Rigid analytification and uniformization

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1 Analytification

1.1 Over the complex numbers

Definition 1.1. *A complex analytic space is a locally ringed space $(X, \mathcal{O}_X)$ such that for each $x \in X$ there exists an open $U \subset X$ with the property that $(U, \mathcal{O}_X|_U)$ is isomorphic to $\{v \in V \mid f_1(v) = \cdots = f_k(v) = 0\}$, where V is an open subset of $\mathbb{C}^n$ and f_i are complex analytic functions (in other words, holomorphic functions) defined on V, endowed with its sheaf of holomorphic functions.*

Let X be a finite type scheme over $\mathbb{C}$. Then X is locally given by $\operatorname{Spec} \mathbb{C}[t_1, \dots, t_n]/(f_1, \dots, f_k)$. It follows that $X^{an} := X(\mathbb{C})$, endowed with the analytic topology and analytic structure sheaf, is a complex analytic space, called the analytification of X. Moreover, in the category of locally ringed spaces (in which both schemes and complex analytic spaces fully faithfully embed), we have a morphism $\epsilon\colon X^{an} \to X$ induced by the inclusion $X^{an} \subset X$. Given any other complex analytic space Z and morphism $Z \to X$, it uniquely factors through $X^{an} \to X$. We therefore find the following characterization of the analytification:

Lemma 1.2. *Let X be a finite type scheme over $\mathbb{C}$. Then the functor of points of X^{an} is given by*

$$\begin{aligned} (\textit{Complex analytic spaces}) &\to (\textit{Sets}) \\ Z &\mapsto \operatorname{Hom}(Z, X). \end{aligned}$$

The Hom*-set is taken in the category of locally ringed spaces.*

This categorical description is nice because it shows that analytification is indepedent of the choice of affine covering of X.

This defines a functor

$$(-)^{an}\colon (\text{Finite type schemes}/\mathbb{C}) \to (\text{Complex analytic spaces})$$

with the following fundamental properties:

1. X is connected/reduced/smooth/separated/proper if and only if the same is true for X^{an}.

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2. If X is proper, the functor $\mathscr{F} \mapsto \mathscr{F}^{an} = \epsilon^*\mathscr{F}$ from coherent sheaves on X to coherent sheaves on X^{an} is an equivalence and induces an isomorphism on sheaf cohomology groups. ('GAGA')

3. The functor $(-)^{an}$ is fully faithful when restricted to proper schemes over $\mathbb{C}$. (Follows from GAGA using the ideal sheaf of the graph of a function.)

4. If Z is a compact Riemann surface, there exists a smooth projective connected curve $X/\mathbb{C}$ with $X^{an} \simeq Z$. ('Riemann existence theorem')

1.2 In the rigid setting

Let k be a non-archimedean field. Then the story is very similar.

Definition 1.3. *Let X be a finite type scheme over k. An* analytification *X^{an} of X is a rigid analytic space over k with functor of points*

$$\begin{aligned} (\textit{Rigid analytic spaces}) &\to (\textit{Sets}) \\ Z &\mapsto \operatorname{Hom}(Z, X). \end{aligned}$$

The Hom*-set is taken in the category of locally G-ringed spaces.*

By definition, an analytification of X is unique up to unique isomorphism if it exists and comes with a natural map $\epsilon \colon X^{an} \to X$.

We now prove that X^{an} exists for any X/k of finite type.

Lemma 1.4. *Let X be an affine scheme of finite type over k and Z a rigid space. Then the natural map $\operatorname{Hom}(Z, X) \to \operatorname{Hom}(\mathcal{O}(X), \mathcal{O}(Z))$ is an isomorphism.*

Proof sketch. We construct an inverse to the above natural map. Since $\mathcal{O}_Z$ is a sheaf, we may assume that Z is affinoid. So let $X = \operatorname{Spec} A$ and $Z = \operatorname{Sp} B$, where A is a finitely generated k-algebra and B is a Tate k-algebra. Then the inverse is given by sending $\phi \colon A \to B$ to $\operatorname{Sp} B \to \operatorname{Spec} A, \mathfrak{m} \mapsto \phi^{-1}(\mathfrak{m})$. □

Let's start with constructing the analytification of $\mathbb{A}^1_k$ explicitly. Let $c \in k$ be an element with $|c| > 1$. Let $T(c^n) = k\langle c^{-n}x\rangle$, seen as a sub-algebra of $T(1) = k\langle x\rangle$. The chain

$$\cdots \subset T(c^2) \subset T(c) \subset T(1)$$

gives rise to a chain of inclusions of closed balls

$$D(1) \subset D(c^1) \subset D(c^2) \subset \cdots$$

By glueing rigid analytic spaces, we obtain the rigid space $\mathbb{A}^{1,rig}_k$, the rigid analytic affine line.

Lemma 1.5. *The rigid space $\mathbb{A}^{1,rig}_k$ is the analytification of $\mathbb{A}^1_k$*

Proof. By Lemma 1.4, we have to show that $\operatorname{Hom}(Z, \mathbb{A}^{1,rig}_k) = \mathcal{O}_Z(Z)$ for any rigid space Z. It suffices to check this for $Z = \operatorname{Sp} A$ affinoid. Since $\operatorname{Sp} A$ is affinoid and the cover $\cup_{n\geq 0} D(c^n)$ of $\mathbb{A}^{1,rig}_k$ is admissible, any morphism $\operatorname{Sp} A \to \mathbb{A}^{1,rig}_k$ lands in $D(c^n)$ for some $n \geq 0$. To such a morphism corresponds an element $f \in A$ of norm $|f| \leq |c|^n$. Conversely, given an element $f \in A$ of norm $|f| \leq |c|^n$, there exists an associated morphism $\operatorname{Sp} A \to \mathbb{A}^{1,rig}_k$ mapping into $D(c^n)$. Since any element of A is of norm $\leq |c|^n$ for some $n \geq 0$, this completes the proof. □

Lemma 1.6. *Let $X = \operatorname{Spec} k[t_1, \dots, t_n]/(f_1, \dots, f_k)$. Then X^{an} exists.*

Proof. Let $c \in k$ be an element with $|c| > 1$. Let $A_m = k\langle c^{-m}x_1, \dots, c^{-m}x_n\rangle/(f_1, \dots, f_k)$ and $U_m = \operatorname{Sp} A_m$. By the same reasoning as Lemma 1.6, $X^{an} = \cup_{m \geq 0} U_m$ is the analytification of X. □

Proposition 1.7. *Let X/k be of finite type. Then X^{an} exists.*

Proof. If X is affine, this follows from the previous lemma. To glue the affine patches, one needs the following fact: if X has an analytification $\epsilon \colon X^{an} \to X$ and $U \subset X$ is an open subscheme, $U^{an} = \epsilon^{-1}(U)$ is an analytification of U. The fact allows us to then glue analytifications of affine opens along their intersections. □

This defines a functor:

$$(-)^{an} \colon (\text{Finite type schemes}/k) \to (\text{Rigid analytic spaces}/k).$$

Theorem 1.8. *The functor $(-)^{an}$ satisfies the same properties 1-4 as in the complex case.*

Note that we have not defined what it means for a rigid space to be connected/reduced/smooth/separated or proper. These definitions are straightforward, except for defining properness which requires some work. I think the proof is very similar!

2 Uniformization

Recall that every elliptic curve over $\mathbb{C}$ can be uniformized by $\mathbb{C}$, i.e. is the quotient of $\mathbb{C}$ by a lattice (a discrete and cocompact subgroup). When suitably interpreted, there's an analogous story for rigid spaces, using the theory of the Tate curve. In higher genus, recall that every projective curve of genus ≥ 2 can be uniformized by the upper half plane, i.e. is the quotient of $\mathcal{H}$ by a discrete and cocompact subgroup. Mumford has shown that there's a similar story in the rigid setting, using the p-adic upper half plane. It is important to emphasize (as we will see below) that in the rigid world *not every curve can be uniformized*!

2.1 The Tate curve over $\mathbb{C}$

Let τ be an element of the upper half plane $\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$ and let $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$. Then $\mathbb{C}/\Lambda_\tau$ is the analytification of the elliptic curve $E_\tau/\mathbb{C}$ with Weierstrass equation

$$y^2 = 4x^3 - g_2(\tau)x - g_3(\tau).$$

Here

$$g_2(\tau) = 60 \sum_{\lambda \in \Lambda_\tau \setminus \{0\}} \frac{1}{\lambda^4},$$
$$g_3(\tau) = 140 \sum_{\lambda \in \Lambda_\tau \setminus \{0\}} \frac{1}{\lambda^6}.$$

Explicitly, the map $\mathbb{C} \to E_\tau(\mathbb{C})$ is given by $z \mapsto (\wp(z,\tau), \wp'(z,\tau))$, where $\wp(z,\tau)$ is the Weierstrass $\wp$-function

$$\wp(z,\tau) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda_\tau \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

and where $\wp'(z,\tau)$ is the derivative of $\wp(z,\tau)$ with respect to z.

Since $\Lambda_\tau = \Lambda_{\tau+n}$ for all $n \in \mathbb{Z}$, we may introduce $q = e^{2\pi i \tau}$ and write g_2, g_3 as functions in q. After some substitutions, we see that E_τ is isomorphic to

$$E_q\colon y^2 + xy = x^3 - a_4 x - a_6 \tag{2.1.1}$$

Where

$$\begin{aligned} a_4 &= 5 \sum_{n \geq 1} \frac{n^3 q^n}{1-q^n} = 5q + 45q^2 + 140q^3 + \dots \\ a_6 &= \sum_{n \geq 1} \frac{7n^5 + 5n^3}{12} \cdot \frac{q^n}{1-q^n} = q + 23q^2 + 154q^3 + \dots \end{aligned}$$

Miracle: the power series $a_4(q), a_6(q)$ lie in $\mathbb{Z}[[q]]$.

Let $\Delta(q)$ be the discriminant of the above equation. It turns out that $\Delta(q) = q\prod(1-q^n)^{24}$. (What is Ramanujan doing here?)

Definition 2.1. *We call the curve with Equation* (2.1.1) *the* Tate curve. *It is an elliptic curve over* $\mathbb{Z}[[q]][\Delta(q)^{-1}]$.

Using the exponential map we see that $\mathbb{C}/\Lambda_\tau \simeq \mathbb{C}^\times/q^{\mathbb{Z}}$. Moreover $|q| < 1$ since $\tau \in \mathcal{H}$. Writing $u = e^{2\pi i z}$ and writing $\wp(z,\tau)$ and $\wp'(z,\tau)$ in terms of u and q, we obtain:

Proposition 2.2. *Let $q \in \mathbb{C}^\times$ with $|q| < 1$. Then the complex analytic space $\mathbb{C}^\times/q^{\mathbb{Z}}$ is the analytification of the elliptic curve E_q given by Equation* (2.1.1)*. The isomorphism is given by $\mathbb{C}^\times \to E_q(\mathbb{C}), u \mapsto (X(u,q), Y(u,q))$, where*

$$\begin{aligned} X(u,q) &= \sum_{n \in \mathbb{Z}} \frac{q^n u}{(1-q^n u)^2} - 2 \sum_{n \geq 1} \frac{n q^n}{1-q^n} \\ Y(u,q) &= \sum_{n \in \mathbb{Z}} \frac{(q^n u)^2}{(1-q^n u)^3} + \sum_{n \geq 1} \frac{n q^n}{1-q^n} \end{aligned}$$

Moreover, every elliptic curve over $\mathbb{C}$ arises in this way.

2.2 The non-archimedean case

Let k be a p-adic field and let E be an elliptic curve over k. We cannot possibly expect that $E(k)$ is the quotient of k by a discrete and cocompact subgroup, because no such subgroups exist! However, $k^\times$ has plenty such subgroups, and it is Proposition 2.2 that will generalise well to this setting.

The main point properties of the Tate curve are encapsulated in the following theorem, which does not need rigid spaces to state.

Theorem 2.3. *Let $q \in k^\times$ be an element satisfying $|q| < 1$. Then $a_4(q), a_6(q)$ (defined in the previous section) are elements of k, and E_q is an elliptic curve over k. There exists an isomorphism*

$$\bar{k}^\times / q^{\mathbb{Z}} \simeq E_q(\bar{k})$$

compatible with the Galois action on both sides. The curve E_q has split multiplicative reduction. Conversely, every elliptic curve over k with split multiplicative reduction arises in this way, i.e. is of the form E_q for some q.

Sketch of proof. Since $a_4, a_6 \in \mathbb{Z}[[q]]$ and $|q| < 1$, it is clear that $a_4(q), a_6(q)$ converge to elements of k. To write down an isomorphism $\bar{k}^\times / q^{\mathbb{Z}} \simeq E_q(\bar{k})$, one uses the same formulae as Proposition 2.2 and see (amazingly!) that they still work. The curve E_q has split multiplicative reduction by reducing Equation (2.1.1). Moreover the j-invariant of E_q is the *actual* j-function $j(q) = q^{-1} + 744 + \ldots$. It is elementary that $q \mapsto j(q)$ induces a bijection

$$\{q \in k^\times \mid |q| < 1\} \to \{j \in k \mid |j| > 1\}.$$

Let E/k be an elliptic curve with split multiplicative reduction. Then $|j(E)| > 1$ so E is isomorphic to E_q over $\bar{k}$ for some $q \in k$ with $|q| < 1$. Moreover $|j(E)| > 1$ implies that $j(E) \neq 0$ or 1728. Therefore E and E_q are quadratic twists of each other. Since there is a unique twist where the reduction is split, we conclude that $E \simeq E_q$ over k, as claimed. □

The class of elliptic curves over k that can be uniformized is hence restricted. On the other hand, we get a lot of information about those that *do* admit a uniformization. For example, it is very useful that it is compatible with the Galois action. This quickly leads to:

Corollary 2.4. *Let E/k be an elliptic curve with split multiplicative reduction. Let l be a prime different from the residue characateristic of k and let $T_l E$ be the l-adic Tate module of E. Then there is a short exact sequence of Galois modules*

$$1 \to \mathbb{Z}_l(1) \to T_l E \to \mathbb{Z}_l \to 1.$$

In other words, $T_l E$ has a $\mathbb{Z}_l$-basis in which the Galois action is of the form $\begin{pmatrix} \chi & * \\ 0 & 1 \end{pmatrix}$, where χ is the cyclotomic character.

2.3 The Tate curve as a rigid space

Let $q \in k$ with $|q| < 1$. We will now introduce a rigid space $X_q = \mathbb{G}_{m,k}^{an} / q^{\mathbb{Z}}$ and upgrade Theorem 2.3 to an isomorphism $X_q \simeq E_q^{an}$. We first need to talk about quotients of rigid spaces.

Definition 2.5. *Let Γ be a group acting freely and continuously on a rigid space X. We say the action is properly discontinuous if there exists an admissible covering of the form $\{\gamma \cdot U_i\}_{\gamma \in \Gamma, i \in I}$ of X with $\gamma \cdot U_i \cap U_i = \varnothing$ unless $\gamma = 1$ and such that the sets $\cup_{\gamma \in \Gamma} \gamma \cdot U_i$ are admissible for all $i \in I$.*

In the above situation, we can form the quotient $Y = X/\Gamma$. Indeed, to construct Y, by glueing it suffices to give a cover by rigid spaces V_i, admissible opens $V_{ij} \subset V_i$ and comparison isomorphisms $V_{ij} \xrightarrow{\sim} V_{ji}$ satisfying the usual cocycle compatibility. We may take $V_i = U_i$ and $V_{ij} = U_i \cap (\cup \gamma \cdot U_j)$ and $V_{ij} \to V_{ji}$ the natural isomorphism. One can check that this rigid space comes with a Γ-invariant map $X \to Y$ (where Γ acts trivially on Y), and that any Γ-invariant map $X \to Z$ factors through $X \to Y$.

We first explicitly write down $\mathbb{G}_{m,k}^{an}$. Since $\mathbb{G}_{m,k} = \operatorname{Spec} k[u,v]/(uv-1)$, we know by construction that $\mathbb{G}_{m,k}^{an}$ has an admissible cover

$$\cup_{n\geq 0}\{|u|,|v| \leq |q|^{-n}\} = \cup_{n\geq 0}\{|q|^n \leq |u| \leq |q|^{-n}\}.$$

More generally, if $a \leq b$ are rational numbers let $X[a,b]$ be the subset of $\mathbb{G}_{m,k}^{an}$ given by $\{|q|^b \leq u \leq |q|^a\}$. Then $X[a,b]$ is open affinoid. Moreover the above cover is $\cup X[-n,n]$.

Let $t_q\colon \mathbb{G}_{m,k}^{an} \to \mathbb{G}_{m,k}^{an}$ be the multiplication by q map. It sends $X[a,b]$ to $X[a+1,b+1]$. This defines a $\mathbb{Z}$-action on $\mathbb{G}_{m,k}^{an}$ which is free and continuous.

Lemma 2.6. *This action is properly discontinuous.*

Proof. We may take $U_0 = X[0,1/2]$ and $U_1 = X[1/2,1]$. Then the cover $\{\gamma \cdot U_i\} = \{X[n,n+1/2]\} \cup \{X[n+1/2,n+1]\}$ is an admissible cover that satisfies the requirements for being properly discontinuous. □

It follows that the quotient $X_q = \mathbb{G}_{m,k}^{an}/q^{\mathbb{Z}}$ exists as a rigid space; write $\pi\colon \mathbb{G}_{m,k}^{an} \to X_q$ for the quotient map. Explicitly, it is obtained by glueing $X[0,1/2]$ and $X[1/2,1]$ along their boundary: $X[0,0] \sqcup X[1/2,1/2] \xrightarrow{t_q \sqcup \mathrm{Id}} X[1,1] \sqcup X[1/2,1/2]$. Using this description or the universal property of quotients, we see that the structure sheaf of X_q is given by

$$\mathcal{O}_{X_q}(U) = \{f \in \mathcal{O}_{\mathbb{G}_{m,k}^{an}}(\pi^{-1}U) \mid t_q^* f = f\}$$

For example, $\mathcal{O}_{X_q}(X_q)$ consists of those element $\sum_{n\in\mathbb{Z}} a_n t^n$ with $|a_n|\rho^n \to 0$ as $|n| \to +\infty$ for all $\rho > 0$ satisfying $a_n = q^n a_n$. Therefore $a_n = 0$ if $n \neq 0$, so $\mathcal{O}_{X_q}(X_q) = k$.

It is clear that for every finite field extension l/k we have $X_q(l) = l^\times/q^{\mathbb{Z}}$.

Proposition 2.7. *There exists an isomorphism $X_q \simeq E_q^{an}$ of rigid spaces.*

Proof. This quickly follows, since the isomorphism of Theorem 2.3 is given by analytic functions, namely the ones displayed in Proposition 2.2. □

2.4 Mumford curves

We briefly mention the story in higher genus. There exists a rigid analytic space $\mathcal{H}_p$ over $\mathbb{Q}_p$, called the p-adic upper half plane, whose $\mathbb{C}_p$-points are in bijection with $\mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{P}^1(\mathbb{Q}_p)$. (Note, by analogy, that $\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$ is the disjoint union of the upper and lower half plane.) The rigid space $\mathcal{H}_p$ comes equipped with an action of $\mathrm{PGL}_2(\mathbb{Q}_p)$, and given a discrete, finitely generated and free subgroup $\Gamma \subset \mathrm{PGL}_2(\mathbb{Q}_p)$, the quotient $\mathcal{H}_p/\Gamma$ is isomorphic to the analytification of a unique smooth projective algebraic curve X_Γ over $\mathbb{Q}_p$, called a *Mumford curve.* The smooth projective curves over $\mathbb{Q}_p$ obtained in this way are precisely the ones with split stable degenerate reduction: those curves that have a stable model, of which the irreducible components of the special fibre have normalization isomorphic to $\mathbb{P}^1_{\mathbb{F}_p}$ and of which all the singular points and branches are $\mathbb{F}_p$-rational. Note that this is a generalization of an elliptic curve with split multiplicative reduction!

If you want to read more, we refer to

`http://www.math.columbia.edu/~chaoli/docs/MumfordCurves.html`

for a more complete and entertaining account.