}(\mathrm{Top}_0)$). The natural map $\mathbf{Et}(X) \to \mathbf{Et}^\sharp(X)$ is always a weak equivalence, and an equivalence if and only if $\pi^{et}_n(X) = 0$ for $n >> 0$. We call a map $\mathbf{X} \to \mathbf{Y}$ in $\mathrm{pro}\,\mathrm{Ho}(\mathrm{Top})$ a $\sharp$-isomorphism if the induced map of pro-objects $\mathbf{X}^\sharp \to \mathbf{Y}^\sharp$ is an isomorphism. ($\mathbf{X}^\sharp$ is the pro-object $\{\mathrm{cosk}_k \mathbf{X}_i\}$ indexed over $I \times \mathbb{N}$.)

Theorem 3.2 (pro-Whitehead's theorem)**.** *A morphism in* $\mathrm{pro}\,\mathrm{Ho}(\mathrm{Top})$ *is a weak equivalence if and only if it is a $\sharp$-isomorphism.*

Often when we want to work with a pro-object $\mathbf{X}$ in $\mathrm{Ho}(\mathrm{Top})$, we first try to understand the $\sharp$-isomorphic object $\mathbf{X}^\sharp$. If it turns out that $\pi_n(\mathbf{X}^\sharp) = \pi_n(\mathbf{X}) = 0$ for n large enough (true in most examples below), then $\mathbf{X}$ is homotopy equivalent to $\mathbf{X}^\sharp$.

3.2 Comparison over $\mathbb{C}$

Let X be a connected finite type normal $\mathbb{C}$-scheme[1]. We write X_{cx} for the homotopy type of the complex analytic space $X(\mathbb{C})$. By the Riemann existence theorem, the natural map $\pi_1(X_{cx}) \to \pi_1^{et}(X)$ induces an isomorphism on profinite completions. We would like to have an analogous theorem for higher homotopy groups. There will always be a natural map

$$\pi_n(X_{cx}) \to \pi_n(\mathbf{Et}(X)),$$

but in general it will not induce an isomorphism on profinite completions. In fact something better is true: we can compare X_{cx} with the homotopy type of $\mathbf{Et}(X)$.

Define Top_0^{fin} as the full subcategory of Top_0 (pointed CW-complexes) of objects whose homotopy groups are finite.

Proposition 3.3. *The inclusion* $\mathrm{pro}\,\mathrm{Ho}(\mathrm{Top}_0^{fin}) \to \mathrm{pro}\,\mathrm{Ho}(\mathrm{Top}_0)$ *has a left-adjoint, denoted by* $\mathbf{X} \mapsto \widehat{\mathbf{X}}$. *We call* $\widehat{\mathbf{X}}$ *the profinite completion of* $\mathbf{X}$.

Theorem 3.4 (Generalized Riemann Existence Theorem)**.** *Let X be a connected, pointed, normal finite type $\mathbb{C}$-scheme. Then there is a canonical map $X_{cx} \to \mathbf{Et}(X)$ in* $\mathrm{pro}\,\mathrm{Ho}(\mathrm{Top}_0)$ *which realizes $\mathbf{Et}(X)$ as the profinite completion of X_{cx}*

This implies that $\mathbf{Et}(X)$ only depends on the homotopy type of $X(\mathbb{C})$. For example if X is a smooth projective curve then the étale homotopy type only depends on the genus (and will be computed below).

3.3 Further properties

We collect a few more properties of the étale homotopy type. First of all, the étale homotopy type only depends on the site X_{et}. Moreover any universal homeomorphism $X \to Y$ induces an equivalence of étale sites. This implies that $\mathbf{Et}(X)$ only depends on the reduced subscheme X_{red}, and if X is of finite type over a separably closed field k then $\mathbf{Et}(X) \simeq \mathbf{Et}(X_K)$ for any seperably closed field extension K/k. So if k is of characteristic zero we may equally well assume that $k = \mathbb{C}$ and then we can use the comparison theorem. If k has positive characteristic, we may approximate $\mathbf{Et}(X)$ by choosing a lift over $W(k)$ and using the following proposition (cf.[AM69, Corollary 12.13]):

Proposition 3.5. *Let R be a discrete valuation ring with separably closed residue field of characteristic $p \geq 0$. Let $X \to \operatorname{Spec} R$ be a smooth proper morphism with geometrically connected fibres. Then there is an isomorphism of prime-to-p étale homotopy types*

$$\widehat{\mathbf{Et}(X_s)}^{prime-p} \simeq \widehat{\mathbf{Et}(X_\eta)}^{prime-p}.$$

Of course, this only gives information about the prime-to-p part of $\mathbf{Et}(X)$. But we should bear in mind that even innocuous looking objects like $\pi_1(\mathbb{A}^1_{\overline{\mathbb{F}}_p})$ are huge! In particular, even though the étale homotopy type of $\mathbb{A}^1_k$ is contractible if $\mathrm{char}\, k = 0$ (by the comparison theorem), this is very far from true in positive characteristic.

Just like in topology, higher homotopy groups are invariant under covering spaces:

[1] In what follows we can replace normal by geometrically unibranch, i.e. the normalization of every local ring of X is again local.

Proposition 3.6. *Let $X \to Y$ be a finite étale cover between pointed connected locally noetherian schemes. Then for every $n \geq 2$ it induces an isomorphism*

$$\pi_n^{et}(X) \to \pi_n^{et}(Y).$$

We now discuss varieties over $\mathbb{C}$ whose analytification are Eilenberg-Maclane spaces. We need the following definition due to Serre [Ser02, Chapter 1, Section 2.6].

Definition 3.7. *We say a group G is* good *if the natural maps in group cohomology $H^i(\hat{G}, M) \to H^i(G, M)$ are isomorphisms for every finite continuous $\hat{G}$-module M.*

Here are some examples of good groups (no reference given): free groups, surface groups, successive extensions of finitely generated free groups (e.g. braid groups), groups containing a finitely generated good group with finite cohomology of finite index (e.g. $\mathrm{SL}_2(\mathbb{Z})$).

Proposition 3.8. *Let X be a normal, connected finite type $\mathbb{C}$-scheme such that $G := \pi_1(X(\mathbb{C}))$ is good and the universal cover of $X(\mathbb{C})$ is contractible. Then $\mathbf{Et}(X)$ is weakly equivalent to $K(\widehat{G}, 1)$. In particular, $\pi_n^{et}(X) = 0$ for $n \geq 2$.*

3.4 Examples

We may use the results of the previous subsection to calculate some examples.

Example 3.9 (Homotopy type of a complex curve)**.** *Let $X/\mathbb{C}$ be a smooth projective curve of genus $g \geq 1$. (The case of the projective line will be treated Example 3.14.) By the uniformization theorem, the universal cover of $X(\mathbb{C})$ is $\mathbb{C}$ or the hyperbolic plane hence contractible. Moreover its fundamental group is good. Hence the higher homotopy groups of $\mathbf{Et}(X)$ vanish.*

We now turn to the homotopy types of fields. We start with a warmup:

Example 3.10 (Homotopy type of $\operatorname{Spec}\mathbb{R}$)**.** *The étale homotopy type of $\operatorname{Spec}\mathbb{R}$ is $B(\mathbb{Z}/2\mathbb{Z})$, or the infinite real projective space $\mathbb{RP}^\infty$. Indeed, the (Čech nerve of the) cover $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$ is cofinal among all hypercoverings of $\operatorname{Spec}\mathbb{R}$, and the geometric realization of its connected components is precisely the simplicial model for $B(\mathbb{Z}/2\mathbb{Z})$.*

The same reasoning applies in general if we use the truncated homotopy type:

Example 3.11 (Homotopy type of a field)**.** *Let k be a field. The étale homotopy type of $\operatorname{Spec} k$ is $\sharp$-isomorphic to the pro-object $\{B(\mathrm{Gal}(K/k))\}$ where K/k runs over the system of all finite Galois extensions K/k. Indeed the Čech nerve of the system of Galois coverings $\operatorname{Spec} K \to \operatorname{Spec} k$ is cofinal among all hypercoverings of $\operatorname{Spec} k$ in the truncated type $\mathbf{Et}_n^\sharp(\operatorname{Spec} k)$. So $\mathbf{Et}^\sharp(\operatorname{Spec} k)$ is isomorphic to the pro-object $\{B(\mathrm{Gal}(K/k))\}$. This implies that all the higher homotopy groups vanish so $\mathbf{Et}(\operatorname{Spec} k) \simeq \mathbf{Et}^\sharp(\operatorname{Spec} k)$ is isomorphic to $\{B(\mathrm{Gal}(K/k))\}$ too.*

Example 3.12 (Homotopy type of $\operatorname{Spec}\mathbb{Z}$)**.** *We know that $\pi_1^{et}(\operatorname{Spec}\mathbb{Z}) = 0$. Moreover $H^n_{et}(\operatorname{Spec}\mathbb{Z}, \mathbb{Z}_l) = 0$ for all $n \geq 1$ and primes l by arithmetic duality theorems. By an analogue of Hurewicz' theorem* [AM69, Corollary 4.15]*, this implies that $\pi_n^{et}(\operatorname{Spec}\mathbb{Z}) = 0$ too. This implies that $\mathbf{Et}(\operatorname{Spec}\mathbb{Z}) \simeq \mathbf{Et}^\sharp(\operatorname{Spec}\mathbb{Z})$ is contractible, in accordance with the heuristic that it should be the punctured three-sphere.*

In all of the above examples the higher étale homotopy groups vanished. If this is not the case then in special situations we might still be able to compute them (cf. [AM69, Theorem 6.7]):

Proposition 3.13. *Let* $\mathbf{X}$ *be a pro-object in* $\mathrm{Ho}(\mathrm{Top}_0)$*. Suppose that* $\pi_1(\mathbf{X}) = 0$ *and* $\pi_q(\mathbf{X})$ *is good for all* $q < n$*. Then*

$$\pi_n(\widehat{\mathbf{X}}) \simeq \widehat{\pi_n(\mathbf{X})}.$$

The definition of goodness in the case of a pro-group is exactly the same.

Example 3.14 (Homotopy type of projective space)**.** *Let* $n \geq 1$ *with* $X = \mathbb{P}^n_{\mathbb{C}}$*. If* $n = 1$*, then* $X(\mathbb{C})$ *is the 2-sphere* S^2*. Topologists tell us that the group* $\pi_n(S^2)$ *is isomorphic to* $\mathbb{Z}$ *if* $n = 2, 3$ *and finite if* $n \geq 4$*. We may therefore apply the above proposition, and see that*

$$\pi_n^{et}(X) = \widehat{\mathbb{Z}}, \widehat{\mathbb{Z}}, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/12, \mathbb{Z}/2, \ldots \qquad \text{if } n = 2, 3, 4, 5, 6, 7, \ldots$$

If $n \geq 2$*, the homotopy type of* $X(\mathbb{C})$ *can be computed using the fibration*

$$\mathbb{C}^\times \hookrightarrow \mathbb{C}^{n+1} \setminus \{0\} \to X(\mathbb{C})$$

and the étale homotopy groups of X *may be computed similarly in terms of homotopy groups of spheres.*

Example 3.15 (Homotopy type of a K3 surface)**.** *Let* X *be a complex* $K3$ *surface. Then* $X(\mathbb{C})$ *is simply connected, its higher homotopy groups are good and can be computed in terms of homotopy groups of spheres; see* [BB15]*. Therefore we have for example:*

$$\pi_n^{et}(X) = \widehat{\mathbb{Z}}^{22}, \widehat{\mathbb{Z}}^{252}, \widehat{\mathbb{Z}}^{3520} \oplus (\mathbb{Z}/2)^{42}, \ldots \qquad \text{if } n = 2, 3, 4, \ldots$$

In fact, any two complex $K3$ *surfaces are diffeomorphic so have the same (étale) homotopy type. This means that the homotopy type does not detect the Picard number of a variety.*

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