A survey of six ADE classifications

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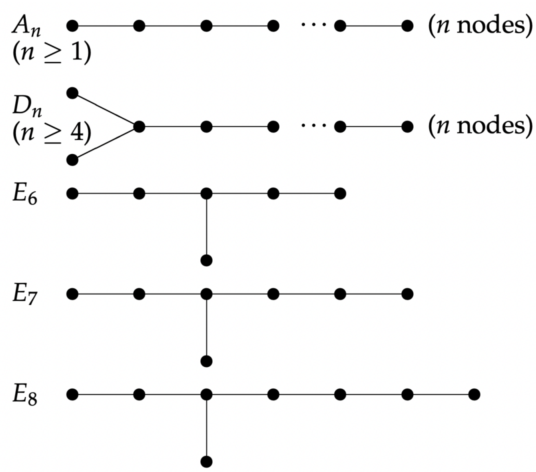
We say a collection of mathematical objects admits an ADE classification if there exists a 'natural' bijection between isomorphism classes of such objects and the following collection of Dynkin diagrams:



These diagrams show up surprisingly often in various areas of mathematics, ranging from combinatorics, algebra, geometry and representation theory. We hope to convince the reader of this ubiquity by giving a quick tour of six ADE classifications. This list is far from exhaustive: we completely omit the role ADE classifications play in other fields such as catastrophe theory and conformal field theory, because we know nothing about them. We apologise for our incompetence! Corrections and comments are very welcome at `jcsl5@cam.ac.uk`.

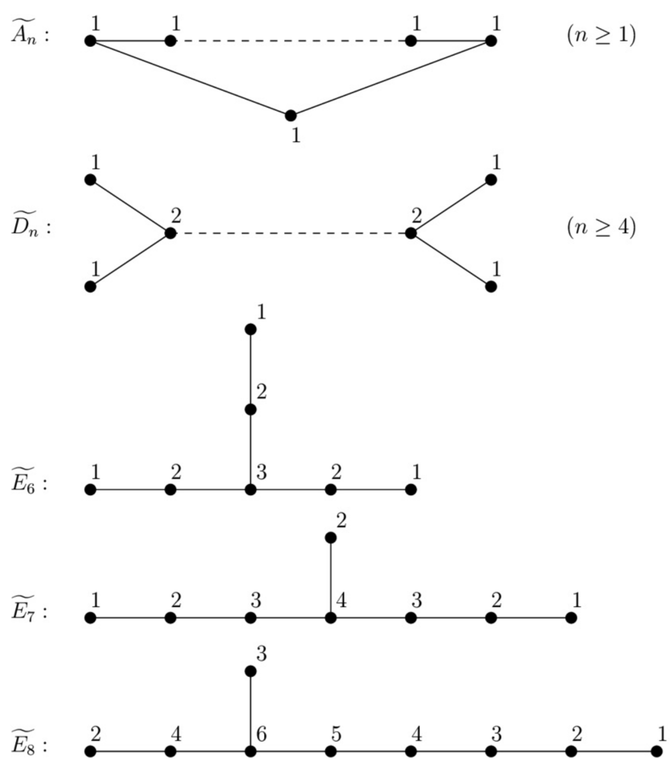
1. Graphs

Let G be a graph with vertex set V with no loops or multi-edges. Recall that the adjacency matrix of G is the matrix $A_G = (a_{ij})_{i,j \in V}$, where $a_{ij} = 1$ if i, j are distinct and $\{i, j\}$ is an edge, and 0 otherwise. Since A_G is a symmetric matrix, all of its eigenvalues are real. Given a real number X, can we classify graphs G such that the largest eigenvalue of A_G is $< X$? The ADE diagrams answer this question for $X = 2$.

Proposition 1.1. *The following are equivalent for a connected graph G:*

(1) The largest eigenvalue of A_G is < 2;
(2) $2I - A_G$ is positive definite;
(3) G is of type ADE.

Proof. We give the proof because it is short and sweet. The equivalence between the first two properties is clear, and one can check that the ADE diagrams have largest eigenvalue < 2. It therefore suffices to prove that if G has largest eigenvalue < 2, then G is of type ADE. If H is a subgraph of G induced by a subset of vertices, then H must also have largest eigenvalue < 2. Consider the following labeled graphs, called 'extended Dynkin diagrams':



Since twice the sum of a number on a vertex equals the sum of the numbers on the neighbouring vertices, these numbers are the coordinates of an eigenvector of the corresponding adjacency matrices with eigenvalue 2. It follows that G cannot contain any of these extended diagrams as a subgraph. Since G doesn't contain $\widetilde{A}_n$, G is a tree. Since it doesn't contain $\widetilde{D}_4$, every vertex has degree ≤ 3, and there is at most one having degree exactly 3 since it doesn't contain any $\widetilde{D}_n$. Therefore either G is A_n or has a unique degree 3 vertex with three branches. The diagrams $\widetilde{E}_n$ restrict the lengths of these branches, leaving only D_n and $E_{6,7,8}$. $\square$

Proposition 1.1 can be considered as the most basic ADE classification, and in fact many others can be seen to rely on this one.

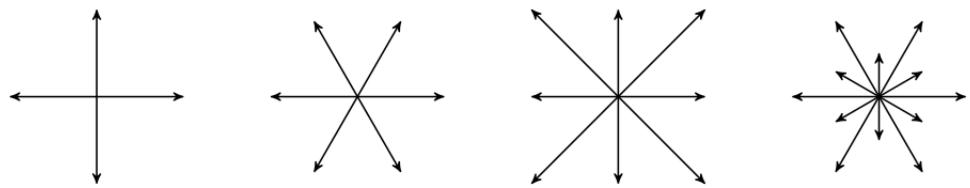
2. Root systems

Let E be a finite dimensional $\mathbb{R}$-vector space with an inner product $(\cdot,\cdot)\colon E \times E \to \mathbb{R}$. For every nonzero $v \in E$, write $w_v(x) = x - \frac{2(x,v)}{(v,v)}v$ for the reflection through the hyperplane orthogonal to v.

Definition 2.1. *A root system in E is a finite subset $\Phi \subset E \setminus \{0\}$ with the property that:*

(1) the $\mathbb{R}$-span of Φ is E;
(2) if $\alpha \in \Phi$ and $c \in \mathbb{R}$, then $c\alpha \in \Phi$ if and only if $c = \pm 1$;
(3) $w_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$;
(4) for all $\alpha, \beta \in \Phi$, the quantity $2(\alpha,\beta)/(\alpha,\alpha)$ is an integer.

We will consider $\mathbb{R}^n$ with its standard inner product. The simplest root system is the A_1 root system: $\Phi = \{\pm 1\} \subset \mathbb{R}$. The root systems in $\mathbb{R}^2$ are (up to a suitable notion of isomorphism) given by the following four pictures:



$A_1 \times A_1$ $\quad$ A_2 $\quad$ $B_2 \cong C_2$ $\quad$ G_2

Root systems can be classified by Dynkin diagrams, as follows. Given a root system (E, Φ), let $v \in E$ be any nonzero element such that the hyperplane $H = \{x \in E \mid (x, v) = 0\}$ is disjoint from Φ. Using this choice, we may define the subsets of positive roots $\Phi^+ = \{\alpha \in \Phi \mid (v, \alpha) > 0\}$ and analogously negative roots Φ^-, giving rise to a decomposition $\Phi = \Phi^+ \sqcup \Phi^-$. One can prove that there exists a unique subset $\Delta \subset \Phi^+$ (called simple roots) such that every element of Φ^+ is expressible as an integer linear combination of simple roots with nonnegative coordinates. Define the graph $D(\Phi)$ as follows. The vertices of $D(\Phi)$ are indexed by the elements of Δ. Between vertices α and β we draw $4(\alpha, \beta)^2/(\alpha, \alpha)(\beta, \beta)$ (a positive integer by the fourth axiom of a root system) edges; if α and β do not have equal lengths we draw an arrow on the edges pointing towards the smaller vector among α and β. This procedure produces a graph which does not depend on the choice of hyperplane H, called the Dynkin diagram of Φ.

Say a root system Φ is simply laced if all the roots have equal length, in other words the value of (α, α) is the same for all $\alpha \in \Phi$. In the above picture, the simply laced root systems are exactly $A_1 \times A_1$ and A_2. For such a root system, the graph $D(\Phi)$ is undirected and has no multiple edges.

To state the classification, note that $(E \oplus E', \Phi \sqcup \Phi')$ is a root system when (E, Φ) and (E', Φ') are, and we say Φ is irreducible if it is not the sum of two nonzero root systems.

Theorem 2.2. *The map $\Phi \mapsto D(\Phi)$ induces a bijection between isomorphism classes of irreducible, simply laced root systems and ADE diagrams.*

Proof sketch. We first show that $D(\Phi)$ is an ADE diagram. The irreducibility condition guarantees that $D(\Phi)$ is connected. Since Φ is simply laced, we may scale the inner product such that $(\alpha, \alpha) = 2$ for all $\alpha \in \Phi$. The simply laced assumption implies that $(\alpha, \beta) \in \{0, -1\}$ for all $\alpha, \beta \in \Delta$. If $A_{D(\Phi)}$ denotes the adjacency matrix of $D(\Phi)$, it follows that $2I - A_{D(\Phi)}$ is the Gram matrix of the positive definite form $(\cdot, \cdot)$. By Proposition 1.1, it follows that $D(\Phi)$ is of type ADE!

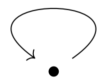
Conversely, let D be an ADE diagram. Let E be the $\mathbb{R}$-vector space with basis indexed by $\alpha \in \Delta$ and define the bilinear form $(\cdot, \cdot)$ with Gram matrix $2I - A_D$. The fact that D is an ADE diagram implies that $(\cdot, \cdot)$ is positive definite. Let $\Lambda \subset E$ be the $\mathbb{Z}$-lattice generated by Δ, and let $\Phi = \{\alpha \in \Lambda \mid (\alpha, \alpha) = 2\}$. Then it is straightforward to check that Φ is a simply laced root system with Dynkin diagram D. $\square$

3. Quivers of finite representation type

Definition 3.1. *A quiver $\vec{Q}$ is a directed finite graph, possibly with multi-edges and loops. A representation of $\vec{Q}$ is a collection of finite-dimensional $\mathbb{C}$-vector spaces $\{V_i\}$, one for each vertex i of $\vec{Q}$, together with a collection of linear maps $V_i \to V_j$, one for each arrow $i \to j$.*

There is a notion of isomorphism of representations, direct sums, irreducibility and indecomposability which we won't spell out explicitly.

Example 3.2. *Representations of the quiver*



are pairs $(V, \alpha\colon V \to V)$*, where* V *is a vector space and* α *a linear map. Up to isomorphism, they are classified by the Jordan normal form. There are therefore infinitely many isomorphism classes of indecomposable representations.*

Example 3.3. *Representations of the quiver*

$$\bullet \longrightarrow \bullet$$

correspond to triples $(V, W, \alpha\colon V \to W)$*. Every such representation is isomorphic to a direct sum of one of the following three:* $(\mathbb{C}, 0, \mathbb{C} \xrightarrow{0} 0)$*,* $(0, \mathbb{C}, 0 \xrightarrow{0} \mathbb{C})$ *and* $(\mathbb{C}, \mathbb{C}, \mathbb{C} \xrightarrow{\mathrm{Id}} \mathbb{C})$*.*

These two examples behave very differently. We say $\vec{Q}$ is of finite representation type if the set

$$\mathrm{Indec}(\vec{Q}) = \begin{Bmatrix} \text{Indecomposable representations} \\ \text{of } \vec{Q} \text{ up to isomorphism} \end{Bmatrix}$$

is finite.

Theorem 3.4 (Gabriel)**.** *Let* $\vec{Q}$ *be a quiver such that the underlying undirected group* Q *is connected.*

(1) $\vec{Q}$ *is of finite representation type if and only if* Q *is of type ADE.*
(2) Suppose Q *is of type ADE, and let* Φ *be the corresponding root system under Theorem 2.2. Then there is a bijection between* $\mathrm{Indec}(\vec{Q})$ *and a subset of positive roots of* Φ*.*

The proof makes crucial use of Proposition 1.1 again. Note that the property of being of finite representation type only depends on the underlying graph of $\vec{Q}$, a fact which is already non-obvious. The bijection in the second part of Theorem 3.4 can be made very explicit: we leave it to the reader to experiment with A_2 and D_4 quivers to see this beautiful construction for themselves!

4. Lie algebras

Definition 4.1. *A Lie algebra (always over* $\mathbb{C}$ *in this note) is a finite dimensional vector space* $\mathfrak{g}$ *together with an alternating bilinear pairing* $[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *(the Lie bracket) that satisfies one of the following two equivalent conditions:*

(1) Every $x, y, z \in \mathfrak{g}$ *satisfy the Jacobi identity*

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

(2) Write $\mathrm{ad}_x\colon \mathfrak{g} \to \mathfrak{g}$ *for the linear map* $y \mapsto [x, y]$*. Then the Lie bracket transforms into the commutator pairing of linear maps: for all* $x, y \in \mathfrak{g}$ *we have*

$$\mathrm{ad}_{[x,y]} = \mathrm{ad}_x \circ \mathrm{ad}_y - \mathrm{ad}_y \circ \mathrm{ad}_x .$$

Definition 4.2. *An ideal of a Lie algebra* $\mathfrak{g}$ *is a subspace* I *with the property that* $[x, I] \subset I$ *for all* $x \in \mathfrak{g}$*. A Lie algebra is said to be simple if the Lie bracket is not identically zero and it has no nonzero proper ideal, and semisimple if it is isomorphic to a direct sum of simple ones.*

Example 4.3. *Let* $\mathfrak{g} = \mathfrak{gl}_n$ *be the set of* $n \times n$ *matrices, equipped with the commutator pairing. Then* $\mathfrak{g}$ *is a Lie algebra. The subset of trace zero matrices is a semisimple subalgebra denoted* $\mathfrak{sl}_n$*.*

We briefly explain how to classify semisimple Lie algebras in terms of root systems. Let $\mathfrak{g}$ be a semisimple Lie algebra. There exists a subalgebra $\mathfrak{t} \subset \mathfrak{g}$ with the property that ad_x is semisimple for each $x \in \mathfrak{t}$ and the dimension of $\mathfrak{t}$ is maximal with respect to this property; such a subalgebra is called a Cartan subalgebra or CSA. For example, if $\mathfrak{g} = \mathfrak{sl}_n$ then a choice for $\mathfrak{t}$ is the subset of diagonal matrices (of trace zero). Fix a choice of CSA $\mathfrak{t} \subset \mathfrak{g}$. It turns out that $[x, y] = 0$ for all $x, y \in \mathfrak{t}$, so the ad_x are mutually commuting semisimple linear maps for $x \in \mathfrak{t}$. It follows that they can be simultaneously diagonalised, i.e. there is a decomposition

$$\mathfrak{g} = \bigoplus_{f\colon \mathfrak{g}\to\mathbb{C}} \{x \in \mathfrak{g} \mid [t, x] = f(t)x \,\forall t \in \mathfrak{t}\}, \tag{4.0.1}$$

where the sum runs over all linear functionals of $\mathfrak{g}$. It turns out that the part corresponding to the zero functional is exactly $\mathfrak{t}$, i.e. the only elements of $\mathfrak{g}$ commuting with $\mathfrak{t}$ are the elements of $\mathfrak{t}$ itself. Writing $\Phi \subset \mathfrak{t}^* = \{f\colon \mathfrak{g} \to \mathbb{C}\}$ for the set of nonzero functionals for which the corresponding eigenspace $\mathfrak{g}_f$ is nonzero, we may write

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha .$$

There exists a canonical perfect pairing on $\mathfrak{g}$, called the Killing form, which induces a bilinear form on $\mathfrak{t}^*$. If E denotes the $\mathbb{R}$-span of Φ inside $\mathfrak{t}^*$, then equipped with the restriction of this pairing it turns out that (E, Φ) is a root system!

Theorem 4.4 (Cartan–Killing)**.** *The association $\mathfrak{g} \mapsto (E, \Phi)$ is, up to isomorphism, independent of any choices and induces a bijection between simple complex Lie algebras and irreducible root sytems.*

We say a simple Lie algebra $\mathfrak{g}$ is simply laced if the associated root system under Theorem 4.4 is simply laced, i.e. corresponds to an ADE diagram.

Example 4.5. *Let $\mathfrak{g} = \mathfrak{sl}_2$ and let $\mathfrak{t} \subset \mathfrak{g}$ be the subspace of diagonal matrices. Then $\Phi = \{\pm\alpha\}$, where $\alpha\colon \mathfrak{t} \to \mathbb{C}$ sends $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ to 2. The associated root system is A_1. More generally, $\mathfrak{sl}_{n+1}$ corresponds to the A_n root sytem.*

The Lie algebras corresponding to D_n can be constructed using symmetric bilinear forms and are denoted by $\mathfrak{so}_{2n}$. The exceptional Lie algebras of type E are denoted by $\mathfrak{e}_6, \mathfrak{e}_7$ and $\mathfrak{e}_8$. These Lie algebras are harder to describe using matrices. For example, $\dim \mathfrak{e}_8 = 248$, but if $\mathfrak{e}_8 \to \mathfrak{gl}_n$ is a nonzero Lie algebra homomorphism, then $n \geq 248$. So every embedding of $\mathfrak{e}_8$ inside spaces of matrices is bound to be very inefficient!

5. Finite subgroups of $\mathrm{SL}_2(\mathbb{C})$

Theorem 5.1. *Up to conjugation, every finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ is equal to one of the following*

(A_n) *A cyclic group of order $n + 1$:*

$$\left\langle \begin{pmatrix} \zeta_{n+1} & 0 \\ 0 & \zeta_{n+1}^{-1} \end{pmatrix} \right\rangle \subset \mathrm{SL}_2(\mathbb{C}).$$

(D_n) *A dihedral group of order $4(n - 2)$:*

$$\left\langle \begin{pmatrix} \zeta_{2(n-2)} & 0 \\ 0 & \zeta_{2(n-2)}^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle \subset \mathrm{SL}_2(\mathbb{C}).$$

(E_6) *The binary tetrahedral group:*

$$\left\langle \begin{pmatrix} \zeta_4 & 0 \\ 0 & \zeta_4^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta_8^7 & \zeta_8^7 \\ \zeta_8^5 & \zeta_8 \end{pmatrix} \right\rangle \subset \mathrm{SL}_2(\mathbb{C}).$$

(E_7) *The binary octahedral group: generated by the binary tetrahedral group together with the element*

$$\begin{pmatrix} \zeta_8 & 0 \\ 0 & \zeta_8^{-1} \end{pmatrix}.$$

(E_8) *Binary icosahedral group:*

$$\left\langle -\begin{pmatrix} \zeta_5^3 & 0 \\ 0 & \zeta_5^2 \end{pmatrix}, \frac{1}{\zeta_5^2 - \zeta_5^3}\begin{pmatrix} \zeta_5 + \zeta_5^4 & 1 \\ 1 & -\zeta - \zeta^4 \end{pmatrix} \right\rangle \subset \mathrm{SL}_2(\mathbb{C}).$$

(Here $\zeta_n = e^{2\pi i/n}$.)

Proof sketch. The group of unitary transformations $\mathrm{SU}_2 \subset \mathrm{SL}_2(\mathbb{C})$ is a maximal compact subgroup, so every finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ can be conjugated into SU_2. Using the double cover $\mathrm{SU}_2 \to \mathrm{SO}_3$, the theorem is reduced to classifying finite subgroups of SO_3. But the finite subgroups of SO_3 are classical and given by symmetries of one of the following objects in $\mathbb{R}^3$: a cone over a regular polygon (type A), a double cone over a regular polygon (type D), a tetrahedron (E_6), cube (E_7) or dodecahedron (E_8). □

The proof sketch should explain the names given to these groups, and their connection to platonic solids. Note that the symmetry group of a cube is the same as that of an octahedron, and the symmetry group of a dodecahedron is the same as that of an icosahedron: this is a consequence of duality between platonic solids.

It appears at first sight that the ADE labels of the finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ are somewhat arbitrary, but John McKay showed us that there is more going on! He discovered an amazing direct connection between these subgroups and the coresponding graphs. To describe it, let $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. Let $V_1, \ldots, V_r$ be representatives of the set of isomorphism classes of nontrivial irreducible representations of Γ, and let $W = \mathbb{C}^2$ be the defining representation. For $1 \le i, j \le r$, define the integer $a_{ij} \in \mathbb{Z}_{\ge 0}$ by the formula

$$V_i \otimes W \simeq (\text{trivial}) \bigoplus_{j=1}^{r} V_j^{\oplus a_{ij}}.$$

Theorem 5.2 (McKay correspondence)**.** *The matrix A is the adjacency matrix of the Dynkin diagram of the corresponding type!*

Remark 5.3. *The McKay correspondence says more: If we include the trivial representation in the above picture, we get the affine Dynkin diagram of the corresponding type. We have encountered these already in the proof of Proposition 1.1. Moreover the dimension of a representation corresponding to a simple root corresponds to the coefficient of the highest root vector at that simple root.*

Remark 5.4. *There is also an inverse to this construction (that is, a construction of the group from the ADE diagram), as follows: let D be an ADE diagram with adjacency matrix A_D and set $C = (c_{ij})_{1 \le i,j \le r} = 2I - A_D$. Let Γ be the group with generators $e_1, \ldots, e_r$ and relations*

$$e_1^{c_{i1}} e_2^{c_{i2}} \ldots e_r^{c_{ir}} = 1$$

for all $1 \le i \le r$. Then Γ is isomorphic to a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ of type D.

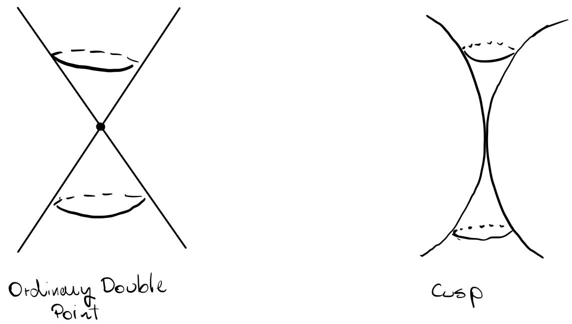
6. Simple surface singularities

For a polynomial $f \in \mathbb{C}[x, y, z]$, let $S_f \subset \mathbb{C}^3$ be the subscheme cut out by $(f = 0)$. Let $\mathcal{P}$ be the set of nonzero polynomials $f \in \mathbb{C}[x, y, z]$ such that S_f contains the origin and is smooth in a punctured open neighbourhood of the origin. If $f, g \in \mathcal{P}$, we say $f \sim g$ if there is an isomorphism of complete $\mathbb{C}$-algebras $\mathbb{C}[[x, y, z]]/(f) \simeq \mathbb{C}[[x, y, z]]/(g)$. (Note the double brackets!) In that case we say f and g define the same singularity and we call an element of $\mathcal{S} = \mathcal{P}/\sim$ a singularity.[1]

[1] More precisely, these are exactly the isolated surface singularities of embedding dimension 3. We will content ourselves with this subclass in this note.

Example 6.1 (Smooth point)**.** *All $f \in \mathbb{C}[x,y,z]$ which are smooth at the origin (in other words, contain a linear term) are $\sim$-equivalent, and define the smooth point.*

Example 6.2 (Ordinary double point)**.** *Any polynomial having the same singularity as $x^2+y^2+z^2=0$ is called an* ordinary double point*: it is the simplest nontrivial singularity.*



Example 6.3 (Cusp)**.** *The singularity represented by $x^2+y^2+z^3$ is called a* cusp*.*

It turns out that classifying singularities (that is, understanding the set $\mathcal{S}$) is rather hard. However, there is a class of singularities that admits a very neat ADE classification, called simple singularities. Intuitively, these are the singularities which can only change in finitely many ways when you perturb the coefficients a little bit. To make this precise, define for a polynomial $f = \sum_I f_I x^I \in \mathbb{C}[x,y,z]$ its norm as $||f|| = \sum |f_I|^2$.

Definition 6.4. *We say $f \in \mathcal{P}$ is a* simple singularity *if there exists an $\varepsilon > 0$ such that*

$$\begin{Bmatrix} g \in \mathcal{P} \text{ with} \\ ||f-g|| < \varepsilon \end{Bmatrix}$$

represents only finitely many $\sim$-equivalence classes, in other words has finite image in $\mathcal{S}$.

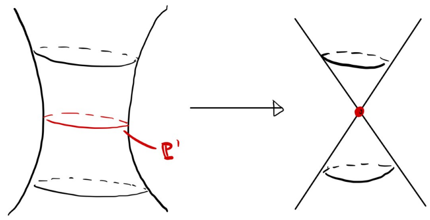
It is easy to see that a smooth point is a simple singularity.

Example 6.5. *Let's show that an ordinary double point $f = x^2+y^2+z^2$ is a simple singularity. For any $g \in \mathcal{P}$ write $g = g_1 + g_2 + \dots$ into its decomposition in homogeneous parts. Then g is smooth at the origin if and only if $g_1 \neq 0$, and is an ordinary double point if and only if $g_1 = 0$ and g_2 is a nondegenerate quadratic form. It follows that if g_2 is nondegenerate then g is smooth or an ordinary double point. Since g_2 being nondegenerate is an open condition and f_2 is nondegenerate, the same will be true for any $g \in \mathcal{P}$ with $||f-g|| < \varepsilon$ for $\varepsilon > 0$ sufficiently small.*

Theorem 6.6. *The following is a complete list of simple singularities:*

$$\begin{cases} z^2+y^2+x^{n+1}=0 & (A_n) \\ z^2+xy^2+x^{n-1}=0 & (D_n) \\ z^2+y^3+x^4=0 & (E_6) \\ z^2+y^3x+x^3=0 & (E_7) \\ z^2+y^3+x^5=0 & (E_8) \end{cases}$$

Just like with the finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ in Theorem 5.1, the ADE labels corresponding to the different singularities seem somewhat arbitrary at this point, but there is more going on! Suppose $f \in \mathcal{P}$ defines a simple singularity, and let $\tilde{S}_f \to S_f$ be the minimal resolution of this singularity, given by successively blowing up the singular locus. For example, an ordinary double point can be resolved by one blowup:



Moreover the exceptional fibre is isomorphic to a projective line $\mathbb{P}^1_{\mathbb{C}}$. The general story is similar: the exceptional fibre will be a union of finitely many projective lines intersecting transversally.

Proposition 6.7. *The exceptional fibre of $\tilde{S}_f \to S_f$ is a union of finitely many projective lines intersecting transversally. The intersection diagram is exactly the ADE diagram of type corresponding to the one in Theorem 6.6!*

7. Connections between ADE classifications

In the above examples we have seen that there is often a direct and surprising construction of the Dynkin diagram starting from an object obeying an ADE classification. (See for example Theorem 5.2 and Proposition 6.7.) What is perhaps even more surprising is that there exist highly nontrivial bijections between different types of objects satisfying ADE classifications, without passing through the Dynkin diagrams. We briefly mention two of these.

7.1. **From subgroups of $\mathrm{SL}_2(\mathbb{C})$ to simple surface singularities.** Given a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$, we can construct a simple surface singularity of the corresponding ADE type, as follows. The algebra of invariant polynomials $\mathbb{C}[u,v]^\Gamma$ is generated by three elements x, y, z satisfying one relation $f(x,y,z) \in \mathbb{C}[x,y,z]$. (This is an algebraic way of saying that the surface S_f is isomorphic to the quotient $\mathbb{C}^2/\Gamma$.) The polynomial f then defines a simple singularity of the corresponding type!

Example 7.1 (Type A_1)**.** *If $\Gamma = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$, then $\mathbb{C}[u,v]^\Gamma = \mathbb{C}[u^2, uv, v^2]$ and $x = u^2$, $y = UV$, $z = V^2$ satisfy the relation $xz = y^2$, which is an ordinary double point.*

7.2. **From Lie algebras to simple singularities.** It was conjectured by Grothendieck, announced by Brieskorn in his 1970 ICM address and fully proven by Slodowy that one can find a singularity inside a Lie algebra in a very explicit way: this is nowadays called the Grothendieck–Brieskorn correspondence. It turns out that the 'generic singularity of the nilpotent cone' of a simply laced Lie algebra is a simple singularity of the corresponding ADE type. Instead of defining the previous sentence carefully, we will content ourselves here with a simple example.

Example 7.2 (Type A_1)**.** *Let $\mathfrak{g} = \mathfrak{sl}_2$, a simple Lie algebra of type A_1. Then the set of nilpotent matrices in $\mathfrak{sl}_2$ is given by the set of all $\begin{pmatrix} x & y \\ z & -x \end{pmatrix}$ with determinant zero, i.e. $x^2 + yz = 0$. This is exactly the equation of an ordinary double point, which is a simple singularity of type A_1.*

This is not the end of the story of the Grothendieck–Brieskorn correspondence. In fact an arguably more exciting part is that we can find all 'neighbouring fibres' of the singularity in the Lie algebra too.