The Bloch-Kato conjecture on special values of L-functions

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March 10, 2021

We state the Bloch-Kato conjecture, following the formulation of Fontaine and Perrin-Riou [FPR94], mostly following [Fon92]. Then we specialize to the case of pure motives and 1-motives. For simplicity, we will only consider motives over $\mathbb{Q}$ and do not treat more general coefficients. Please consider this document as a rough summary. Happy to receive corrections at: `jcsl5@cam.ac.uk`.

1 Statement of the Bloch-Kato conjecture

1.1 Systems of realizations

Instead of working with a not yet existing abelian category of mixed motives, we phrase our conjecture in a certain subcategory $\mathcal{M}$ of a category of 'realizations' $\mathcal{C}$, see Convention 1.10.

Definition 1.1. *The category $\mathcal{C}_0$ has as objects tuples $(M_{dR}, M_B, \{M_\ell\}, \{i_\ell\}, j_\infty, \{j_\ell\})$ consisting of:*

1. *A finite-dimensional $\mathbb{Q}$-vector space M_{dR}, equipped with a decreasing, exhaustive and separated filtration $(F^i M_{dR})_{i\in\mathbb{Z}}$.*
2. *A $\mathbb{Q}$-vector space M_B, equipped with a $\mathbb{Q}$-linear involution ϕ_∞. We set $M_B^+ = M_B^{\phi_\infty=1}$.*
3. *For each prime number ℓ, a $\mathbb{Q}_\ell$-vector space M_ℓ equipped with a continuous $G_\mathbb{Q}$-action. There exists a finite set of primes S such that for all ℓ and all $p \notin S$, M_ℓ is unramified at p if $p \neq \ell$ and crystalline at p if $p = \ell$.*

together with comparison isomorphisms:

1. *(Betti-etale) For each prime ℓ, an isomorphism $i_\ell \colon M_B \otimes \mathbb{Q}_\ell \xrightarrow{\sim} M_\ell$, identifying $\phi_\infty \otimes \mathrm{Id}$ with complex conjugation on M_ℓ.*
2. *(Betti-de Rham) An isomorphism $j_\infty \colon M_B \otimes \mathbb{C} \to M_{dR} \otimes \mathbb{C}$, identitying $\phi_\infty \otimes c$ with $\mathrm{Id}_{M_{dR}} \otimes c$, where c denotes complex conjugation on $\mathbb{C}$.*
3. *(Etale-de Rham) For each prime ℓ, an isomorphism*

$$j_l \colon B_{dR,\ell} \otimes_{\mathbb{Q}_\ell} M_\ell \to B_{dR,\ell} \otimes_{\mathbb{Q}} M_{dR}. \tag{1.1.1}$$

Here $B_{dR,\ell}$ is the field of ℓ-adic periods, equipped with a $G_{\mathbb{Q}_\ell}$-action and a decreasing filtration. We equip M_{dR} with the trivial $G_{\mathbb{Q}_\ell}$-action and M_ℓ with the trivial filtration $F^0 = M_\ell, F^1 = 0$. We therefore obtain $G_{\mathbb{Q}_p}$-actions and filtrations on both sides of (1.1.1), and we insist that they are respected by j_ℓ.

The morphisms are the evident ones.

Note that for any object M of $\mathcal{C}_0$, the $\mathbb{Q}$-vector space M_B carries a Hodge structure, with (p, q) part the subspace of $M_B \otimes \mathbb{C}$ given by $j_\infty^{-1}(F^p M_{dR}) \cap \phi_\infty(j_\infty^{-1}(F^q M_{dR}))$.

Example 1.2. *We define the Tate object $\mathbb{Q}(1)$ as follows:*

- $\mathbb{Q}(1)_{dR} = \mathbb{Q}$ *with filtration* $F^{-1} = \mathbb{Q}(1)_{dR}$ *and* $F^0 = 0$.
- $\mathbb{Q}(1)_B = (2\pi i)\mathbb{Q}$ *(seen as a subset of the same copy of $\mathbb{C}$ where $\mathbb{Q}(1)_{dR}$ also lives), with ϕ_∞ acting as* -1.
- $\mathbb{Q}(1)_\ell = \mathbb{Q}_\ell(1)$*, the ℓ-adic Tate module of $\mathbb{G}_m$.*
- *The Betti-etale and Betti-de Rham isomorphisms are standard. For the etale-de Rham isomorphism, we need to produce an isomorphism of filtered $G_{\mathbb{Q}_\ell}$-modules $B_{dR} \otimes \mathbb{Q}_\ell(1) \simeq B_{dR} \otimes \mathbb{Q}(1)$ where, rather confusingly, the $\mathbb{Q}_\ell(1)$ on the left twists the Galois action and leaves the filtration unchanged while the $\mathbb{Q}(1)$ on the right only twists the filtration. To define such an isomorphism, recall (or accept) that B_{dR} is the fraction field of the discrete valuation ring B_{dR}^+. There is a distinguished uniformizer $t \in B_{dR}^+$ such that $F^k B_{dR} = t^k B_{dR}^+$ and $G_{\mathbb{Q}_\ell}$ acts on t through the ℓ-adic cyclotomic character. It follows that multiplication by t^{-1} will do the job.*

For the next definition, if ℓ, p are primes and M an object of $\mathcal{C}_0$ we write $D_p(M_\ell) = M_\ell^{I_p}$ if $\ell \neq p$ and $D_p(M_p) = (B_{cris} \otimes_{\mathbb{Q}_p} M_p)^{G_{\mathbb{Q}_p}}$ otherwise.

Definition 1.3. *We say an object M of $\mathcal{C}_0$ has an L-function if for each prime p and ℓ (not necessarily distinct), the local L-factor $L_p(M, s) := \det(1 - p^{-s} \operatorname{Frob}_p \mid D_p(M_\ell))$ is independent of ℓ, and the product $L(M, s) = \prod_p L_p(M, s)$ converges for $Re(s)$ large enough.*

Definition 1.4. *The category $\mathcal{C}$ of systems of realizations has as objects a pair $(M, W_\bullet M)$ where*

- *M is an object of $\mathcal{C}_0$.*
- *$W_\bullet M$ is an increasing, separated and exhaustive filtration by subobjects of M such that the Hodge structure $Gr_i^W M_B$ is pure of weight i.*

The category $\mathcal{C}$ has a notion of a tensor product and dual, and is a $\mathbb{Q}$-linear neutral tannakian category. Our definition of $\mathcal{C}$ differs from Fontaine in that we do not require objects to have an L-function.

Example 1.5. *Let M be an object of $\mathcal{C}_0$ (which has an L-function) such that M_B is pure of weight i. Then by setting $W_{i-1} = 0$ and $W_i = M$, we may view M as an object of $\mathcal{C}$. In particular, the weight -2 Tate object $\mathbb{Q}(1)$ defines an object of $\mathcal{C}$.*

1.2 Working category

Let $\mathcal{M}$ be a full abelian subcategory of $\mathcal{C}$ containing the unit object $\mathbb{Q}$ and which is closed under $M \mapsto M^*(1)$. Write

$$\mathrm{H}^0(M) = \mathrm{Hom}_{\mathcal{M}}(\mathbb{Q}, M) = \mathrm{Hom}_{\mathcal{C}}(\mathbb{Q}, M),$$
$$\mathrm{H}^1(M) = \mathrm{Ext}^1_{\mathcal{M}}(\mathbb{Q}, M).$$

We define the subspace $\mathrm{H}^1_f(M) \subset \mathrm{H}^1(M)$ consisting of those elements with the property that for all ℓ its ℓ-adic realization lies in the Bloch-Kato Selmer group $\mathrm{H}^1_f(\mathbb{Q}, M_\ell)$, i.e. is unramified at $p \neq \ell$ and crystalline at ℓ. For simplicity, write $\mathrm{H}^0_f(M) = \mathrm{H}^0(M)$. Although we suppress $\mathcal{M}$ from the notation of $\mathrm{H}^1(M)$, this group may certainly depend on the choice of $\mathcal{M}$.

Remark 1.6. *These definitions are motivated by the fact that if $\mathcal{M}$ would be isomorphic to the conjectural category of mixed motives over $\mathbb{Q}$ and $M = \mathrm{H}^i(X)(n)$, then (for most values of i and n) $\mathrm{H}^1(M)$ should be isomorphic to the motivic cohomology group $\mathrm{H}^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(m))$ and $\mathrm{H}^1_f(M)$ should correspond to the 'integral part' of $\mathrm{H}^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(m))$, i.e. those classes coming from $\mathrm{H}^{i+1}_{\mathcal{M}}(\mathcal{X}, \mathbb{Q}(m))$ where $\mathcal{X}/\mathbb{Z}$ is a regular model for X.*

Definition 1.7. *We say $\mathcal{M}$ is f-admissible if for every object M of $\mathcal{M}$, the following are satisfied:*

1. *For every prime p, the p-adic regulator map $\mathrm{H}^i_f(M)_{\mathbb{Q}_p} \to \mathrm{H}^i_f(M_p)$ is an isomorphism for $i = 0, 1$.*
2. *If $\mathrm{H}^0_f(M^*(1)) = \mathrm{H}^1_f(M^*(1)) = 0$, the archimedean regulator $\mathrm{H}^i_f(M)_{\mathbb{R}} \to \mathrm{Ext}^i_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}, (M_B)_{\mathbb{R}})$ is an isomorphism for $i = 0, 1$.*

Here $\mathrm{MHS}^+_{\mathbb{R}}$ denotes the category of $\mathbb{R}$-mixed Hodge structures which are defined over $\mathbb{R}$ (i.e. carry an involution ϕ_∞).

Here's a rather imprecise conjecture.

Conjecture 1.8. *There exists a contravariant functor*

$$\mathsf{real}\colon \left\{ \begin{matrix} \textit{Reduced, irreducible,} \\ \textit{finite type } \mathbb{Q}\textit{-schemes} \end{matrix} \right\} \to \mathcal{C}$$

extending the realization functor for smooth projective varieties. For any variety X, $\mathsf{real}(X)_B$ is isomorphic to $\mathrm{H}^*_B(X(\mathbb{C}), \mathbb{Q})$ with its mixed Hodge structure constructed by Deligne [Del74, §8.2].

Such a functor is known to exist on the subcategory of smooth quasi-projective varieties [Fon92, §6.7].

Definition 1.9. *An object of $\mathcal{C}$ lying in the Tannakian subcategory generated by the image of* real *is called of motivic origin.*

Strictly speaking, this definition only makes sense when Conjecture 1.8 holds. However since we know how to define real in many cases (e.g. smooth quasi-projective varieties, 1-motives), we are able to identify many objects of $\mathcal{C}$ of motivic origin.

Convention 1.10. *For the remainder of this note, fix an f-admissible subcategory $\mathcal{M}$ of $\mathcal{C}$ such that every object of $\mathcal{M}$ is of motivic origin.*

1.3 The Deligne period map

Let M be an object of $\mathcal{M}$. Let $t_M = M_{dR}/F^0$ be its tangent space. Since $j_\infty \colon M_B \otimes \mathbb{C} \to M_{dR} \otimes \mathbb{C}$ sends $\phi_\infty \otimes c$ to $\mathrm{Id} \otimes c$, the subspace $M_B^+ \otimes \mathbb{R}$ lands in $M_{dR} \otimes \mathbb{R}$. Quotienting out $F^0 M_{dR}$ gives rise to the Deligne period map

$$\alpha_M \colon M_B^+ \otimes \mathbb{R} \to t_M \otimes \mathbb{R}.$$

Lemma 1.11. *There are canonical isomorphisms* $\ker(\alpha_M) \simeq \operatorname{coker}(\alpha_{M^*(1)})^*$ *and* $\operatorname{coker}(\alpha_M) \simeq \ker(\alpha_{M^*(1)})^*$.

Proof. First note that $M_{dR} \otimes \mathbb{R} \simeq (M_B^+)_\mathbb{R} \otimes (M_B^-)_\mathbb{R}$. Therefore $(M_{dR})_\mathbb{R}^* \simeq (M^*(1)_B^-)_\mathbb{R} \oplus (M^*(1)_B^+)_\mathbb{R}$. On the other hand, the filtration $F^i M^*(1)_{dR}$ is given by $\ker(M_{dR}^* \to (F^{-i})^*)$. Therefore $(F^0 M_{dR})^* \simeq M_{dR}^* / \ker(M_{dR}^* \to (F^{-i})^*) \simeq M^*(1)_{dR}/F^0$. It follows that the exact sequence

$$0 \to \ker \alpha_M \to (F^0 M_{dR} \otimes M_B^+)_\mathbb{R} \to (M_{dR})_\mathbb{R} \to \operatorname{coker}(\alpha_M) \to 0$$

dualizes to

$$0 \to \operatorname{coker}(\alpha_M)^* \to (M^*(1)_B^-)_\mathbb{R} \oplus (M^*(1)_B^+)_\mathbb{R} \to (M^*(1)_B^-)_\mathbb{R} \oplus M^*(1)_{dR}/F^0 \to \ker(\alpha_M)^* \to 0.$$

When omitting the red terms, the middle map is exactly $\alpha_{M^*(1)}$. □

We now relate α_M to $\mathrm{MHS}_\mathbb{R}^+$. Write $\mathrm{H}^i(\mathbb{R}, M_B)$ for $\mathrm{Ext}^i_{\mathrm{MHS}_\mathbb{R}^+}(\mathbb{R}, (M_B)_\mathbb{R})$ for $i = 0, 1$. If $W_0 M = M$ (i.e. all weights are ≤ 0), $\mathrm{H}^0(\mathbb{R}, M_B) \simeq \ker(\alpha_M)$ and $\mathrm{H}^1(\mathbb{R}, M_B) \simeq \operatorname{coker}(\alpha_M)$. In general:

Lemma 1.12. *There is an exact sequence*

$$0 \to \mathrm{H}^0(\mathbb{R}, M_B) \to \ker \alpha_M \to \ker \alpha_{M/W_0 M} \to \mathrm{H}^1(\mathbb{R}, M) \to \operatorname{coker} \alpha_M \to 0. \qquad (1.3.1)$$

If $M = \mathrm{H}^i(X)(n)$ with $i - 2n < 0$, then $\mathrm{H}^1(\mathbb{R}, M_B)$ is isomorphic to Deligne cohomology $\mathrm{H}^{i+1}_\mathcal{D}(X_{/\mathbb{R}}, \mathbb{R}(n))$.

1.4 The fundamental exact sequence

We define the fundamental exact sequence of M. It has the form:

$$0 \to \mathrm{H}^0(M)_\mathbb{R} \xrightarrow{u_M} \ker \alpha_M \xrightarrow{v'_M} \mathrm{H}^1_f(M^*(1))^*_\mathbb{R} \xrightarrow{\delta_M} \mathrm{H}^1_f(M)_\mathbb{R} \xrightarrow{v_M} \operatorname{coker}(\alpha_M) \xrightarrow{u'_M} \mathrm{H}^0(M^*(1))_\mathbb{R} \to 0. \qquad (1.4.1)$$

The morphism u_M is the composite of the inclusion $\mathrm{H}^0(M)_\mathbb{R} \hookrightarrow \mathrm{H}^0(\mathbb{R}, M_B)$ and the inclusion $\mathrm{H}^0(\mathbb{R}, M_B) \to \ker \alpha_M$ of (1.3.1). Similarly v_M is the composite $\mathrm{H}^1_f(M) \to \mathrm{H}^1(\mathbb{R}, M_B) \to \operatorname{coker}(\alpha_M)$. The maps u'_M and v'_M are the duals of $u_{M^*(1)}$ and $v_{M^*(1)}$, using the identification of Lemma 1.11.

The middle map δ_M is defined via a (slightly intricate) procedure in $\mathcal{M}$ using computations with extensions, see [FPR94, Proposition III.3.2.5]. It generalizes the Neron-Tate height pairing on elliptic curves. That same proposition also proves:

Proposition 1.13. *Under the assumption that $\mathcal{M}$ is f-admissible, the sequence (1.4.1) is exact.*

1.5 The fundamental line

For a finite-dimensional vector space V, write $\det V$ for its top exterior power. Write $L_f(M) = \det \mathrm{H}^0(M) \otimes (\det \mathrm{H}^1_f(M))^*$.

The fundamental line is defined as

$$\Delta_f(M) = L_f(M) \otimes L_f(M^*(1)) \otimes \det t_M \otimes \det(M_B^+)^*. \tag{1.5.1}$$

As written, this is just a one-dimensional $\mathbb{Q}$-vector space. The point is that:

- There exists a canonical isomorphism $i_M : \Delta_f(M)_{\mathbb{R}} \simeq \mathbb{R}$.
- For every prime p, there exists a canonical measure $|\cdot|_p$ on $\Delta_f(M)_{\mathbb{Q}_p}$. (In other words, there exists a canonical $\mathbb{Z}_p$-lattice inside $\Delta_f(M)_{\mathbb{Q}_p}$.)

The first isomorphism follows immediately from the exact sequence (1.3.1) and the definition of the Deligne period map. The second assertion follows from Galois cohomology computations and p-adic Hodge theory, see [FPR94, §II.4] for details.

1.6 Statement of the Bloch-Kato conjecture

Let M be an object of $\mathcal{M}$. To state the conjectures that follow, assume that M has an L-function (Definition 1.3.).

Conjecture 1.14 (Weak Bloch-Kato). *For every object M of $\mathcal{M}$, the order of vanishing of $L(M,s)$ at $s=0$ equals* $\dim \mathrm{H}^1_f(M^*(1)) - \dim \mathrm{H}^0(M^*(1))$.

Write $L^*(M,0)$ for the leading term of $L(M,s)$ at $s=0$.

Conjecture 1.15 (Beilinson-Deligne). *There exists (a necessarily unique) $\delta_f(M) \in \Delta_f(M)$ such that $i_M(\delta_f(M) \otimes 1)L^*(M,0) = 1$.*

Conjecture 1.16 (The Bloch-Kato conjecture). *Conjecture 1.15 holds and moreover $|\delta_f(M) \otimes 1|_p = 1$ for all p.*

Remark 1.17. *The Bloch-Kato conjecture determines the real number $L^*(M,0)$ up to sign, which can be easily determined. Indeed, let $r_M = \mathrm{ord}_{s=0} L(M,s)$. Then $r_{M(i)} = 0$ for $i \geq 0$ large enough and the sign of $L^*(M,0)$ is given by $(-1)^{\sum_{i>0} r_{M(i)}}$.*

Remark 1.18. *The formulation of the Bloch-Kato conjecture given here differs from the original one given in* [BK90]*, which puts an emphasis on Tamagawa numbers. The connection between the two is discussed in* [Fon92, §11].

1.7 The f-closed case

There is a special class of mixed motives for which the conjecture is easy to state.

Definition 1.19. *We say an object M of $\mathcal{M}$ is f-closed if $\mathrm{H}^1_f(M) = \mathrm{H}^1_f(M^*(1)) = 0$.*

Definition 1.20. *Define the equivalence relation* $\sim_f$ *(called* f*-equivalence) on (isomorphism classes of) the objects of* $\mathcal{M}$ *as the equivalence relation generated by:*

1. *If* N *is an extension of* M *by* $\mathbb{Q}$ *defining an element of* $\mathrm{H}^1_f(M)$*, then* $M \sim_f N$.
2. *If* $M \sim_f N$*, then* $M^*(1) \sim_f N^*(1)$.

Assuming a functional equation between $L(M,s)$ and $L(M^*(1),-s)$, the conjectures for M of §1.6 are true if and only if they hold for any object f-equivalent to M. It turns out that every element of $\mathcal{M}$ is f-equivalent to an f-closed motive M which even satisfies $\mathrm{H}^0(M) = \mathrm{H}^0(M^*(1)) = 0$ (an explicit recipe is given in [Sch94, §7.8]). In that case, the fundamental line is trivial and Conjectures 1.14 and 1.15 reduce to:

$$L(M,0) \neq 0 \text{ and } L(M,0)/c^+(M) \in \mathbb{Q}.$$

Here $c^+(M)$ is the determinant of the Deligne period map when $\mathbb{Q}$-bases of M_B^+ and M_{dR}/F^0 are chosen.

Therefore, in principle, it is enough to consider the f-closed case.

Example 1.21. *Let* $E/\mathbb{Q}$ *be an elliptic curve of rank zero with finite Tate-Shafarevich group. Let* $M = \mathrm{H}^1(E)(1)$*. Then* $(M_B^+)_\mathbb{R} \to (M_{dR}/F^0)_\mathbb{R}$ *is an isomorphism between one-dimensional vector spaces, and choosing suitable* $\mathbb{Q}$*-generators is given by* $1 \mapsto \Omega_E = \int_{E(\mathbb{R})} \omega$*, where* ω *is a holomorphic differential. It follows that the Beilinson-Deligne conjecture in this case asserts that* $L(M,0)/\Omega_E \in \mathbb{Q}^\times$.

Example 1.22. *Let* M *be the Artin motive associated to the one-dimensional Galois representation* $G_\mathbb{Q} \to \{\pm 1\}$ *which cuts out the extension* $\mathbb{Q}(i)/\mathbb{Q}$*. It corresponds to the unique nontrivial Dirichlet character* $\chi\colon (\mathbb{Z}/4\mathbb{Z})^\times \to \{\pm 1\}$*. One can check that* $M(1)$ *is* f*-closed and* $\mathrm{H}^1_f(M(1)) = \mathrm{H}^1_f(M^*) = 0$*. Therefore* $L(\chi,1)$ *is a nonzero rational multiple of* $c^+(M) = \pi$*. Indeed:*

$$1 - 1/3 + 1/5 - 1/7 + \cdots = \frac{\pi}{4}.$$

Remark 1.23. *We say* M *is critical if* α_M *is an isomorphism. If* M *is* f*-closed and* $\mathrm{H}^0(M) = \mathrm{H}^0(M^*(1)) = 0$*, then* M *is critical. Therefore the above remarks are very much related to the conjectures made by Deligne in the critical case.*

2 The pure case

If M is an object of $\mathcal{M}$, its weights are those integers n such that $Gr_n^W M \neq 0$.

Lemma 2.1.

1. $\mathrm{H}^0(M) = 0$ *if* 0 *is not a weight of* M.
2. $\mathrm{H}^1_f(M) = 0$ *if all weights are* > -1.
3. $\mathrm{H}^0(M^*(1)) = 0$ *if* -2 *is not a weight of* M.
4. $\mathrm{H}^1_f(M^*(1)) = 0$ *if all weights are* < -1.

Proof. We only need to show the first two, which follow from the assumption that the p-adic regulator map is an isomorphism and the corresponding properties of Bloch-Kato Selmer groups. ∎

Assume now that M is pure of weight w, of the form $\mathrm{H}^i(X)(n)$ for some smooth projective variety $X/\mathbb{Q}$ and $w = i - 2n$. Then the conjecture can be simplified according to different values of w. Since it is expected that there exists a functional equation relating $L(M,s)$ and $L(M^*(1),-s)$ and we know (by Poincaré duality and hard Lefschetz) that $M^*(1) \simeq M(w+1)$, we will only consider the case $w \leq -1$. See [Nek94, §6] for the same case distinction but where the conjectures are formulated in terms of motivic cohomology.

2.1 Weight ≤ -3

Lemma 2.1 implies that $\mathrm{H}^0(M) = \mathrm{H}^0(M^*(1)) = \mathrm{H}^1_f(M^*(1)) = 0$. It follows that the fundamental line is given by

$$\Delta_f(M) = \det \mathrm{H}^1_f(M)^* \otimes \det t_M \otimes \det(M_B^+)*$$

The Beilinson regulator $v_M \colon \mathrm{H}^1_f(M) \to \operatorname{coker} \alpha_M$ and the Deligne period map $\alpha_M \colon (M_B^+)_{\mathbb{R}} \hookrightarrow (M_{dR}/F^0)_{\mathbb{R}}$ combine to give an isomorphism $\Delta_f(M)_{\mathbb{R}} \simeq \mathbb{R}$. The weak Bloch-Kato conjecture predicts that $L(M,0) \neq 0$, and the Beilinson-Deligne conjecture predicts that $L(M,0) \in (\text{regulator})(\text{period})\mathbb{Q}$.

It is expected that $\mathrm{H}^1_f(M) \simeq \mathrm{H}^{i+1}_{\mathcal{M},\mathbb{Z}}(X,\mathbb{Q}(n))$ (the integral part of motivic cohomology), $\operatorname{coker} \alpha_M \simeq \mathrm{H}^{i+1}_{\mathcal{D}}(X_{/\mathbb{R}},\mathbb{R}(n))$ (Deligne cohomology defined over $\mathbb{R}$) and the map v_M corresponds to the Beilinson regulator. Modulo these identifications, we recover Beilinson's conjecture in weight ≤ -3.

2.2 Near central point: weight -2

Lemma 2.1 implies that $\mathrm{H}^0(M) = \mathrm{H}^1_f(M^*(1)) = 0$, but $\mathrm{H}^0(M^*(1))$ and $\mathrm{H}^1_f(M)$ may be nonzero. Weak Bloch-Kato predicts that $L(M,s)$ has a pole of order $\dim \mathrm{H}^0(M^*(1))$. The fundamental exact sequence reduces to

$$0 \to \mathrm{H}^1_f(M)_{\mathbb{R}} \to \operatorname{coker} \alpha_M \to \mathrm{H}^0(M^*(1))^*_{\mathbb{R}} \to 0.$$

Since $w = -2$, $M = \mathrm{H}^{2n-2}(X)(n)$ for some smooth projective variety, it is expected that $\mathrm{H}^0(M^*(1))$ is isomorphic to $(\mathrm{CH}^{n-1}(X)/\mathrm{CH}^{n-1}(X)_{hom\sim 0}) \otimes \mathbb{Q}$. Assuming this, the weak Bloch-Kato conjecture is equivalent to the Tate conjecture.

Example 2.2. *Let $X/\mathbb{Q}$ be a smooth projective surface. Then the Tate conjecture predicts that $L(\mathrm{H}^2(X),s)$ has a pole of order $\dim_{\mathbb{Q}} NS(X)_{\mathbb{Q}}$ at $s=2$.*

2.3 Central point: weight -1

This is the only case where both $\mathrm{H}^1_f(M)$ and $\mathrm{H}^1_f(M^*(1))$ may be nonzero. Lemma 2.1 implies that $\mathrm{H}^0(M) = \mathrm{H}^0(M^*(1)) = 0$. Since α_M is injective by weight reasons and $M^*(1) \simeq M$, Lemma 1.11 shows that α_M is an isomorphism. Therefore the fundamental exact sequence reduces to a perfect pairing

$$\mathrm{H}^1_f(M)_{\mathbb{R}} \times \mathrm{H}^1_f(M)_{\mathbb{R}} \to \mathbb{R}.$$

Weak Bloch-Kato predicts that $\mathrm{ord}_{s=0} L(M,s) = \dim \mathrm{H}^1_f(M)$; the Beilinson-Deligne conjecture predicts that $L^*(M,0) \in (\text{period})(\text{determinant of pairing})\mathbb{Q}$.

Since w has weight -1, $M = \mathrm{H}^{2n+1}(X)(n)$. Moreover $\mathrm{H}^1_f(M)$ is expected to be isomorphic to $\mathrm{CH}^{n+1}(X)_{hom\sim 0} \otimes \mathbb{Q}$. Therefore we recover the generalized Birch-Swinnerton-Dyer conjecture.

3 The case of 1-motives

3.1 Definition of 1-motives

Slogan: A 1-motive is a mixed motive associated to an open semistable curve $X/\mathbb{Q}$.

Slogan: A 1-motive is a mixed motive whose Hodge numbers belong to $\{(0,0),(-1,0),(0,-1),(-1,-1)\}$.

Example 3.1. *Let X be the curve obtained by identifying the points 1 and $t \in \mathbb{Q}^\times$ on $\mathbb{G}_m$ transversally. It embeds in the curve $\overline{X}$, obtained by identifying two points of $\mathbb{P}^1$ transversally. We have $\operatorname{Pic}^0 X \simeq \operatorname{Pic}^0 \overline{X}/\langle[0]-[\infty]\rangle$, and $\operatorname{Pic}^0 \overline{X} \simeq \mathbb{G}_m$. Therefore $\operatorname{Pic}^0 X$ is the cokernel of the morphism $\mathbb{Z} \xrightarrow{u} \mathbb{G}_m$ sending 1 to t.*

The previous example motivates the following definition [Del74, §10]:

Definition 3.2. *A 1-motive over $\mathbb{Q}$ is a morphism $X \to G$ of commutative group schemes over $\mathbb{Q}$, where G is a semi-abelian variety (i.e. an extension of an abelian variety by a torus) and X is a locally constant group scheme of finite free $\mathbb{Z}$-modules.*

In other words, $X(\overline{\mathbb{Q}})$ is a finite free $\mathbb{Z}$-module equipped with a $G_{\mathbb{Q}}$-action with open kernel, and the morphism $X(\overline{\mathbb{Q}}) \to G(\overline{\mathbb{Q}})$ is Galois-equivariant. With an obvious definition of morphisms between 1-motives, we obtain an additive category $\mathcal{MM}_1$.

Definition 3.3. *Let $\mathcal{MM}_1(\mathbb{Q})$ be the isogeny category of 1-motives: the objects of $\mathcal{MM}_1(\mathbb{Q})$ are those of $\mathcal{MM}_1$ and the morphisms are given by*

$$\operatorname{Hom}_{\mathcal{MM}_1(\mathbb{Q})}(M_1, M_2) = \mathbb{Q} \otimes \operatorname{Hom}_{\mathcal{MM}_1}(M_1, M_2).$$

It can be useful to view an object of $\mathcal{MM}_1(\mathbb{Q})$ as a morphism $[X \to G]$, where $G/\mathbb{Q}$ is a semi-abelian variety, X is a finite dimensional $\mathbb{Q}$-vector space with a $G_{\mathbb{Q}}$-action with open kernel and $u\colon X \to G(\overline{\mathbb{Q}})$ a Galois equivariant morphism. Just as is the case with abelian varieties, $\mathcal{MM}_1(\mathbb{Q})$ is a $\mathbb{Q}$-linear abelian category; however it is not semisimple. It contains the category of abelian varieties and $\mathbb{Q}$-valued Galois representations.

3.2 Realizations

In [Del74, §10], a realization functor

$$\mathsf{Real}_1 \colon \mathcal{MM}_1(\mathbb{Q}) \to \mathcal{C}$$

is described, landing in the subcategory of $\mathcal{C}$ of objects of Hodge numbers $\subset \{(0,0),(-1,0),(0,-1),(-1,-1)\}$. (We follow the convention of using homology, so the weights are ≤ 0.) The part $Gr_i^W \mathsf{Real}_1(M)$ for $i = -2,-1,0$ depends only on the toric, abelian and discrete part respectively. Let's briefly describe the different realizations of $M = [X \xrightarrow{u} G]$.

Betti realization: Consider $G(\mathbb{C})$ as a complex manifold and let $\operatorname{Lie}(G(\mathbb{C})) \to G(\mathbb{C})$ be the exponential map. Then M_B is defined as the pullback of this map along u, tensorized with $\mathbb{Q}$. It fits in an exact sequence

$$0 \to \mathrm{H}_1(G(\mathbb{C}), \mathbb{Q}) \to M_B \to X \to 0.$$

Etale realization: $M_\ell = \varprojlim M[\ell^n]$, where $M[n] = \{(x,g) \in X \times G \mid u(x) = ng\}/\{(nx, u(x)) \mid x \in X\}$. When $X = 0$, we recover the Tate module of G. When $G = 0$, we recover $X \otimes \mathbb{Z}_\ell$.

de Rham realization: M_{dR} is the Lie algebra of the universal vectorial extension of M; we omit the details.

The Tate conjecture for abelian varieties implies:

Proposition 3.4. *The functor* Real_1 *is fully faithful.*

We denote its essential image by $\mathcal{M}_1$. It is an abelian subcategory of $\mathcal{C}$ containing the unit object $\mathbb{Q} = \mathsf{Real}_1([\mathbb{Q} \to 0])$ and is stable under $M \mapsto M^*(1)$.

Proposition 3.5. *The category* $\mathcal{M}_1$ *is* f*-admissible if and only if for every abelian variety over* $\mathbb{Q}$ *and every* p*, the* p*-primary part of the Tate-Shafarevich group of* A *is finite.*

3.3 Example

We examine what our conjectures say for L-functions of 1-motives. Since we have already discussed the case of pure motives in the previous section, we consider a mixed example.

Example 3.6. *Let* p *be a prime number and* $M = \mathsf{Real}_1([\mathbb{Z} \xrightarrow{1 \mapsto p} \mathbb{G}_m])$*. This* M *arises from Example 3.1, and defines an element* $M \in \mathrm{Ext}^1_{\mathcal{M}_1}(\mathbb{Q}, \mathbb{Q}(1))$*. It turns out that* M *is critical and* f*-closed, and the Deligne period is* $\log p$*. We have* $L(M,s) = \zeta(s)\zeta(s+1)(1-p^{-s})$*, so* $L(M,0) = -\frac{1}{2}\log p$*, as predicted by the Deligne-Beilinson conjecture.*

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