Raising the level and symmetric power functoriality, II

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1 Introduction

This paper is devoted to the study of a specific instance of Langlands' functoriality for GL_2. Let us begin with a conjecture. For any notation with which the reader is unfamiliar, we refer to §3 below. Let K be a finite Galois extension of the field $\mathbb{Q}$ of rational numbers.

Conjecture 1.1 ($\mathrm{SP}_{n+1}(K)$)**.** *Let F be a totally real field, linearly disjoint from K over $\mathbb{Q}$. Let (π, χ) be a RAESDC (regular algebraic, essentially self-dual, cuspidal) representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Suppose that π does*

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not have CM, i.e. is not the automorphic induction of an algebraic Hecke character from a quadratic CM extension. Let $n \geq 1$ be an integer.

Then the n^{th} symmetric power lifting of π exists, in the following sense: there exists a RAESDC automorphic representation (Π, ψ) of $\mathrm{GL}_{n+1}(\mathbb{A}_F)$ such that for any isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$, there is an isomorphism of associated Galois representations $\mathrm{Sym}^n r_\iota(\pi) \cong r_\iota(\Pi)$. [1]

Our main result is the following.

Theorem 1.2. *Let $l \geq 5$ be a prime. Then the following implication holds:*

$$\mathrm{SP}_{l-1}(K(\zeta_l)) \Rightarrow \mathrm{SP}_{l+1}(K(\zeta_l)).$$

Corollary 1.3. *Conjectures $\mathrm{SP}_6(\mathbb{Q}(\zeta_5))$ and $\mathrm{SP}_8(\mathbb{Q}(\zeta_{35}))$ are true.*

Proof. Indeed, $\mathrm{SP}_4(\mathbb{Q})$ is known to be true, cf. [KS02]. Now use that 5 and 7 are primes. $\square$

For more discussion of Conjecture 1.1, we refer to the paper [CT], of which this one is a sequel. In that paper we outlined a strategy for proving some cases of $\mathrm{SP}_{n+1}(K)$ by reducing it to two other conjectures about automorphic forms, relating to the existence of automorphic tensor products and the construction of so-called 'level-raising' congruences between automorphic representations on unitary groups.

In this paper we carry this strategy out in the first non-trivial case. Namely, we prove a level-raising result for automorphic representations on unitary groups with certain local data, and use this to establish the main theorem above. Our techniques for raising the level seem quite different to previous results in this direction.

2 Admissible representations of p-adic groups

Let F be a finite extension of $\mathbb{Q}_p$, with residue field k_F, ring of integers $\mathcal{O}_F$, uniformizer ϖ_F, and set $q = \#k_F$. In this section we will consider various algebraic F-groups G. We will abuse notation slightly by writing G both for the group and for its group $G(F)$ of F-points. We will use the paper [Mor99] as a convenient reference for the facts about Bruhat-Tits theory that we require here; see [Tit79] or [BT72] for more information.

Let G be a connected reductive group over F. Let $P = MN$ be a parabolic subgroup of G with Levi subgroup M and unipotent radical N, and let π be an admissible $\mathbb{C}[G]$-module. The (unnormalized) Jacquet module π_N of π with respect to N is by definition the space of N-coinvariants, equipped with its natural M-action. We will write $\pi_N^{\mathrm{norm}} = \pi_N \otimes \delta_P^{-1/2}$ for the normalized Jacquet module.

Let $S \subset G$ be a maximal F-split torus. Associated to the pair (G, S) is the apartment $A = A(G, S)$, affine space under the vector space $V = (X^*(S) \otimes_{\mathbb{Z}} \mathbb{R})^*$. We write $\Phi \subset V^*$ for the set of roots with respect to the pair (G, S), and Σ for the set of affine roots, which are affine functions on A. We fix a choice of Iwahori subgroup $\mathfrak{B} \subset G$. This corresponds to a choice of chamber $C \subset A$, and a set of simple affine roots $\Pi \subset \Sigma$. To a choice of parahoric subgroup $\mathfrak{P}$ containing $\mathfrak{B}$, we can associate a subset $J \subset \Pi$, namely the set of simple affine roots which vanish on the facet $\mathcal{F}$ fixed by $\mathfrak{P}$. We associate to $\mathfrak{P}$ the root subsystem Φ_J of Φ consisting of the vector parts of the affine roots in J.

We associate to $\mathfrak{P}$ a standard Levi subgroup of G, as follows. First, let $K \subset G$ denote the reductive subgroup generated by S and the root subgroups $U_\alpha \subset G$ for α in the $\mathbb{Z}$-closure of Φ_J inside Φ. Let Y denote the maximal F-split torus in the center of K. Then the associated Levi subgroup is $M = Z_G(Y)$.

Proposition 2.1. *With notation as in the preceding paragraph, let π be an admissible representation of G, and let P be any parabolic subgroup of G containing L as Levi subgroup. Let N denote the unipotent radical of P. Then there is an isomorphism $\pi^{\mathfrak{P}} \cong \pi_N^{\mathfrak{P} \cap M}$.*

1. We caution the reader that this conjecture differs slightly in its statement to the conjecture $\mathrm{SP}_{n+1}(\mathbf{K})$ of [CT].

Proof. By [Mor99, Theorem 2.1] and [Mor99, Lemma 2.4], $\mathfrak{P}$ has an Iwahori decomposition with respect to P, in the sense of [Cas, §1.4]. Similarly, $\mathfrak{B}$ has an Iwahori decomposition with respect to the minimal parabolic $P_0 = M_0 N_0 \subset G$ containing $Z_G(S)$ and contained inside P. There is a commutative diagram

$$\begin{tikzcd} \pi^{\mathfrak{B}} \arrow[r] & \pi_{N_0}^{\mathfrak{B}\cap M_0} \\ \pi^{\mathfrak{P}} \arrow[r] \arrow[u] & \pi_N^{\mathfrak{P}\cap M}. \arrow[u] \end{tikzcd}$$

The bottom arrow is surjective, by [Cas, Theorem 3.3.3]. On the other hand, the top arrow is injective, by [Mor99, Lemma 3.6]. The commutativity of the diagram now implies that the bottom arrow is also injective, and this completes the proof. $\square$

2.1 A ramified unitary group

Now suppose that E/F is a ramified quadratic extension, and that the residue characteristic of F is not 2. In this case we define a unitary group as follows. Let $n = 2m \geq 6$ be an even integer, and define a matrix by

$$J = \begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{pmatrix},$$

and an $\mathcal{O}_F$-group by the formula $\mathcal{G}(R) = \{g \in \mathrm{GL}_n(R \otimes_{\mathcal{O}_F} \mathcal{O}_E) \,|\, {}^t g J g^c = J\}$. Then $\mathcal{G}$ is smooth over $\mathcal{O}_F$, and its generic fiber $G = U_n$ is a quasi-split unitary group in n variables. The group $K = \mathcal{G}(\mathcal{O}_F)$ is a special maximal parahoric subgroup, and there is a surjective map $K \to \mathrm{Sp}(J, k_F)$ to the symplectic group over the residue field of the symplectic form represented by J. The group $\mathcal{G}$ is the integral model of G associated to the maximal parahoric subgroup K as in [Tit79, §3.4]. An Iwahori subgroup $\mathfrak{B}$ is the inverse image in K under this map of the subgroup of upper-triangular matrices. A maximal F-split torus S of G is the subgroup of matrices of the form

$$\mathrm{diag}(t_1, \ldots, t_m, t_m^{-1}, \ldots, t_1^{-1}),\ t_i \in F^\times.$$

It naturally extends to an $\mathcal{O}_F$-split torus $\mathcal{S} \subset \mathcal{G}$.

Let $\mathcal{P} \subset \mathcal{G}$ denote the subgroup consisting of matrices whose (i, j) entries vanish if $i > j$ and $(i, j) \neq (m+1, m)$. Define $\mathfrak{P}$ to be the parahoric subgroup of G which is the pre-image of $\mathcal{P}(k_F) \subset \mathrm{Sp}(J, k_F)$ inside K. The parabolic subgroup associated to $\mathfrak{P}$ by the recipe of the previous section is just P, the generic fiber of $\mathcal{P}$. We write M for its standard Levi subgroup, isomorphic to $(E^\times)^{m-1} \times U_2$. The base extension of P to E is, under the canonical isomorphism $G(E) \cong \mathrm{GL}_n(E)$, the standard parabolic corresponding to the partition

$$n = \underbrace{1 + \cdots + 1}_{(n-2)/2} + 2 + \underbrace{1 + \cdots + 1}_{(n-2)/2}.$$

Proposition 2.1 now implies the following.

Corollary 2.2. *Let π be an admissible representation of G, and let $P = MN$ denote the Levi decomposition with respect to the maximal F-split torus S. Then projection induces an isomorphism $\pi^{\mathfrak{P}} \cong \pi_N^{\mathfrak{P}\cap M}$.*

Lemma 2.3. *The pro-order of $\mathfrak{B}$ is $q^\infty (q-1)^m$.*

Proof. Arguing as in [Tit79, §3.7], we see that the prime-to-q part of the pro-order of $\mathfrak{B}$ is the order of $Z_{\mathcal{G}}(\mathcal{S})(k_F)$, namely $(q-1)^m$. $\square$

We now introduce the Iwahori-Hecke algebra $H_{\mathfrak{B}}$ of G. By definition, this is the convolution algebra of $\mathfrak{B}$-biinvariant functions $f : G \to \mathbb{Z}$. If R is a ring, we write $H_{\mathfrak{B},R} = H_{\mathfrak{B}} \otimes_{\mathbb{Z}} R$. If M is a smooth $R[G]$-module, then $H_{\mathfrak{B},R}$ acts on $M^{\mathfrak{B}}$ on the left. The algebra $H_{\mathfrak{B}}$ is non-commutative and has a canonical anti-involution $\jmath$ given on double cosets by $\jmath : [\mathfrak{B}g\mathfrak{B}] \mapsto [\mathfrak{B}g^{-1}\mathfrak{B}]$. It is useful to recall the following facts.

Proposition 2.4. *Let K be a field of characteristic zero.*

1. *The assignment $\pi \mapsto \pi^{\mathfrak{B}}$ induces an equivalence of categories between the category of admissible $K[G]$-modules which are generated by their $\mathfrak{B}$-invariant vectors and the category of left $H_{\mathfrak{B},K}$-modules which are finite-dimensional as K-vector spaces.*
2. *Let π be an admissible $K[G]$-module which is generated by its $\mathfrak{B}$-invariant vectors. Then $\pi^\vee$ corresponds, under the above equivalence, to the module $\mathrm{Hom}_K(\pi^{\mathfrak{B}}, K)$, which we make into a left $H_{\mathfrak{B},K}$-module using the anti-involution $\jmath$.*

Proof. These facts are proved in [Bor76] for semisimple p-adic groups, but the arguments easily extend to our case. □

Let us now say a little more about the structure of the algebra $H_{\mathfrak{B}}$. Fix an element $\varpi \in E$ such that $\varpi^2 \in F$ is a uniformizer of F. With respect to the torus S a choice of set of positive roots is

$$\{t_i/t_j \mid 1 \leq i < j \leq m\} \cup \{t_i t_j \mid 1 \leq i \leq j \leq m\}.$$

The corresponding simple roots are the elements

$$\alpha_i = t_i/t_{i+1}, i = 1, \ldots, m-1 \text{ and } \alpha_m = t_m^2.$$

This root system is of type C_m. We write W_0 for its Weyl group. If $\alpha \in \Phi$ is a root, we write $s_\alpha \in W_0$ for the corresponding reflection. We can identify $W_0 \cong \{\pm 1\}^m \rtimes \mathfrak{S}_m$. Here $\mathfrak{S}_m$, the symmetric group on the set $\{1, \ldots, m\}$, acts on S by permutation of $t_1, \ldots, t_m$, and a vector $\mu = (\mu_i)_{i=1}^m$ in $\{\pm 1\}^m$ sends t_i to $t_i^{\mu_i}$. We write $w_0 \in W_0$ for the longest element. It is $(-1, \ldots, -1)$, and is central.

Let $Z = Z_G(S)$, the maximal torus of G consisting of elements

$$\mathrm{diag}(t_1, \ldots, t_m, \overline{t_m}^{-1}, \ldots, \overline{t_1}^{-1}),\ t_i \in E^\times.$$

Let $Z_c \subset Z$ denote the maximal compact subgroup, and set $\Lambda = Z/Z_c \cong \mathbb{Z}^m$. A basis of Λ is given by the elements

$$\epsilon_i = \mathrm{diag}(1, \ldots, \varpi, \ldots, -1/\varpi, \ldots, 1),\ 1 \leq i \leq m,$$

where ϖ occupies the i^{th} position. Let $N = N_G(S)$. The triple $(G, \mathfrak{B}, N)$ is a generalized Tits system, cf. [Cas80], [Iwa66], and the algebra $H_{\mathfrak{B}}$ admits the following presentation. The extended affine Weyl group $W = \Lambda \rtimes W_0$ admits a natural length function $l : W \to \mathbb{N}$; on the other hand, it has a subgroup, the affine Weyl group $W^{\mathrm{af}} \subset W$ generated by the reflections in the affine roots, cf. [Tit79, §1.7]. We may write $G = \coprod_{w \in W} \mathfrak{B}w\mathfrak{B}$, where the union is disjoint. Writing $G^0 = \coprod_{w \in W^{\mathrm{af}}} \mathfrak{B}w\mathfrak{B}$, $G^0 \subset G$ is a normal subgroup, and $(G^0, \mathfrak{B}, N \cap G^0)$ is a Tits system. We write $H^0 \subset H_{\mathfrak{B}}$ for the subalgebra of elements supported in G^0.

Let $\Psi \subset W$ denote the subgroup of elements of length zero. There is a decomposition $W = W^{\mathrm{af}} \rtimes \Psi$, and $G/G^0 \cong \Psi$. In our case, the group Ψ has order two, the non-trivial element being represented by the matrix

$$\omega = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1/\varpi \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \varpi & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

It is easy to check that ω normalizes $\mathfrak{B}$. For each $i = 1, \ldots, m$, let $s_i = s_{\alpha_i}$. Let s_0 denote the conjugate of s_1 by ω. Then the elements $s_0, \ldots, s_m \in W^{\mathrm{af}}$ are the reflections corresponding to the set of simple affine roots induced by $\mathfrak{B}$. Let B_W denote the group generated by the elements $T_w, w \in W$, subject to the relations $T_w T_{w'} = T_{ww'}$ if $l(w) + l(w') = l(ww')$, and define $B_{W^{\mathrm{af}}}$ similarly. Then there is a canonical isomorphism between $H_{\mathfrak{B}}$ and the quotient of the group algebra $\mathbb{Z}[B_W]$ by the relations $(T_{s_i} - 1)(T_{s_i} + q) = 0$, $i = 0, \dots, m$, which takes $\mathfrak{B}w\mathfrak{B}$ to T_w. Similarly, H^0 is canonically isomorphic to the quotient of the group algebra $\mathbb{Z}[B_{W^{\mathrm{af}}}]$ by the same set of relations, and there is an isomorphism

$$H_{\mathfrak{B}} \cong \mathbb{Z}[\Psi] \widetilde{\otimes} H^0,$$

where the twisted tensor product is as in [Iwa66, §5].

We now introduce the Bernstein presentation of the algebra $H_{\mathfrak{B},\mathbb{C}}$, following [Lus89]. This is defined in terms of a root system $(X, Y, R, \check{R}, \Pi)$. Here we take $X = \Lambda$ and $Y = \mathrm{Hom}(\Lambda, \mathbb{Z})$. The set $R \subset X$ of roots is taken to consist of the elements

$$\{\pm\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq m\} \cup \{\pm 2\epsilon_i \mid 1 \leq i \leq m\},$$

the simple roots in $\Pi \subset R$ being given by the formulae

$$\beta_i = \epsilon_i - \epsilon_{i+1},\ 1 \leq i \leq m-1,\ \beta_m = 2\epsilon_m.$$

Writing $e_1, \dots, e_m$ for the basis of Y dual to $\epsilon_1, \dots, \epsilon_m$, the set $\check{R}$ of coroots is

$$\{\pm e_i \pm e_j \mid 1 \leq i < j \leq m\} \cup \{\pm e_i \mid 1 \leq i \leq m\}.$$

This root system is isomorphic to that of the group $\mathrm{Sp}_m(\mathbb{C})$. It is now easy to check that the extended affine Weyl group defined in [Lus89, §1] is just our W above, and the set S of simple reflections constructed there is equal to $\{s_0, \dots, s_m\}$. (The main point to check is as follows. Let $\beta_0 \in R$ be such that $\check{\beta}_0 \in \check{R}$ is the lowest root. Then $s_0 = s_{\beta_0}\beta_0 \in W_0 \ltimes \Lambda = W$.) Comparing the above discussion with [Lus89, §3] shows that the algebra H constructed by Lusztig in terms of the data $(X, Y, R, \check{R}, \Pi)$ is canonically identified with our algebra $H_{\mathfrak{B},\mathbb{C}}$, once (in the notation there) v is specialized to $q^{1/2}$ and the function $L : S \to \mathbb{N}$ takes the constant value 1.

Lusztig defines a presentation, the Bernstein presentation, of the algebra $H_{\mathfrak{B},\mathbb{C}}$ as a twisted tensor product

$$H_{\mathfrak{B},\mathbb{C}} \cong H_0 \widetilde{\otimes}_{\mathbb{C}} \mathbb{C}[X],$$

where $H_0 \subset H_{\mathfrak{B},\mathbb{C}}$ is the $\mathbb{C}$-subalgebra spanned by the elements $T_w, w \in W_0$, and $\mathbb{C}[X]$ is the coordinate ring of the complex algebraic torus $\mathrm{Hom}(\Lambda, \mathbb{C}^\times)$. If $\beta \in \Pi$ is a simple root and $s = s_\beta \in W_0$ is the corresponding simple reflection, then $T_s \in H_0$ and writing $B_s = T_s - q$, we have the following relation for all $\theta \in \mathbb{C}[X]$:

$$\theta B_s = B_s \theta^s + (\theta^s - \theta)\zeta_\beta,$$

where $\zeta_\beta = (q - e_\beta)/(1 - e_\beta)$. Here we write $e_\beta \in \mathbb{C}[X]$ for the element corresponding to $\beta \in X$, and W_0 acts on $\mathbb{C}[X]$ by its natural right action.

Finally, we relate this presentation to parabolic induction. Let $\tau \in \mathrm{Hom}(\Lambda, \mathbb{C}^\times)$. Then τ defines a module $\mathbb{C}_\tau$ for the group algebra $\mathbb{C}[X]$, which is one-dimensional as $\mathbb{C}$-vector space. Following [Ree97], we define $M(\tau) = H_{\mathfrak{B},\mathbb{C}} \otimes_{\mathbb{C}[X]} \mathbb{C}_\tau$.

Proposition 2.5. *1. Let V be a left $H_{\mathfrak{B},\mathbb{C}}$-module, finite-dimensional as $\mathbb{C}$-vector space. There are functorial isomorphisms*

$$\mathrm{Hom}_{H_{\mathfrak{B},\mathbb{C}}}(M(\tau), V) \cong \mathrm{Hom}_{\mathbb{C}[X]}(\mathbb{C}_\tau, V) \text{ and } \mathrm{Hom}_{H_{\mathfrak{B},\mathbb{C}}}(V, M(w_0\tau)) \cong \mathrm{Hom}_{\mathbb{C}[X]}(V, \mathbb{C}_\tau).$$

2. Let $I(\tau)$ denote the normalized induction of the character $\tau : \Lambda \to \mathbb{C}^\times$, an admissible $\mathbb{C}[G]$-module. Then there is a canonical isomorphism of left $H_{\mathfrak{B},\mathbb{C}}$-modules $I(\tau)^{\mathfrak{B}} \cong M(w_0\tau)$.

Proof. The first part follows immediately from [Ree97, (3.7)] and the proof of [Ree97, (3.8), Lemma]. For the second part, let π be an admissible $\mathbb{C}[G]$-module, generated by its Iwahori-fixed vectors. By Frobenius reciprocity, [Cas80, Proposition 2.5], and the first part of the proposition, there are functorial isomorphisms

$$\mathrm{Hom}_G(\pi, I(\tau)) \cong \mathrm{Hom}_{\mathbb{C}[X]}(\pi_N^{\mathrm{norm}}, \mathbb{C}_\tau) \cong \mathrm{Hom}_{\mathbb{C}[X]}(\pi^{\mathfrak{B}}, \mathbb{C}_\tau) \cong \mathrm{Hom}_{H_{\mathfrak{B},\mathbb{C}}}(\pi^{\mathfrak{B}}, M(w_0\tau)).$$

On the other hand, by Proposition 2.4, there is a functorial isomorphism $\mathrm{Hom}_G(\pi, I(\tau)) \cong \mathrm{Hom}_{H_{\mathfrak{B},\mathbb{C}}}(\pi^{\mathfrak{B}}, I(\tau)^{\mathfrak{B}})$. The result now follows from Yoneda's lemma. □

2.2 Unitary groups and functoriality

Now suppose that n is an even integer, and let E/F be a quadratic extension. Let U_n denote the quasi-split unitary group in n variables associated to this extension. Let $L_F = W_F \times \mathrm{SU}_2(\mathbb{R})$. The L-group of U_n is a semidirect product

$$^L U_n = \widehat{G} \rtimes \mathrm{Gal}(E/F) = \mathrm{GL}_n(\mathbb{C}) \rtimes \mathrm{Gal}(E/F),$$

where the non-trivial element $c \in \mathrm{Gal}(E/F)$ acts on $\mathrm{GL}_n(\mathbb{C})$ by the automorphism

$$\alpha(g) = \Phi_n {}^t g^{-1} \Phi_n^{-1},\ \Phi_n = \begin{pmatrix} & & & & & 1 \\ & & & & -1 & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{pmatrix}.$$

We define an admissible parameter to be a homomorphism $L_F \to {}^L U_n$ such that the projection $L_F \to {}^L U_n \to \mathrm{Gal}(E/F)$ is the canonical homomorphism, and $\Phi(U_n)$ to be the set of admissible parameters taken up to $\mathrm{GL}_n(\mathbb{C})$-conjugation. If $n = a + b$ is a partition into even integers, then there is an L-homomorphism $\xi : {}^L(U_a \times U_b) \to {}^L U_n$, given by formulae

$$\xi(g_1, g_2, w) = \left(\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}, w \right) \ (w \in W_E),$$

$$\xi(w_c) = \left(\begin{pmatrix} \Phi_a & 0 \\ 0 & \Phi_b \end{pmatrix} \Phi_n^{-1}, w_c \right),$$

where $w_c \in W_F \backslash W_E$. On the other hand, there is an injective map $\Phi(U_n) \to \Phi(\mathrm{GL}_n(E))$ given by restriction of parameters to L_E. If $G = U_n$ or $\mathrm{GL}_n(E)$ we write $\Phi_{\mathrm{bdd}}(G)$ for the subset of parameters φ such that $\varphi(W_E)$ is a bounded subset of $\widehat{G}$, and $\Pi_{\mathrm{temp}}(G)$ for the set of isomorphism classes of irreducible admissible representations of G which are tempered.

Lemma 2.6. *This map induces a bijection between $\Phi(U_n)$ and the subset of $\Phi(\mathrm{GL}_n(E))$ consisting of those parameters which are conjugate symplectic, in the sense of [Mok, §2.2].*

Proof. This follows from [Mok, Lemma 2.2.1]. It uses that n is even. □

Given $\varphi \in \Phi(G)$ we define groups

$$S_\varphi = Z_{\widehat{G}}(\operatorname{im} \varphi),\ \overline{S}_\varphi = S_\varphi / Z(\widehat{G})^{\Gamma_F},\ \mathcal{S}_\varphi = \pi_0(\overline{S}_\varphi).$$

In [Mok, Theorem 2.5.1] is associated to each $\varphi \in \Phi_{\mathrm{bdd}}(U_n)$ a finite set Π_φ of isomorphism classes of tempered irreducible admissible representations of U_n, and a bijective mapping $\Pi_\varphi \to \mathrm{Hom}(\mathcal{S}_\varphi, \mathbb{C}^\times)$. This set is characterized by certain character identities. The set $\Pi_{\mathrm{temp}}(U_n)$ is the disjoint union of the sets Π_φ for $\varphi \in \Phi_{\mathrm{bdd}}(U_n)$. We refer to Π_φ as the L-packet associated to φ. If $\pi \in \Pi_\varphi$ and φ is a bounded parameter, then we define the stable base change $\mathrm{BC}(\pi)$ to be the irreducible admissible representation of $\mathrm{GL}_n(E)$ corresponding to the restriction of φ in $\Phi_{\mathrm{bdd}}(\mathrm{GL}_n(E))$.

We will be in interested in a particular L-packet. Suppose once more that E/F is ramified, and that the residue characteristic of F is not 2. We write $\mathrm{St}_{n,E}$ for the Steinberg representation of $\mathrm{GL}_n(E)$. The following description of our L-packet of interest was explained to us by Moeglin, who has kindly written up a proof in the appendix to this paper.

Theorem 2.7. *The representation $\Pi_E = \mathrm{St}_{2,E} \boxplus \mathrm{St}_{n-2,E}$ of $\mathrm{GL}_n(E)$ is in the image of the stable base change map. The corresponding L-packet of U_n contains exactly two elements X, Y which may be characterized as follows.*

– $\langle \operatorname{tr} X + \operatorname{tr} Y, f\rangle = \langle \operatorname{tr} \Pi_E \times I_c, f_E\rangle$, *where the intertwining operator* $I_c : \Pi_E \cong \Pi_E^c$ *is Whittaker normalized, cf. §3.7 below.*
– $\dim X^{\mathfrak{P}} = \dim X^{\mathfrak{B}} = 1$ *and* $\dim Y^{\mathfrak{B}} = n/2 + 1$.
– $X_{N_0}^{norm} = [n-3, n-5, \dots, 1, -1]$ *and* $(Y_{N_0}^{norm})^{ss} = [1, n-3, n-5, \dots, 1] + \sum_{i=1}^{n/2-3}[n-3, \dots, n-1-2i, 1, n-3-2i, \dots, 1] + 2[n-3, n-5, \dots, 1, 1] + [n-3, n-5, \dots, 1, -1]$.

Here we write $[a_1, \dots, a_{n/2}]$ for the character $|\cdot|^{a_1/2} \otimes \dots \otimes |\cdot|^{a_{n/2}/2}$ of $E^\times \times \dots \times E^\times$, the F-points of the standard Levi subgroup of the minimal parabolic $P_0 \subset U_n$. We remark that the character $[n-3, n-5, \dots, 1, 1]$ occurs in $(Y_{N_0}^{\mathrm{norm}})^{\mathrm{ss}}$ with multiplicity two, while every other character occurs with multiplicity one.

Proof. The only assertion which is not in the appendix below is the statement on $X^{\mathfrak{P}}$. By Proposition 2.1 it suffices to check that $X_N^{\mathfrak{P}\cap M} \neq 0$. By Frobenius reciprocity and transitivity of the Jacquet modules, X_N is sent non-trivially to $[n-3, \dots, 1] \otimes \operatorname{Ind}_{P_0^2}^{U_2} |\cdot|^{-1/2}$, P_0^2 being the Borel subgroup in U_2. The induced representation has the trivial representation as its only submodule. □

Proposition 2.8. *Let* $\varphi : H_{\mathfrak{B}} \to \mathbb{C}$ *denote the homomorphism giving the action of* $H_{\mathfrak{B}}$ *on* $X^{\mathfrak{B}}$. *Then* $\varphi \circ \jmath = \varphi$.

Proof. This is an immediate consequence of the fact that X is self-dual, which may be checked as follows. Given the structure of the Jacquet modules of X and Y, it suffices to show that $X + Y$ is self-dual in the Grothendieck group of admissible $\mathbb{C}[U_n]$-modules. The correspondence $f \rightsquigarrow f_E$ is compatible with the anti-involutions $g \mapsto g^{-1}$ (on both groups), so it suffices to check that there is an isomorphism $\Pi_E \cong \Pi_E^\vee$, compatible with the Whittaker functional. However $\Pi_E^\vee$ is isomorphic to the representation $\Pi_E(\Phi_n {}^t g^{-1} \Phi_n^{-1}) = \rho_E$, say, and an isomorphism $\Pi_E \to \rho_E$ respects the Whittaker functional. □

Fix an odd prime l and an isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$, and let $K \subset \overline{\mathbb{Q}}_l$ be a finite extension of $\mathbb{Q}_l$, with ring of integers $\mathcal{O}$ and residue field k. We suppose that K contains a square root of q.

Proposition 2.9. *1.* $\iota^{-1}X$ *and* $\iota^{-1}Y$ *are defined over* K. *We write* X_K, Y_K *for a choice of admissible* $K[U_n]$*-modules satisfying*

$$X_K \otimes_{K,\iota} \mathbb{C} \cong X,\ Y_K \otimes_{K,\iota} \mathbb{C} \cong Y.$$

2. Suppose that $l \nmid q(q+1)\prod_{i=1}^{n/2-2}(q^i - 1)$. *Then there exist* $H_{\mathfrak{B},\mathcal{O}}$*-submodules* $X_{\mathcal{O}}^{\mathfrak{B}} \subset X_K^{\mathfrak{B}}$ *and* $Y_{\mathcal{O}}^{\mathfrak{B}} \subset Y_K^{\mathfrak{B}}$ *such that the natural maps*

$$X_{\mathcal{O}}^{\mathfrak{B}} \otimes_{\mathcal{O}} K \to X_K^{\mathfrak{B}} \text{ and } Y_{\mathcal{O}}^{\mathfrak{B}} \otimes_{\mathcal{O}} K \to Y_K^{\mathfrak{B}}$$

are isomorphisms, and $X_{\mathcal{O}}^{\mathfrak{B}} \otimes_{\mathcal{O}} k$ *and* $Y_{\mathcal{O}}^{\mathfrak{B}} \otimes_{\mathcal{O}} k$ *have no Jordan-Hölder factors as* $H_{\mathfrak{B},k}$*-modules in common.*

Proof. We give a proof by explicit calculation, using the results of Reeder [Ree97], [Ree00]. We use the notation for the algebra $H_{\mathfrak{B}}$ established in the previous section. If M is a $\mathbb{C}[\Lambda]$-module and $\tau : \Lambda \to \mathbb{C}^\times$ is a homomorphism, we write $M[\tau^\infty]$ for the subspace which is annihilated by some power of the ideal $\mathfrak{m}_\tau \subset \mathbb{C}[\Lambda]$, kernel of the associated homomorphism $\mathbb{C}[\Lambda] \to \mathbb{C}$.

Let τ_0 denote the character $[n-3, n-5, \dots, 1, 1]$ of Λ. As observed above, $Y^{\mathfrak{B}}[\tau_0^\infty]$ has dimension 2. Let $\tau = w_0\tau_0 = [3-n, 5-n, \dots, -1, -1]$. We claim that $Y^{\mathfrak{B}}$ is isomorphic, as left $H_{\mathfrak{B},\mathbb{C}}$-module, to the submodule of $M(\tau)$ generated by $M(\tau)[\tau_0^\infty]$. Indeed, by Proposition 2.5, there is an injection of $H_{\mathfrak{B},\mathbb{C}}$-modules $Y^{\mathfrak{B}} \hookrightarrow M(\tau)$. As $M(\tau)[\tau_0^\infty]$ also has dimension 2, this inclusion induces an isomorphism $Y^{\mathfrak{B}}[\tau_0^\infty] \cong M(\tau)[\tau_0^\infty]$, implying the claim. We now use this to compute a model for $Y^{\mathfrak{B}}$, and then calculate its reduction modulo l.

The non-trivial characters of Λ occurring in $Y^{\mathfrak{B}}$ are τ_0, $s_m\tau_0$, and $s_{m-2}\tau_0$, $s_{m-3}s_{m-2}\tau_0$, $\dots$, $s_1 \dots s_{m-3}s_{m-2}\tau_0$. Using [Ree00, Proposition 2.1], we can calculate bases for the weight spaces of these characters in $Y^{\mathfrak{B}}$ and the matrices of the operators T_{s_i}. Let us treat first the case $n = 6$, $\tau_0 = [3, 1, 1]$. (The module $Y^{\mathfrak{B}}$ then corresponds to the module V_{01} of [Ree97, §13.2]; note that our $[3, 1, 1]$ is Reeder's

$[-3,-1,-1]$.) The stabilizer of τ in W_0 is $W_{0,\tau} = \{1, s_{m-1}\} = \{1, s_2\}$. One calculates using [Ree00, Proposition 2.1] and [Ree00, Proposition 2.2] that a basis for $Y^{\mathfrak{B}} \subset M(\tau)$ is given by the vectors (Reeder's notation)

$$\{H_{s_m w_0 s_{m-1}} \otimes 1, H_{w_0} \otimes 1, H_{w_0 s_{m-1}} \otimes 1, H_{s_{m-2} w_0 s_{m-1}} \otimes 1\} = \{H_{s_3 w_0 s_2} \otimes 1, H_{w_0} \otimes 1, H_{w_0 s_2} \otimes 1, H_{s_1 w_0 s_2} \otimes 1\}.$$

With respect to this basis, the operators T_{s_i} are given by the matrices

$$T_{s_1} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & q & -1 & 1 \\ 0 & q(q+1) & 0 & q \end{pmatrix},$$

$$T_{s_2} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$T_{s_3} = \begin{pmatrix} q & 2q(q+1) & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 2q & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The group Λ is freely generated by the elements ϵ_i, $i = 1, 2, 3$, and these elements act on $Y^{\mathfrak{B}}$ by the matrices

$$\epsilon_1 = \begin{pmatrix} q^{-3/2} & 0 & 0 & 0 \\ 0 & q^{-3/2} & 0 & 0 \\ 0 & 0 & q^{-3/2} & 0 \\ 0 & 0 & 0 & q^{-1/2} \end{pmatrix},$$

$$\epsilon_2 = \begin{pmatrix} q^{-1/2} & 0 & 0 & 0 \\ 0 & q^{-1/2} & 0 & 0 \\ 0 & q^{1/2} - q^{-1/2} & q^{-1/2} & 0 \\ 0 & 0 & 0 & q^{-3/2} \end{pmatrix},$$

$$\epsilon_3 = \begin{pmatrix} q^{1/2} & 0 & 0 & 0 \\ 0 & q^{-1/2} & 0 & 0 \\ 0 & q^{-1/2} - q^{1/2} & q^{-1/2} & 0 \\ 0 & 0 & 0 & q^{-1/2} \end{pmatrix}.$$

(We remark that $q^{1/2}$ is a canonically defined element of $\mathbb{R} \subset \mathbb{C}$.) Let $Y_{\mathbb{Z}[q^{1/2}, q^{-1/2}]}$ denote the free $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-module spanned by the above basis elements. Then

$$Y^{\mathfrak{B}}_{\mathbb{Z}[q^{1/2}, q^{-1/2}]} \subset Y^{\mathfrak{B}}$$

is a $H_{\mathfrak{B}, \mathbb{Z}[q^{1/2}, q^{-1/2}]}$-submodule and the natural map $Y^{\mathfrak{B}}_{\mathbb{Z}[q^{1/2}, q^{-1/2}]} \otimes_{\mathbb{Z}[q^{1/2}, q^{-1/2}]} \mathbb{C} \to Y^{\mathfrak{B}}$ is an isomorphism. The choice of ι induces a homomorphism $\mathbb{Z}[q^{1/2}, q^{-1/2}] \to \mathcal{O}$. We set

$$Y^{\mathfrak{B}}_{\mathcal{O}} = Y^{\mathfrak{B}}_{\mathbb{Z}[q^{1/2}, q^{-1/2}]} \otimes_{\mathbb{Z}[q^{1/2}, q^{-1/2}]} \mathcal{O},$$

and choose $X^{\mathfrak{B}}_{\mathcal{O}} \subset X^{\mathfrak{B}}_K$ arbitrarily. The proposition now follows in this case from the fact that the above matrices generate, after reduction mod λ, the whole algebra $\mathrm{End}_k(Y^{\mathfrak{B}}_{\mathcal{O}} \otimes_{\mathcal{O}} k) = M_4(k)$. Indeed, it is easy to see that the matrices $\epsilon_1, \epsilon_2, \epsilon_2$ generate a subalgebra of $M_4(k)$ containing the diagonal matrices $\mathrm{diag}(\lambda, \mu, \mu, \nu)$, $\lambda, \mu, \nu \in k$. Multiplying the matrices T_{s_1}, T_{s_2} and T_{s_3} on the left and right by matrices of this form, and using that $2q(q^2-1)$ is non-zero in k, we obtain the elementary matrices $E_{3,4}$, $E_{4,3}$, $E_{1,2}$ and $E_{3,1}$, where

$E_{i,j}$ is the matrix with exactly one non-zero entry in the (i,j) spot, which is equal to 1. Using the matrices ϵ_2 and T_{s_2}, we obtain all block diagonal matrices with blocks of size $1+2+1=4$. It is now easy to check that the algebra generated by all of these operators is $M_4(k)$.

We treat the general case by induction on $n \geq 8$. Suppose the proposition to be true for the group U_{n-2}. We again choose $X_{\mathcal{O}}^{\mathfrak{B}} \subset X_K^{\mathfrak{B}}$ arbitrarily. We identify U_{n-2} as the subgroup of U_n consisting of block diagonal matrices, corresponding to the partition $n = 1 + (n-2) + 1$. We write Y_{n-2} for the corresponding representation of U_{n-2}. Similarly we write $\mathfrak{B}_{n-2} \subset U_{n-2}$ for the Iwahori subgroup of this group. We can view $H_{\mathfrak{B}_{n-2},\mathbb{C}}$ as a subalgebra of $H_{\mathfrak{B},\mathbb{C}}$, namely the one generated by the elements $T_{s_2}, \ldots, T_{s_m} \in H_0$ and $\epsilon_2^{\pm 1}, \ldots, \epsilon_m^{\pm 1} \in \mathbb{C}[\Lambda]$. One calculates using [Ree00, Proposition 2.2] that a basis for $Y^{\mathfrak{B}}$ is given by the elements

$$\{H_{s_m w_0 s_{m-1}} \otimes 1, H_{w_0} \otimes 1, H_{w_0 s_{m-1}} \otimes 1, H_{s_{m-2} w_0 s_{m-1}} \otimes 1, H_{s_{m-3} s_{m-2} w_0 s_{m-1}} \otimes 1, \ldots, H_{s_1 \ldots s_{m-3} s_{m-2} w_0 s_{m-1}} \otimes 1\}.$$

We first show that the $\mathcal{O}$-submodule $Y_{\mathcal{O}}^{\mathfrak{B}}$ of $\iota^{-1} Y^{\mathfrak{B}}$ spanned by these elements is $H_{\mathfrak{B},\mathcal{O}}$-invariant. Indeed, the $\mathcal{O}$-submodule spanned by the first m of these elements is preserved by the subalgebra $H_{\mathfrak{B}_{n-2},\mathcal{O}}$, and is isomorphic to the module $Y_{n-2,\mathcal{O}}^{\mathfrak{B}}$ (in the obvious notation). The operator T_{s_1} preserves the subspace spanned by the vectors

$$H_{s_2 \ldots s_{n-3} s_{n-2} w_0 s_{n-1}} \otimes 1, H_{s_1 \ldots s_{n-3} s_{n-2} w_0 s_{n-1}} \otimes 1,$$

and the matrix of its restriction to this subspace is

$$\begin{pmatrix} -\frac{q-1}{q^{m-2}-1} & 1 \\ \frac{q(q^{m-3}-1)(q^{m-1}-1)}{(q^{m-2}-1)^2} & \frac{q^{m-2}(q-1)}{q^{m-2}-1} \end{pmatrix}.$$

It acts as multiplication by q on the other basis vectors. It is now easy to see that the algebra $H_{\mathfrak{B},\mathcal{O}}$ preserves $Y_{\mathcal{O}}^{\mathfrak{B}}$. The character of $\mathcal{O}[\Lambda]$ afforded by $X_{\mathcal{O}}^{\mathfrak{B}}$ is distinct from the other characters of $\mathcal{O}[\Lambda]$ appearing in $Y_{\mathcal{O}}^{\mathfrak{B}}$, even modulo λ. If $Y_{\mathcal{O}}^{\mathfrak{B}} \otimes_{\mathcal{O}} k$ and $X_{\mathcal{O}}^{\mathfrak{B}} \otimes_{\mathcal{O}} k$ have a common Jordan-Hölder constituent as $H_{\mathfrak{B},k}$-modules, then they must also have a common Jordan-Hölder constituent as $H_{\mathfrak{B}_{n-2},k}$-modules, contradicting the induction hypothesis. This completes the proof of the proposition. □

Corollary 2.10. *Suppose that M an $H_{\mathfrak{B},\mathcal{O}}$-module which is finite flat as an $\mathcal{O}$-module, and such that $M \otimes_{\mathcal{O},\iota} \mathbb{C} \cong (X^{\mathfrak{B}})^a \oplus (Y^{\mathfrak{B}})^b$ for some integers $a, b \geq 0$. Suppose that $l \nmid q(q+1)\prod_{i=1}^{n/2-2}(q^i - 1)$. Let M_X denote the intersection of M with the $X_K^{\mathfrak{B}}$-isotypic component of $M \otimes_{\mathcal{O}} K$, and similarly for M_Y. Then there is a direct sum decomposition of $H_{\mathfrak{B},\mathcal{O}}$-modules*

$$M = M_X \oplus M_Y.$$

Proof. Consider the map $M_X \oplus M_Y \to M$. It is injective, and surjective after tensoring with K. To show that it is surjective, it suffices to show that the induced map

$$M_X \otimes_{\mathcal{O}} k \oplus M_Y \otimes_{\mathcal{O}} k \to M \otimes_{\mathcal{O}} k$$

is injective. However, the kernel of this map can be viewed as a submodule of $M_X \otimes_{\mathcal{O}} k$ and as a submodule of $M_Y \otimes_{\mathcal{O}} k$. By Proposition 2.9, these two spaces have no simple subquotients as $H_{\mathfrak{B},k}$-modules in common. Therefore the kernel must be trivial, and this implies the result. □

3 Automorphic representations

3.1 GL_n

Let p be a prime, and let K be a finite extension of $\mathbb{Q}_p$. Let Ω denote an algebraically closed field of characteristic zero. There is a bijection

$$\mathrm{rec}_K : \mathrm{Adm}_{\mathbb{C}} \mathrm{GL}_n(K) \leftrightarrow \mathrm{WD}_{\mathbb{C}}^n W_K,$$

characterized by a certain equality of epsilon- and L-factors on either side, cf. [HT01], [Hen02]. When $n = 1$, it is induced by the local Artin map, normalized to take uniformizers to geometric Frobenius elements. Here we write $\mathrm{Adm}_\Omega \mathrm{GL}_n(K)$ for the set of isomorphism classes of irreducible admissible representations of this group over Ω, and $\mathrm{WD}^n_\Omega W_K$ for the set of Frobenius-semisimple Weil-Deligne representations (r, N) of W_K valued in $\mathrm{GL}_n(\Omega)$. We define $\mathrm{rec}^T_K(\pi) = \mathrm{rec}_K(\pi|\cdot|^{(1-n)/2})$. This is the normalization of the local Langlands correspondence with good rationality properties; in particular, for any $\sigma \in \mathrm{Aut}(\mathbb{C})$ and any $\pi \in \mathrm{Adm}_\mathbb{C} \mathrm{GL}_n(K)$ there is an isomorphism

$$\mathrm{rec}^T_K({}^\sigma\pi) \cong {}^\sigma \mathrm{rec}^T_K(\pi).$$

This can be seen using, for example, the characterization of rec_K and the description given in [Tat79, §3] of the action of Galois on local ϵ- and L-factors. It follows that for any Ω we can define a canonical bijection

$$\mathrm{rec}^T_K : \mathrm{Adm}_\Omega \mathrm{GL}_n(K) \leftrightarrow \mathrm{WD}^n_\Omega W_K.$$

Suppose instead that K is a finite extension of $\mathbb{R}$. Then there is a bijection (Langlands' normalization):

$$\mathrm{rec}_K : \mathrm{Adm}_\mathbb{C} \mathrm{GL}_n(K) \leftrightarrow \mathrm{Rep}^n_\mathbb{C} W_K.$$

Here we write $\mathrm{Adm}_\mathbb{C} \mathrm{GL}_n(K)$ for the set of infinitesimal equivalence classes of irreducible admissible representations of $\mathrm{GL}_n(K)$ and $\mathrm{Rep}^n_\mathbb{C} W_K$ for the set of continuous semisimple representations of W_K into $\mathrm{GL}_n(\mathbb{C})$. We define $\mathrm{rec}^T_K(\pi) = \mathrm{rec}_K(\pi|\cdot|^{(1-n)/2})$.

Now let E be an imaginary CM field with totally real subfield F, and let $c \in \mathrm{Gal}(E/F)$ denote the non-trivial element.

Definition 3.1. *1. We say that an automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_E)$ is RACSDC (regular algebraic, conjugate self-dual, cuspidal) if it satisfies the following conditions:*
- *It is conjugate self-dual: $\pi^c \cong \pi^\vee$.*
- *It is cuspidal.*
- *It is regular algebraic. By definition, this means that for each place $v|\infty$ of E, the representation $\mathrm{rec}^T_{E_v}(\pi_v)$ is a direct sum of pairwise distinct algebraic characters.*

2. We say that a pair (π, χ) of an automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_E)$ and a character $\chi : F^\times\backslash\mathbb{A}_F^\times \to \mathbb{C}^\times$ is RAECSDC (regular algebraic, essentially conjugate self-dual, cuspidal) if it satisfies the following conditions:
- *It is essentially conjugate self-dual: $\pi^c \cong \pi^\vee \otimes \chi \circ \mathbb{N}_{E/F}$.*
- *π is cuspidal.*
- *π is regular algebraic.*
- *χ is an algebraic character such that $\chi_v(-1) = (-1)^n$ for each place $v|\infty$.*

3. We say that an pair (π, χ) of an automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ and a character $\chi : F^\times\backslash\mathbb{A}_F^\times \to \mathbb{C}^\times$ is RAESDC (regular algebraic, essentially self-dual, cuspidal) if it satisfies the following conditions:
- *It is essentially self-dual: $\pi \cong \pi^\vee \otimes \chi$.*
- *π is cuspidal.*
- *π is regular algebraic. By definition, this means that for each place $v|\infty$, the representation $\mathrm{rec}^T_{E_v}(\pi_v)|_{\mathbb{C}^\times}$ is a direct sum of pairwise distinct algebraic characters.*
- *χ is an algebraic character such that $\chi_v(-1)$ is independent of the place $v|\infty$.*

If π is a regular algebraic automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$, then for each embedding $\tau : E \hookrightarrow \mathbb{C}$, we are given a representation $r_\tau : \mathbb{C}^\times \to \mathrm{GL}_n(\mathbb{C})$, induced by $\mathrm{rec}_{E_v}(\pi_v)$, where v is the infinite place induced by τ, and the isomorphism $E_v^\times \cong \mathbb{C}^\times$ induced by τ. This representation has the form

$$r_\tau(z) = ((z/\overline{z})^{a_{\tau,1}}, \ldots, (z/\overline{z})^{a_{\tau,n}}),$$

where $a_{\tau,i} \in (n-1)/2 + \mathbb{Z}$. We will refer to the tuple $\mathbf{a} = (a_{\tau,1}, \ldots, a_{\tau,n})_{\tau\in\mathrm{Hom}(E,\mathbb{C})}$, where for each τ we have $a_{\tau,1} > a_{\tau,2} > \cdots > a_{\tau,n}$, as the infinity type of π. We also define a tuple $\boldsymbol{\lambda} = (\lambda_\tau)_{\tau\in\mathrm{Hom}(E,\mathbb{C})} =$

$(\lambda_{\tau,1},\dots,\lambda_{\tau,n})_{\tau\in\mathrm{Hom}(E,\mathbb{C})}$, which we call the weight of π, by the formula $\lambda_{\tau,i}=-a_{\tau,n+1-i}+(n-1)/2-(n-i)$. Then for each $\tau:E\hookrightarrow\mathbb{C}$, we have $\lambda_{\tau,1}\geq\dots\geq\lambda_{\tau,n}$, and the irreducible admissible representation of $\mathrm{GL}_n(\mathbb{C})$ corresponding to r_τ has the same infinitesimal character as the dual of the algebraic representation of $\mathrm{GL}_n(\mathbb{C})$ with highest weight λ_τ. If π is a regular algebraic automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, then for each embedding $\tau:F\hookrightarrow\mathbb{C}$, we get a representation $r_\tau=\mathrm{rec}_{F_v}(\pi_v)|_{\mathbb{C}^\times}$, where v is the place of F corresponding to τ. In this case we use the same formulae to define the infinity type and the weight of the pair π.

We will also have cause to consider representations which are not cuspidal. Suppose that σ_1,σ_2 are conjugate self-dual cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_E)$, and that $\Sigma=\sigma_1\boxplus\sigma_2$ is regular algebraic. Then the representations $\sigma_i|\cdot|^{(n_i-n)/2}$ are regular algebraic. We call a representation Σ arising in this way a RACSD sum of cuspidal representations. In this case, define $\mathbf{a}^i=(a^i_\tau)_{\tau\in\mathrm{Hom}(E,\mathbb{C})}$ by the requirement that $(a^i_{\tau,1}+(n_i-n)/2,\dots,a^i_{\tau,n_i}+(n_i-n)/2)$ equal the infinity type of $\sigma_i|\cdot|^{(n_i-n)/2}$, and define $\mathbf{b}=(b_\tau)_{\tau\in\mathrm{Hom}(E,\mathbb{C})}$ by the formula

$$(b_{\tau,1},\dots,b_{\tau,n})=(a^1_{\tau,1},\dots,a^1_{\tau,n_1},a^2_{\tau,1},\dots,a^2_{\tau,n_2}).$$

Then there is a unique tuple $\mathbf{w}=(w_\tau)_{\tau\in\mathrm{Hom}(E,\mathbb{C})}\in\mathfrak{S}_n^{\mathrm{Hom}(E,\mathbb{C})}$ such that for each $\tau\in\mathrm{Hom}(E,\mathbb{C})$, the infinity type of Σ is $(b_{\tau,w_\tau(1)},\dots,b_{\tau,w_\tau(n)})_{\tau\in\mathrm{Hom}(E,\mathbb{C})}$. (Here $\mathfrak{S}_n$ denotes the symmetric group on the set $\{1,\dots,n\}$.)

Theorem 3.2. *1. Let π be a RACSD sum of cuspidals or a RAECSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$, and fix an isomorphism $\iota:\overline{\mathbb{Q}}_l\cong\mathbb{C}$. Then there exists a continuous semisimple representation*

$$r_\iota(\pi):G_E\to\mathrm{GL}_n(\overline{\mathbb{Q}}_l)$$

satisfying the following property: for every finite place v of E not dividing l, there is an isomorphism

$$WD(r_\iota(\pi)|_{G_{E_v}})^{F\text{-}ss}\cong\mathrm{rec}^T_{E_v}(\iota^{-1}\pi_v).$$

For each place v of E dividing l, $r_\iota(\pi)|_{G_{E_v}}$ is de Rham, and if $\tau:E_v\hookrightarrow\overline{\mathbb{Q}}_l$ is an embedding and $\mathbf{a}$ the infinity type of π, then the Hodge-Tate weights with respect to this embedding are

$$\mathrm{HT}_\tau(r_\iota(\pi))=\{-a_{\iota^{-1}\tau,1}+(n-1)/2,\dots,-a_{\iota^{-1}\tau,n}+(n-1)/2\}.$$

2. Let (π,χ) be a RAESDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, and fix an isomorphism $\iota:\overline{\mathbb{Q}}_l\cong\mathbb{C}$. Then there exists a continuous semisimple representation

$$r_\iota(\pi):G_F\to\mathrm{GL}_n(\overline{\mathbb{Q}}_l)$$

satisfying the following property: for every finite place v of F not dividing l, there is an isomorphism

$$WD(r_\iota(\pi)|_{G_{E_v}})^{F\text{-}ss}\cong\mathrm{rec}^T_{F_v}(\iota^{-1}\pi_v).$$

For each place v of F dividing l, $r_\iota(\pi)|_{G_{F_v}}$ is de Rham, and if $\tau:F_v\hookrightarrow\overline{\mathbb{Q}}_l$ is an embedding and $\mathbf{a}$ the infinity type of π, then the Hodge-Tate weights with respect to this embedding are

$$\mathrm{HT}_\tau(r_\iota(\pi))=\{-a_{\iota^{-1}\tau,1}+(n-1)/2,\dots,-a_{\iota^{-1}\tau,n}+(n-1)/2\}.$$

Proof. This theorem is due to many people. We give references for the case of a RACSDC automorphic representation π, from which the others can be deduced. In this case the existence of the representation $r_\iota(\pi)$ is proved in [CH, Theorem 3.2.3]. The strong form of local-global compatibility is proved in [Car12]. □

Lemma 3.3. *Let π be one of the above types of automorphic representations, and fix an isomorphism $\iota:\overline{\mathbb{Q}}_l\cong\mathbb{C}$. Let σ be a continuous automorphism of $\overline{\mathbb{Q}}_l$. Then ${}^{\iota\sigma\iota^{-1}}\pi$ is defined, by [Clo90b, Theorem 3.13]. There are isomorphisms*

$$r_\iota({}^{\iota\sigma\iota^{-1}}\pi)\cong r_{\iota\sigma}(\pi)\cong{}^\sigma r_\iota(\pi).$$

Proof. This follows from local-global compatibility, the rationality of the local Langlands correspondence for GL_n, and the Chebotarev density theorem. $\square$

We will use the following convention for residual representations. If $\rho : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ is a continuous representation, then after choosing an invariant lattice, defined over a finite extension of $\mathbb{Q}_l$, we obtain by reduction modulo l a residual representation valued in $\mathrm{GL}_n(\overline{\mathbb{F}}_l)$. By the principle of Brauer-Nesbitt, the semisimplification of this representation depends, up to isomorphism, only on ρ, and will be denoted $\overline{\rho} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$.

3.2 Ordinary forms

Let $L = E$ or F. If π is a regular algebraic automorphic representation of $\mathrm{GL}_n(\mathbb{A}_L)$ of infinity type $\mathbf{a}$ and weight $\boldsymbol{\lambda}$, we define Hecke operators $U^j_{\boldsymbol{\lambda},v}$ as follows at primes v above l. They depend on a choice of isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$, which we fix for the rest of this section, as well as a choice of uniformizer ϖ_v of $\mathcal{O}_{L_v}$. Define a matrix

$$\alpha_v^j = \mathrm{diag}(\underbrace{\varpi_v, \dots, \varpi_v}_{j}, \underbrace{1, \dots, 1}_{n-j})$$

and set

$$U^j_{\boldsymbol{\lambda},v} = \prod_\tau \iota^{-1}\tau(\varpi_v)^{-\lambda_{\tau,n}+\dots+\lambda_{\tau,n+1-j}} \left[\mathrm{Iw}_c(v)\alpha_v^j \,\mathrm{Iw}_c(v)\right].$$

By definition, the subgroup $\mathrm{Iw}_c(v) \subset \mathrm{GL}_n(\mathcal{O}_{L_v})$ is the subgroup of matrices whose reduction modulo ϖ_v^c is an upper-triangular matrix with 1's on the diagonal, and the product runs over embeddings $\tau : L \hookrightarrow \mathbb{C}$ such that $\iota^{-1}\tau$ induces the place v of L. We note that by [Ger, Lemma 2.3.3], the Hecke operators $U^j_{\boldsymbol{\lambda},v}$ commute with the inclusions $\iota^{-1}\pi_v^{\mathrm{Iw}_c(v)} \to \iota^{-1}\pi_v^{\mathrm{Iw}_{c'}(v)}$ when $c' \geq c$. It therefore makes sense to omit c from the notation defining $U^j_{\boldsymbol{\lambda},v}$. We also write $T_c(v) \subset \mathrm{Iw}_c(v)$ for the group of diagonal matrices with integral entries which are congruent to 1 modulo ϖ_v^c, e_v for the absolute ramification index of $[L_v : \mathbb{Q}_l]$, f_v for the absolute residue degree, and $\mathrm{val} : \overline{\mathbb{Q}}_l^\times \to \mathbb{Q}$ for the valuation such that $\mathrm{val}(l) = 1$.

Definition 3.4. *Let π be a regular algebraic automorphic representation of $\mathrm{GL}_n(\mathbb{A}_L)$ of weight $\boldsymbol{\lambda}$. We say that π is ι-ordinary if for each place v of L dividing l, there is an integer $c \geq 1$ and a line inside $\iota^{-1}\pi_v^{\mathrm{Iw}_c(v)}$ which is invariant under each operator $U^j_{\boldsymbol{\lambda},v}$, and such that the eigenvalues of these operators on this line are all l-adic units.*

The next lemma follows immediately from [CT, Lemma 2.5] and [CT, Lemma 2.6].

Lemma 3.5.

1. *The subspace of $\varinjlim_c \iota^{-1}\pi_v^{\mathrm{Iw}_c(v)}$ where each operator $U^j_{\boldsymbol{\lambda},v}$ acts with eigenvalue a unit has dimension at most one.*
2. *Suppose that π_1, π_2 are cuspidal conjugate self-dual automorphic representations of $\mathrm{GL}_{n_1}(\mathbb{A}_E)$ and $\mathrm{GL}_{n_2}(\mathbb{A}_E)$, respectively. Suppose that $\Pi = \pi_1 \boxplus \pi_2$ is regular algebraic. Then the representations $\pi_i|\cdot|^{(n_i-n)/2}$ are regular algebraic, and Π is ι-ordinary if and only if $\pi_1|\cdot|^{(n_1-n)/2}, \pi_2|\cdot|^{(n_2-n)/2}$ are ι-ordinary and the following condition on infinity types holds. Recall the tuple $\mathbf{w} = (w_\tau)_{\tau\in\mathrm{Hom}(E,\mathbb{C})}$ of permutations associated to the infinity types of π_1, π_2. Then w_τ depends only on the place v of E dividing l induced by the embedding $\iota^{-1}\tau : E \hookrightarrow \overline{\mathbb{Q}}_l$.*

3.3 Definite unitary groups

We now let E be an imaginary CM field with totally real subfield F, and suppose that $[F : \mathbb{Q}]$ is even. Let G be a unitary group in n variables associated to the extension E/F, quasi-split at every finite place, such that $G(\mathbb{R})$ is compact. Such a group exists since $[F : \mathbb{Q}]$ is even, and is uniquely determined up to isomorphism. We can choose the matrix algebra $B = M_n(E)$ and an involution $\dagger$ of B of the second kind, so that G is defined by

$$G(R) = \{g \in (B \otimes_F R)^\times \mid g^\dagger g = 1\}$$

for any F-algebra R. We may choose an order $\mathcal{O}_B \subset B$, stable under $\dagger$, so that $\mathcal{O}_{B,w}$ is maximal for any place w of E split over F. This defines an integral model of G over $\mathcal{O}_F$, and for any place v of F split as $v = ww^c$ in E, we can choose an isomorphism

$$\mathcal{O}_B \otimes_{\mathcal{O}_F} \mathcal{O}_{F_v} \cong M_n(\mathcal{O}_{E_w}) \times M_n(\mathcal{O}_{E_{w^c}}),$$

such that $\dagger$ acts as $(g_1, g_2) \mapsto (g_2, {}^t g_1)$. Projection onto the first factor induces an isomorphism $\iota_w : G(F_v) \to \mathrm{GL}_n(E_w)$ such that $\iota_w(G(\mathcal{O}_{F_v})) = \mathrm{GL}_n(\mathcal{O}_{E_w})$.

Let l be a prime, and suppose that every prime of F above l splits in E. Let S_l denote the set of primes of F above l. We choose a prime $\widetilde{v}$ of E above v for each $v \in S_l$, and let $\widetilde{S}_l$ denote the set of these primes. Then, as above, we are given an isomorphism $\iota_{\widetilde{v}} : G(F_v) \to \mathrm{GL}_n(E_{\widetilde{v}})$. We write I_l for the set of embeddings $F \hookrightarrow \overline{\mathbb{Q}}_l$, and $\widetilde{I}_l$ for the set of embeddings $E \hookrightarrow \overline{\mathbb{Q}}_l$ inducing an element of $\widetilde{S}_l$. These two sets are therefore in canonical bijection.

Let $K \subset \overline{\mathbb{Q}}_l$ be a finite extension of $\mathbb{Q}_l$, with ring of integers $\mathcal{O}$ and residue field k. We suppose that K contains the image of E under every embedding $E \hookrightarrow \overline{\mathbb{Q}}_l$. To a tuple $\boldsymbol{\lambda} = (\lambda_{\tau,1}, \ldots, \lambda_{\tau,n})_{\tau \in \widetilde{I}_l}$ of dominant weights of GL_n, we associate a representation $M_{\boldsymbol{\lambda}}$ of the group $\prod_{v \in S_l} G(\mathcal{O}_{F_v})$ as in [Ger, Definition 2.2.3]. It is an $\mathcal{O}$-lattice inside the representation $W_{\boldsymbol{\lambda}} = \otimes_{\tau \in \widetilde{I}_l}(W_{\lambda_\tau} \otimes_{F_v,\tau} K)$, where W_{λ_τ} is the algebraic representation of $\mathrm{GL}_n(F_v)$ of highest weight λ_τ, and v is the place of F induced by τ.

Fix $\boldsymbol{\lambda}$ and an open compact subgroup $U = \prod_v U_v \subset G(\mathbb{A}_F^\infty)$, such that $U_v \subset G(\mathcal{O}_{F_v})$ for each $v \in S_l$. Let A be an $\mathcal{O}$-algebra. We can then define a space of automorphic forms with A-coefficients as follows. By definition, $S_{\boldsymbol{\lambda}}(U, A)$ is the set of functions $f : G(F)\backslash G(\mathbb{A}_F^\infty) \to M_{\boldsymbol{\lambda}} \otimes_{\mathcal{O}} A$ such that for all $u \in U$, we have $f(gu) = u_l^{-1} \cdot f(g)$. Here u_l denotes the projection of u to its $\prod_{v \in S_l} G(\mathcal{O}_{F_v})$-component. If $\boldsymbol{\lambda} = 0$, then we write $S_{\boldsymbol{\lambda}}(U, A) = S(U, A)$.

The relation with classical automorphic forms is given by the following result. Let $\mathcal{A}$ denote the space of automorphic forms on $G(F)\backslash G(\mathbb{A})$, and let $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$ be an isomorphism. There is an algebraic representation $W_{\iota\boldsymbol{\lambda}}$ of $G(F \otimes_{\mathbb{Q}} \mathbb{R})$, defined by the formula $\otimes_{\tau \in \widetilde{I}_l} W_{\lambda_\tau} \otimes_{F_v, \iota\tau} \mathbb{C}$.

Proposition 3.6. *There is a canonical isomorphism*

$$\left(\varinjlim_U S_{\boldsymbol{\lambda}}(U, K)\right) \otimes_{K,\iota} \mathbb{C} \cong \mathrm{Hom}_{G(F \otimes_{\mathbb{Q}} \mathbb{R})}(W_{\iota\boldsymbol{\lambda}}^\vee, \mathcal{A}).$$

In particular, for any irreducible subrepresentation $\sigma \subset \mathcal{A}$, there is a canonical subspace $\iota^{-1}(\sigma^\infty)^U \subset S_{\boldsymbol{\lambda}}(U, \overline{\mathbb{Q}}_l)$, and $\varinjlim_U S_{\boldsymbol{\lambda}}(U, K)$ is a semisimple admissible representation of $G(\mathbb{A}_F^\infty)$.

Proof. This is proved just as [CHT08, Proposition 3.3.2]. □

If π is an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ and σ is an automorphic representation of $G(\mathbb{A}_F)$, we say that π is the base change of σ if for any finite place w of E, the following condition is satisfied:

- If w is split over the place v of F, then π_w is the standard base change of σ_v.
- If w is inert over the place v of F and σ_v is unramified, then π_w is the standard unramified base change of σ_v (cf. [Mín11, Theorem 4.1]).

Proposition 3.7. *1. Suppose that σ is an automorphic representation of $G(\mathbb{A}_F)$. Then there exist discrete and conjugate self-dual representations $\pi_1, \ldots, \pi_s$ of $\mathrm{GL}_n(\mathbb{A}_E)$ such that $\pi = \pi_1 \boxplus \cdots \boxplus \pi_s$ is the base change of σ in the above sense.*

Proof. This follows from [Lab11a, Corollaire 5.3]. □

Proposition 3.8. *Let σ be an automorphic representation of $G(\mathbb{A}_F)$. Then there exists a unique continuous semisimple representation*

$$r_\iota(\sigma) : G_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$$

satisfying the following condition: for every place w of E split over F, we have

$$WD(r_\iota(\sigma)|_{G_{E_w}})^{F\text{-}ss} \cong \mathrm{rec}_{E_w}^T(\iota^{-1}(\sigma_v \circ \iota_w)).$$

Let $U = \prod_v U_v$ be an open compact subgroup as above, and suppose that there exists an integer $c \geq 1$ such that for each $v \in S_l$, $U_v = \iota_{\widetilde{v}}^{-1} \operatorname{Iw}_c(\widetilde{v})$. For each prime $v \in S_l$, fix a uniformizer $\varpi_{\widetilde{v}}$ of $\mathcal{O}_{E_{\widetilde{v}}}$, and define the matrix

$$\alpha_v^j = \operatorname{diag}(\underbrace{\varpi_{\widetilde{v}}, \ldots, \varpi_{\widetilde{v}}}_{j}, \underbrace{1, \ldots, 1}_{n-j}).$$

We define an endomorphism $U_{\boldsymbol{\lambda},v}^j$ of the space $S_{\boldsymbol{\lambda}}(U, \mathcal{O})$ by the formula

$$U_{\boldsymbol{\lambda},v}^j = \prod_\tau \iota^{-1} \tau(\varpi_{\widetilde{v}})^{-\lambda_{\tau,n} + \cdots + \lambda_{\tau,n+1-j}} \iota_{\widetilde{v}}^{-1} \left[\operatorname{Iw}_c(\widetilde{v}) \alpha_v^j \operatorname{Iw}_c(\widetilde{v}) \right].$$

If $\boldsymbol{\lambda} = 0$, then we write $U_{\boldsymbol{\lambda},v}^j = U_v^j$. If σ is an automorphic representation of $G(\mathbb{A}_F)$, we say that σ is ι-ordinary if there exists an integer $c \geq 1$ and an open compact subgroup U of this form such that these operators on $(\iota^{-1}\sigma^\infty)^U$ have a common line where they all act with eigenvalues which are l-adic units.

Lemma 3.9. *1. Let σ be an automorphic representation of $G(\mathbb{A}_F)$, and let π denote its base change to $\mathrm{GL}_n(\mathbb{A}_E)$. Then σ is ι-ordinary if and only if π is ι-ordinary.*

2. Let $v \in S_l$. Then the subspace $\iota^{-1}\sigma_v^{ord}$ of $\varinjlim_c \iota^{-1}\sigma_v^{\iota_{\widetilde{v}}^{-1} \operatorname{Iw}_c(\widetilde{v})}$ where each operator $U_{\boldsymbol{\lambda},v}^j$ acts with eigenvalues which are l-adic units has dimension at most one. If $\sigma_v^{\iota_{\widetilde{v}}^{-1} \operatorname{Iw}_c(\widetilde{v})} \neq 0$ then we have $\iota^{-1}\sigma_v^{ord} \subset \iota^{-1}\sigma_v^{\iota_{\widetilde{v}}^{-1} \operatorname{Iw}_c(\widetilde{v})}$.

Proof. Since l is split, by assumption, this follows from the corresponding facts for $\mathrm{GL}_n(\mathbb{A}_E)$ and the definition of base change. □

Let σ be an automorphic representation of $G(\mathbb{A}_F)$. We will write $(\iota^{-1}\sigma^\infty)^{\mathrm{ord}}$ for the subspace

$$\iota^{-1}\sigma^{l,\infty} \otimes \bigotimes_{v \in S_l} \iota^{-1}\sigma_v^{\mathrm{ord}} \subset \iota^{-1}\sigma^\infty.$$

This is an admissible representation of $G(\mathbb{A}_F^{l,\infty})$, and is non-zero precisely when σ is ι-ordinary.

Proposition 3.10. *Let σ be an ι-ordinary automorphic representation of $G(\mathbb{A}_F)$ of weight $\boldsymbol{\lambda}$. Let $U = \prod_v U_v \subset G(\mathbb{A}_F^\infty)$ be an open compact subgroup as above, and suppose that $\sigma^U \neq 0$. Let $\boldsymbol{\lambda}'$ be another choice of weight. Then there exists an ι-ordinary automorphic representation σ' of $G(\mathbb{A}_F)$ of weight $\boldsymbol{\lambda}'$ such that $\overline{r_\iota(\sigma)} \cong \overline{r_\iota(\sigma')}$ and for every finite place v of F not dividing l, $(\sigma'_v)^{U_v} \neq 0$.*

Proof. This is an easy consequence of Hida theory, cf. [Ger, Lemma 2.6.4]. In this reference it is assumed that the extension E/F is everywhere unramified, but in our situation this makes no difference. □

3.4 Endoscopic transfer

We continue with the notation and assumptions of the previous section. We make the following further hypotheses:

- $n \geq 6$ is even.
- There exist places $v_0, \ldots, v_s$ of F not dividing $2l$ which are ramified in E. We write $w_0, \ldots, w_s$ for the places of E above $v_0, \ldots, v_s$.
- E/F is unramified at every finite place $v \neq v_0, \ldots, v_s$.
- For each place v of F dividing l, the local degree $[F_v : \mathbb{Q}_l]$ is even.

We fix for each place v of F inert in E a hyperspecial maximal compact subgroup $U_v \subset G(F_v)$. We fix also an isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$. For each $i = 0, \ldots, s$ we have defined an L-packet $\{X_i, Y_i\}$ of representations of $G(F_{v_i})$, cf. Theorem 2.7. Define a function $\epsilon_i : \{X_i, Y_i\} \to \{\pm 1\}$, by

$$\epsilon_i(X_i) = -1, \; \epsilon_i(Y_i) = +1.$$

Theorem 3.11. *1. Let π_1, π_2 be RACSDC automorphic representations of $\mathrm{GL}_2(\mathbb{A}_E)$, $\mathrm{GL}_{n-2}(\mathbb{A}_E)$, respectively. Suppose that $\pi = \pi_1 \boxplus \pi_2$ satisfies the following:*
– π has weight zero.
– π is ι-ordinary.
– If $w \neq w_0, \dots, w_s$ is a place of E at which π_w is ramified, then w is split over F.
– For each $i = 0, \dots, s$ we have $\pi_{1,w_i} \cong \mathrm{St}_{2,E_{w_i}}$ and $\pi_{2,w_i} \cong \mathrm{St}_{n-2,E_{w_i}}$.
Then there are exactly 2^s automorphic representations σ of $G(\mathbb{A}_F)$ which have base change equal to π, and such that if v is a place of F inert in E, then $\sigma_v^{U_v} \neq 0$. They are in bijective correspondence with elements $\mathbf{d} \in \prod_{i=1}^s \{X_i, Y_i\}$, this correspondence being characterized by the relation

$$\sigma(\mathbf{d})_{v_i} \cong d_i, i = 1, \dots, s.$$

These representations each appear with multiplicity one, and satisfy the further condition

$$\epsilon_0(\sigma(\mathbf{d})_{v_0}) \cdot \prod_{i=1}^{s} \epsilon_i(d_i) = 1.$$

2. Suppose that π is a RACSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ satisfying the following:
– If $w \neq w_0, \dots, w_s$ is a place of E at which π_w is ramified, then w is split over F.
– π has weight zero.
– For each $i = 0, \dots, s$, $\pi_{w_i} \cong \mathrm{St}_{2,E_{w_i}} \boxplus \mathrm{St}_{n-2,E_{w_i}}$.
Then there are exactly 2^{s+1} automorphic representations σ of $G(\mathbb{A}_F)$ such that π is the base change of σ and such that if v is a place of F inert in E, then $\sigma_v^{U_v} \neq 0$. They are in bijective correspondence with elements $\mathbf{d} \in \prod_{i=0}^s \{X_i, Y_i\}$, this correspondence being characterized by the relation

$$\sigma(\mathbf{d})_{v_i} \cong d_i, i = 0, \dots, s.$$

These representations each appear with multiplicity one.

3. Suppose that π is a RACSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ satisfying the following:
– If $w \neq w_0, \dots, w_s$ is a place of E at which π_w is ramified, then w is split over F.
– π has weight zero.
– π_{w_0} is an unramified twist of the Steinberg representation.
– For each $i = 1, \dots, s$, $\pi_{w_i} \cong \mathrm{St}_{2,E_{w_i}} \boxplus \mathrm{St}_{n-2,E_{w_i}}$.
Then there are exactly 2^s automorphic representations σ of $G(\mathbb{A}_F)$ such that π is the base change of σ and such that if v is a place of F inert in E, then $\sigma_v^{U_v} \neq 0$. They are in bijective correspondence with elements $\mathbf{d} \in \prod_{i=1}^s \{X_i, Y_i\}$, this correspondence being characterized by the relation

$$\sigma(\mathbf{d})_{v_i} \cong d_i, i = 1, \dots, s.$$

These representations each appear with multiplicity one.

The proof of this theorem depends on the stabilization of the trace formula for the definite unitary group G, and the rest of §3 will be devoted to its proof. We write G^* for the quasi-split inner form of G. The other elliptic endoscopic groups of G are isomorphic to $U(a) \times U(b)$, where $U(m)$ denotes the quasi-split unitary group in m variables attached to the extension E/F. We will be especially interested in the group

$$H = U(2) \times U(n-2).$$

We recall that we have defined an L-embedding $\xi : {}^L H \to {}^L G$ as

$$\xi(g_1, g_2, w) = \left(\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}, w \right) \ (w \in W_E),$$

$$\xi(w_c) = \left(\begin{pmatrix} \Phi_2 & 0 \\ 0 & \Phi_{n-2} \end{pmatrix} \Phi_n^{-1}, w_c \right),$$

where $w_c \in W_F$ is a representative of complex conjugation. Stable base change is associated to the L-group homomorphism

$$ {}^LG = {}^LG^* \to {}^L(\mathrm{Res}_{E/F}\, G) = \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \rtimes W_F, $$
$$ (g, w) \mapsto (g, g, w) $$

(cf. [Mín11, p. 402], [Mok, (2.1.9)]). The analogue of the above theorem with G replaced by its quasi-split inner form G^* has been proved by Mok [Mok, Theorem 2.5.2]. Let π be one of the automorphic representations of $\mathrm{GL}_n(\mathbb{A}_E)$ appearing in the statement of the theorem, and let S be a finite set of places of F containing the archimedean primes and the places below which π is ramified. Let ψ denote the data of the Hecke matrix t_{π_w} for w coprime to S (this is the data used by Arthur [Art12] and Mok). By unramified base change [Mín11], ψ defines an unramified representation σ_v of $G^*(F_v)$ for $v \notin S$, characterized for v inert in E by the property $\sigma_v^{U_v} \neq 0$.

Mok describes the full subspace of $L^2_{\mathrm{disc}}(G^*(F)\backslash G^*(\mathbb{A}_F))$ associated to ψ. At the archimedean places and the places $v_0, \ldots, v_s$, there is an L-packet $\Pi(\psi_v)$ of representations of $G^*(F_v)$ and the choice of a local representation $\sigma_v \in \Pi(\psi_v)$ is subject to a global sign condition, cf. [Mok, Theorem 2.5.2]. If π is cuspidal, then this condition is vacuous and every representation σ in the global L-packet (product of local L-packets) appears with multiplicity one.

We need the analogous result, however, for G and not G^*, and Arthur's description of the spectrum has not been achieved in this case. We will deduce what we need from Mok's results; we apologize for the obvious redundancy of our arguments.

3.5 Geometric transfer factors

Assume $f = \otimes_v f_v$ is a decomposed, smooth, K_∞-finite function on $G(\mathbb{A}_F)$. We will need the associated functions f^H on $H(\mathbb{A}_F)$, where H is an endoscopic group for G (or G^*). This depends on a choice of transfer factors $\Delta(\gamma, \delta)$, where (γ, δ) are associated (strongly regular) elements in $H(F_v), G(F_v)$.

At the finite places, we use the Whittaker normalization of transfer factors [KS99, §5.3], [Mok, §3]. This is possible since G is quasi-split at the finite places. At the archimedean places we will use Kottwitz's transfer factors, explicitly described for unitary groups in [Clo11]. We must check that such choices are compatible, i.e. that they satisfy a product formula for rational elements (γ, δ). The local factors at the finite places are defined by the formula

$$ \Delta(\gamma, \delta) = \epsilon(V, \psi)\Delta_0(\gamma, \delta), $$

where Δ_0 is the Langlands-Shelstad factor in the quasi-split case [KS99, p. 65]. Here $V = V_G - V_H$ is a virtual representation of $\mathrm{Gal}(\overline{F}_v/F_v)$, with $V_G = X_*(\widehat{T}_G) \otimes \mathbb{C}$, and similarly for V_H, and ψ is an additive character. In our case, $V_G \cong V_H$ and $\Delta(\gamma, \delta) = \Delta_0(\gamma, \delta)$.

For $\gamma, \delta \in H(F_\infty), G(F_\infty)$, consider now the product $\Delta_K^\infty(\gamma, \delta)$ of Kottwitz's transfer factors at the real primes. If G is replaced by G^*, then

$$ \Delta_K^{\infty, G^*}(\gamma, \delta) = (\pm(i)^{2(n-2)})^d \Delta_0^\infty(\gamma, \delta) $$

([Lab11b, p. 414]), where $i = \sqrt{-1}$ and $d = [F : \mathbb{Q}]$. We now use Labesse's argument: the groups G, G^*, and H can be chosen so as to contain the diagonal torus $T = U(1)^n$ (compatibly with [Clo11]). For $\gamma \in T$,

$$ \Delta_K^{\infty, G^*}(\gamma, \gamma) = \prod_{v|\infty} (-1)^{q(G^*)-q(G)} \Delta_K^{\infty, G}(\gamma, \gamma), $$

where $q(G)$ is half the real dimension of the symmetric space of G [Lab11b, p. 414]. In our case, then,

$$ \Delta_K^{\infty, G}(\gamma, \gamma) = \epsilon^d \Delta_0^\infty(\gamma, \gamma) $$

on T, where $\epsilon = \pm i^{2(n-2)}(-1)^{n/2}$. The two factors therefore coincide on T. The compatibility of the factors Δ_0 thus implies that of our chosen factors on T and therefore on (G, H) by the essential uniqueness of local transfer factors.

3.6 Spectral transfer factors, real places

Once we have defined the association of f and f^H, there follow identities between (signed) traces of f and f^H in associated L-packets. We describe this in the case of interest to us, namely when the global parameter ψ arises from a regular algebraic automorphic representation $\pi = \pi_1 \boxplus \pi_2$ of weight zero, and π_1, π_2 are RACSDC. Note that the datum of ψ, outside an arbitrary set S, uniquely determines π_1 and π_2, thus their infinity types, by the theorems of Jacquet-Shalika. In particular, it makes sense to consider the induced local parameter ψ_v at an infinite place v of E, cf. [Mok, §2.3].

Let us write $\mathbf{a}$, $\mathbf{b}$ for the infinity types of π_1, π_2, respectively, cf. §3.1. We recall that we have defined a tuple of permutations $\mathbf{w} = (w_\tau)_{\tau : E \hookrightarrow \mathbb{C}}$ in terms of these infinity types. The Langlands parameters of π_1, π_2 are the infinite place induced by an embedding $\tau : E \hookrightarrow \mathbb{C}$ are given by homomorphisms

$$z \mapsto ((z/\overline{z})^{a_{\tau,1}}, (z/\overline{z})^{a_{\tau,2}}),$$

$$z \mapsto ((z/\overline{z})^{b_{\tau,1}}, (z/\overline{z})^{b_{\tau,2}}, \dots, (z/\overline{z})^{b_{\tau,n-2}}).$$

Let $\psi_{v,H} : W_{\mathbb{R}} \to {}^L H$ be the sum of our two parameters, uniquely extended to $W_{\mathbb{R}}$. Let σ_v denote the trivial representation of $G(F_v)$ for $v|\infty$, associated to the parameter

$$\psi_v : z \mapsto ((z/\overline{z})^{(n-1)/2}, \dots, (z/\overline{z})^{(1-n)/2})$$

(extended to $W_{\mathbb{R}}$). There is a spectral transfer factor $\Delta_v(\psi_{v,H}, \sigma_v)$ satisfying the identity, for f_v, f_v^H associated:

$$\langle \Theta_{\psi_{v,H}}, f_v^H \rangle = \Delta_v(\psi_{v,H}, \sigma_v) \langle \Theta_{\sigma_v}, f_v \rangle.$$

In this identity $\Theta_{\psi_{v,H}}$ is the stable character on $H(F_v)$ associated to the L-packet given by $\psi_{v,H}$; Θ_{σ_v} is the character of the trivial representation. (We note once and for all that in the identity

$$SO_\delta(f^H) = \sum_\gamma \Delta(\gamma, \delta) O_\gamma(f)$$

there is an implicit choice of Haar measures on $H(F_v)$ and $G(F_v)$. The same measures are used to define the integrals against $\Theta_{\psi_{v,H}}$ and Θ_{σ_v}.)

Lemma 3.12. *For any $v|\infty$, $\Delta(\psi_{v,H}, \sigma_v) = \det w_\tau$, where $\tau : E \hookrightarrow \mathbb{C}$ is an embedding inducing the place v of F.*

Proof. This follows immediately from the exposition in [Clo11]. ☐

3.7 Spectral transfer factors, p-adic places

We now describe the character identities at the p-adic primes v_i, for the particular representations which will concern us. We first recall the characterization of the L-packet $\{X_i, Y_i\}$ associated to the representation $\Pi_E = \mathrm{St}_{2,E_{w_i}} \boxplus \mathrm{St}_{n-2,E_{w_i}}$. Recall that stable base change associates to $f \in C_c^\infty(G^*(F_{v_i}))$ a function $f_E \in C_c^\infty(\mathrm{GL}_n(E_{w_i}))$, characterized by its stable orbital integrals. Then

$$\langle \operatorname{tr} X_i + \operatorname{tr} Y_i, f \rangle = \langle \Pi_E \times I_c, f_E \rangle$$

where I_c is the intertwining operator $\Pi_E \cong \Pi_E^c$, normalized by the Whittaker model, cf. [Mok, Theorem 3.2.1]. Consider now the endoscopic group $H = U(2) \times U(n-2)$. The parameter ψ_{v_i} can be seen as a parameter $\psi_{v_i,H}$ for H, which defines the tensor product St_H of the two Steinberg representations.

Proposition 3.13. *For any $f \in C_c^\infty(G(F_{v_i}))$, we have*

$$\langle \operatorname{tr} Y_i - \operatorname{tr} X_i, f \rangle = \langle \epsilon_i(X_i) \operatorname{tr} X_i + \epsilon_i(Y_i) \operatorname{tr} Y_i, f \rangle = \langle \operatorname{tr} \mathrm{St}_H, f^H \rangle.$$

Before sketching the proof we note that this is plausible. One property of the signs ϵ_i (for the Whittaker normalization) is that we should have $\epsilon(Z) = 1$ when Z is the representation in the L-packet for G having a Whittaker model. The computation of the Jacquet modules shows that Y_i is the "bigger" representation in the L-packet. Presumably it has a Whittaker model, although we have not checked this.

Proof. We sketch the proof in the case $n = 6$; it will be clear that the proof extends. For the duration of this proof we also simplify notation by removing the dependence of the various objects on the subscript i. Thus

$$X_{N_0}^{\mathrm{norm}} = [3, 1, -1]$$

$$(Y_{N_0}^{\mathrm{norm}})^{\mathrm{ss}} = [3, 1, -1] + 2[3, 1, 1] + [1, 3, 1].$$

Let $T_0 = Z_{U_n}(S)$ denote the maximal torus consisting of elements

$$\mathrm{diag}(t_1, t_2, t_3, \overline{t_3}^{-1}, \overline{t_2}^{-1}, \overline{t_1}^{-1}),\ t_i \in E^\times.$$

The torus T_0 (or a stably conjugate torus in H) has trivial Galois cohomology, so the relation between f and f^H on elements conjugate to this torus is simply:

$$O_\delta(f^H) = \Delta(\delta, \gamma) O_\gamma(f)$$

($\delta \in H, \gamma \in G$ regular semisimple and associated).

We choose f to be supported on the G-conjugates of the following subset of T_0:

$$T_0^+ = \{(t_1, t_2, t_3) \mid |t_1| < |t_2| < |t_3| < 1\}.$$

Similarly in H we have

$$T_0^+(H) = \{(t_1, t_2, t_3) \mid |t_1| < 1, |t_2| < |t_3| < 1\}.$$

Assume that f is such a function on a quasi-split group G' (which may be G or H), and let π be an admissible representation of G'. The identity [Clo90a, (2.4)] (note that G need not be unramified there) yields

$$\langle \mathrm{tr}\,\pi, f\rangle = \int_{t \in T_0^+} \Theta(\pi_{N_0}^{\mathrm{norm}})(t)(\delta_{P_0}^{-1/2}\Delta)(t) O_t(f)\, dt.$$

Here $\pi_{N_0}^{\mathrm{norm}} = \pi_{N_0}\delta_{P_0}^{-1/2}$ is the normalized Jacquet module, and $\Theta(\pi_{N_0}^{\mathrm{norm}})$ is its trace. We have used the identity

$$\overline{f}^{P_0}(t) = \delta_{P_0}^{-1/2}(t)\Delta(t)O_t(f)$$

(t regular in T_0), where $\Delta(t) = |\det(1 - \mathrm{Ad}_{\mathfrak{n}_0}(t))|$, cf. [Clo85, Lemme 1]. Moreover, $\delta_{P_0}^{-1/2}(t)\Delta(t) = D(t) = \prod_{\alpha>0} |t^{\alpha/2} - t^{-\alpha/2}|$.

We first apply this to G, giving for such functions:

$$\langle \mathrm{tr}\, Y - \mathrm{tr}\, X, f\rangle = \int_{t\in T_0^+} (2e_1 + e_2)(t) D_G(t) O_t(f) dt,$$

where $e_1 = [3, 1, 1]$ and $e_2 = [1, 3, 1]$ are characters of T_0.

Consider now f^H on H. The exponent of the normalized Jacquet module of $\mathrm{St}_H = \mathrm{St}_{U(2)} \otimes \mathrm{St}_{U(4)}$ is, with the same notation, $e = [1; 3, 1]$. The orbital integrals of f^H need not be supported in $T_0^+(H)$. However, we can write $f^H = \sum_w \chi_w f^H = \sum f_w^H$, where $w \in W(H, T_0)$ and χ_w is the characteristic function of the set of elements contracting $w \cdot N_0$. An easy calculation then shows that the formula

$$\langle \mathrm{tr}\,\mathrm{St}_H, f^H\rangle = \int_{t\in T_0^+(H)} e(t) D_H(t) O_t(f^H) dt$$

remains true (note that $D_H(t)$ is invariant). Now if $t = (t_1, t_2, t_3) \in T_0^+$ then it has three distinct conjugates in $T_0^+(H)$, namely

$$(t_1, t_2, t_3),\ (t_2, t_1, t_3),\ \text{and}\ (t_3, t_1, t_2).$$

We then see
$$\langle \operatorname{tr}\operatorname{St}_H, f^H\rangle = \int_{t\in T_0^+}(2e_1+e_2)(t)D_H(t_H)O_{t_H}(f^H)dt,$$
where $t \in T_0^+$ comes from $t_H \in T_0^+(H)$, $e(t_1,t_2,t_3) = e_2(t)$ and $e(t_2,t_1,t_3) = e(t_3,t_1,t_2) = e_1(t)$. Thus we have to check the identity
$$D_H(t_H)O_{t_H}(f^H) = D_G(t)O_t(f),$$
where f, f^H are related by the identity $O_{t_H}(f^H) = \Delta(t_H,t)O_t(f)$. Recall that
$$\Delta(t_H,t) = \Delta_I\Delta_{II}\Delta_{III,1}\Delta_{III,2}\Delta_{IV},$$
where Δ_{IV} is simply $D_G(t)/D_H(t_H)$. We check that the other factors are equal to 1, for the Langlands-Shelstad transfer factors (quasi-split case) of [LS87], which coincide with the Whittaker-normalized transfer factors in our case, cf. §3.5.

The factors Δ_I [LS87, p. 241] and $\Delta_{III,1}$ [LS87, p. 245] are defined by a cup-product with $H^1(F,T_{\mathrm{sc}})$; here $T = T_0$, $T_{\mathrm{sc}} = T\cap SU_6$ and $H^1(F,T_{\mathrm{sc}}) = \{1\}$ (apply Hilbert 90 twice). The factor Δ_{II} is defined in [LS87, p. 243]. It requires the choice of data $a_\alpha \in E^\times$ and χ_α (a character of $E^\times$) for the roots α of (G,T_0). In our case it is easily checked that $a_\alpha = 1, \chi_\alpha = 1$ are suitable and then $\Delta_{II} = 1$, by definition.

There remains the term $\Delta_{III,2}$ [LS87, p. 246 –247], possibly the most complicated. However [LS87, Definition p. 247], we have
$$\Delta_{III,2}(t_H,t) = \chi(t_H),$$
where χ is a character of $T_0(H)$ defined by Langlands functoriality for tori. By [Mok, Theorem 3.2.1], we have only to check a sign. Thus we know, after the preceding computations, that
$$\int_{T_0^+}(2e_1+e_2)(t)\varphi_G(t)dt = \pm\int_{T_0^+}(2e_1+e_2)(t)\varphi_G(t)\chi(t_H)dt$$
where however (cf. [LS87]) the character χ may depend on the isomorphism $T_0 \cong T_0(H)$, i.e. on the 'Weyl chamber' in T_0^+. Here we have written $\varphi_G(t) = D_G(t)O_t(f)$. The function $\varphi_G(t)$ can be an arbitrary smooth, compactly supported function in the Weyl chamber. A character χ cannot be equal to -1 in $\{|t_1| < 1, |t_2| < |t_3| < 1\}$. This completes the proof. □

3.8 Transfer

We now note that by [Lab11a, Proposition 5.6] there is an identity
$$T^G_{\mathrm{disc}}(f) = \sum_{\mathcal{E}}\iota(G,\mathcal{E})ST^{\mathcal{E}}_{\mathrm{disc}}(f^{\mathcal{E}})$$
for f, f^H which are associated, the sum being over the elliptic endoscopic data of G. The terms $ST^{\mathcal{E}}_{\mathrm{disc}}(f^{\mathcal{E}})$ have been computed by Mok, cf. [Mok, Theorem 5.1.2]. If $f = \otimes_v f_v = f^\infty f_\infty = f^S f_S$ is chosen so that f_∞ is an Euler-Poincaré function for the trivial representation of $G(F_\infty)$ (e.g. the constant function), then f traces in only finitely many automorphic representations of G. By a separation of eigenvalues argument (cf. [CHL11, p. 487]), we can even choose f^S so that the only non-trivial contributions in the above formula come from the groups G^* and H, and only the parameter ψ contributes in the expression for the stable trace of [Mok, Theorem 5.1.2].

Let us first suppose that the parameter ψ corresponds to a RACSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$, as in the theorem. In this case we obtain a formula (Mok's notation):
$$T^G_{\mathrm{disc}}(f) = f^{G^*}(\psi) = \prod_{v\nmid\infty} f_v^{G^*}(\psi_v).$$
(Strictly speaking we use here the analogue of Proposition 3.12 for the endoscopic group G^* of G, which states that $\Delta_v(\psi_{v,G^*},\sigma_v) = 1$ for each infinite place v of F.) The theorem now follows in this case from the identity
$$f^G_{v_i}(\psi_{v_i}) = \langle \operatorname{tr} X_i + \operatorname{tr} Y_i, f_{v_i}\rangle,$$

when $\pi_{w_i} \cong \mathrm{St}_{2,E_{w_i}} \boxplus \mathrm{St}_{n-2,E_{w_i}}$.

Now suppose that the parameter ψ corresponds to a sum $\pi = \pi_1 \boxplus \pi_2$ of RACSDC automorphic representations of $\mathrm{GL}_2(\mathbb{A}_E)$ and $\mathrm{GL}_{n-2}(\mathbb{A}_E)$, respectively, such that π is ι-ordinary and regular algebraic of weight zero, as in the theorem. It follows that π_1 and π_2 are ι-ordinary and the infinity types $\mathbf{a}$, $\mathbf{b}$ satisfy the following condition (cf. Lemma 3.5):

– Let $\mathbf{w} = (w_\tau)_{\tau:E\hookrightarrow\mathbb{C}} \in \mathfrak{S}_n^{\mathrm{Hom}(E,\mathbb{C})}$ be the tuple of permutations associated to π_1, π_2, cf. §3.1. Then w_τ depends only on the place of E induced by the embedding $\iota^{-1}\tau : E \hookrightarrow \overline{\mathbb{Q}}_l$.

Choose for each place v of F dividing l an embedding $\tau(v) : E \hookrightarrow \mathbb{C}$ such that $\iota^{-1}\tau$ induces the place v. We have by Lemma 3.12 a formula

$$\prod_{v|\infty} \Delta(\psi_{v,H}, \sigma_v) = \prod_{v|l} \det w_{\tau(v)}^{[F_v:\mathbb{Q}_l]}.$$

Since the local degrees $[F_v : \mathbb{Q}_l]$ are even by hypothesis, this product is equal to 1 and we obtain a formula

$$T^G_{\mathrm{disc}}(f) = (f^{G^*}(\psi) + f^H(\psi_H))/2 = \left(\prod_{v\nmid\infty} f_v^G(\psi_v) + \prod_{v\nmid\infty} f_v^H(\psi_{v,H}) \right) /2.$$

The contribution from the places $v_0, \dots, v_s$ is (by Proposition 3.13):

$$\left(\prod_{i=0}^{s} f_{v_i}^G(\psi_{v_i}) + \prod_{i=0}^{s} f_{v_i}^H(\psi_{v_i,H}) \right) /2 = \sum_{\mathbf{d}} \left[(1 + \prod_{i=0}^{s} \epsilon_i(d_i))/2 \prod_{i=0}^{s} \langle \mathrm{tr}\, d_i, f_{v_i} \rangle \right],$$

where the notation $\mathbf{d}$ is as in the statement of the theorem. This completes the proof.

4 Raising the level

Let E be an imaginary CM field with totally real subfield F. We fix a prime $l \geq 5$ and an isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$. Let $n = l + 1$. We make the following hypotheses:

– For each place $v|l$ of F, v splits in E and $[F_v : \mathbb{Q}_l]$ is even. In particular, $[F : \mathbb{Q}]$ is even and there exists a unitary group G in n variables over F such that $G(F_\infty)$ is compact and G is quasi-split at every finite place.

– Let $v_0, \dots, v_s$ be the places of F ramified in E. Then $s \geq 1$, $q_{v_0} \equiv -1 \bmod l$ and for each $i = 1, \dots, s$, l does not divide $q_{v_i}(q_{v_i} + 1) \prod_{j=1}^{n/2-2} (q_{v_i}^j - 1)$. (This will be the case if, for example, q_{v_i} is a primitive root modulo l.)

We write $w_0, \dots, w_s$ for the places of E above $v_0, \dots, v_s$. We fix RACSDC automorphic representations π_2, π_{l-1} of $\mathrm{GL}_2(\mathbb{A}_E)$ and $\mathrm{GL}_{l-1}(\mathbb{A}_E)$, respectively, satisfying the following hypotheses:

– π_2 and π_{l-1} are ι-ordinary. For each embedding $\tau : E \hookrightarrow \mathbb{C}$, the infinity type of π_{l-1} at τ is

$$((n-3)/2, (n-5)/2, \dots, (5-n)/2, (n-3)/2)$$

and the infinity type of π_2 at τ is

$$((n-1)/2, (1-n)/2).$$

– The residual representations $\overline{r}_2 = \overline{r_\iota(\pi_2)}$ and $\overline{r}_{l-1} = \overline{r_\iota(\pi_{l-1})}$ are irreducible.

– For each $i = 0, \dots, s$ we have isomorphisms

$$\pi_{2,w_i} \cong \mathrm{St}_{2,E_{w_i}} \text{ and } \pi_{l-1,w_i} \cong \mathrm{St}_{l-1,E_{w_i}}.$$

The residual representations $\overline{r}_2|_{G_{E_{w_i}}}$ and $\overline{r}_{l-1}|_{G_{E_{w_i}}}$ are ramified and send a generator of tame inertia at each of these places to a regular unipotent element (that is, having a single Jordan block).

– Any finite place $w \neq w_0, \dots, w_s$ of E at which π_2 or π_{l-1} is ramified is split over F.

In this section we intend to prove the following theorem.

Theorem 4.1. *With hypotheses as above, there exists a RACSDC automorphic representation* Π *of* $\mathrm{GL}_n(\mathbb{A}_E)$ *which is* ι*-ordinary of weight 0, such that* $\overline{r_\iota(\Pi)} \cong \overline{r_\iota(\pi_2 \boxplus \pi_{l-1})}$*, and such that* Π_{w_0} *is an unramified twist of the Steinberg representation.*

In §2.2 we introduced L-packets $\{X_i, Y_i\}$ of representations of the groups $G(F_{v_i})$ corresponding to the representations $\mathrm{St}_{2,E_{w_i}} \boxplus \mathrm{St}_{l-1,E_{w_i}}$ of $\mathrm{GL}_n(E_{w_i})$. Let $\mathfrak{B}_{v_i}$ denote an Iwahori subgroup of $G(F_{v_i})$. Then these representations are characterized within their L-packet by the equations $\dim X_i^{\mathfrak{B}_{v_i}} = 1$, $\dim Y_i^{\mathfrak{B}_{v_i}} = n/2+1$, cf. Theorem 2.7. By Theorem 3.11, there exists an automorphic representation σ_0 of $G(\mathbb{A}_F)$ with base change $\pi_2 \boxplus \pi_{l-1}$. We observe that σ_0 is ι-ordinary, by Lemma 3.5.

We define an open compact subgroup $U_1 = \prod_v U_{1,v}$ of $G(\mathbb{A}_F^\infty)$ as follows:

– $U_{1,v_0} = \mathfrak{P}_{v_0}$, the subgroup containing the Iwahori subgroup defined in §2.1.
– For each $i = 1, \ldots, s$, $U_{1,v_i} = \mathfrak{B}_{v_i}$.
– For each place $v|l$ of F, choose a place $\widetilde{v}$ of E above it, and set $U_{1,v} = \iota_{\widetilde{v}}^{-1} \mathrm{Iw}_c(\widetilde{v})$ for some integer $c > 0$.
– For each place v of F inert in E, $U_{1,v}$ is a hyperspecial maximal compact subgroup of $G(F_v)$.
– For some place v, $U_{1,v}$ contains no non-trivial elements of finite order. (This condition is sometimes referred to by saying that U_1 is sufficiently small.)
– $\sigma_0^{U_1} \neq 0$.

We define another open compact subgroup $U = \prod_v U_v$ by the formulae $U_{v_0} = \mathfrak{B}_{v_0}$ and $U_v = U_{1,v}$ if $v \neq v_0$. Thus $U \subset U_1$ and $[U_1 : U] = [U_{1,v_0} : U_{v_0}] = q_{v_0} + 1$.

Fix a finite extension $K \subset \overline{\mathbb{Q}}_l$ of $\mathbb{Q}_l$, with ring of integers $\mathcal{O}$, residue field k, and maximal ideal λ. We write $S(U_1, \mathcal{O})$ for the space of automorphic forms on G with trivial coefficients and level U_1, as defined in §3.3. Let T denote the set of finite places of F above which E of π is ramified or such that $U_{1,v}$ is not hyperspecial maximal compact. (Thus T contains the places dividing l.) We then define the Hecke algebra $\mathbb{T}(U_1, \mathcal{O})$ to be the $\mathcal{O}$-subalgebra of $\mathrm{End}_{\mathcal{O}}(S(U_1, \mathcal{O}))$ generated by the unramified Hecke operators at places of F not in T and split in E, and the operators U_v^j for each $v|l$. It is a finite flat $\mathcal{O}$-algebra. (We recall that the definition of U_v^j depends on a choice of place $\widetilde{v}$ of E above v and a uniformizer of $E_{\widetilde{v}}$, but these choices play no role here.)

The representation σ_0 gives rise to a homomorphism $\mathbb{T}(U_1, \mathcal{O}) \to \overline{\mathbb{F}}_l$, and we write $\mathfrak{m}$ for the kernel of this homomorphism. Then $S(U_1, \mathcal{O})_\mathfrak{m}$ is an $\mathcal{O}$-direct summand of $S(U_1, \mathcal{O})$, and every automorphic representation σ of $G(\mathbb{A}_F)$ which contributes to $S(U_1, \mathcal{O})_\mathfrak{m}$ is ι-ordinary. Moreover, $\mathbb{T}(U_1, \mathcal{O})_\mathfrak{m} \otimes_\mathcal{O} K$ is a semisimple K-algebra.

Now suppose that $V = \prod_v V_v \subset U_1$ is an open compact subgroup such that for each place v of F such that either $v|l$ or $v \notin T$, $V_v = U_{1,v}$. We can define the space $S(V, \mathcal{O})$ and Hecke algebra $\mathbb{T}(V, \mathcal{O})$ and a natural surjective homomorphism $\mathbb{T}(V, \mathcal{O}) \to \mathbb{T}(U_1, \mathcal{O})$. In an abuse of notation, we will write $\mathfrak{m}$ also for the pullback of this maximal ideal to $\mathbb{T}(V, \mathcal{O})$.

Using Theorem 3.11, we see that there is a direct sum decomposition

$$S(U, \mathcal{O})_\mathfrak{m} \otimes_\mathcal{O} \overline{\mathbb{Q}}_l = \bigoplus_\sigma (\iota^{-1}\sigma^\infty)^{U,\mathrm{ord}} = \bigoplus_\pi \bigoplus_{\substack{\sigma \\ \mathrm{BC}(\sigma)=\pi}} (\iota^{-1}\sigma^\infty)^{U,\mathrm{ord}}.$$

Here the first sum in the third term runs over automorphic representations π of $\mathrm{GL}_n(\mathbb{A}_E)$. The second, inner, sum in the third term runs over automorphic representations σ of $G(\mathbb{A}_F)$ which contribute to $S(U, \mathcal{O})_\mathfrak{m}$ and such that π is the base change of σ. We will say that a representation π for which the π-summand in the above expression is non-trivial is *relevant.*

Proposition 4.2. *Let* π *be relevant. Then one of the following is true:*

– $\pi = \pi_a \boxplus \pi_b$*, where* π_a*,* π_b *are RACSDC automorphic representations of* $\mathrm{GL}_2(\mathbb{A}_E)$*,* $\mathrm{GL}_{l-1}(\mathbb{A}_E)$*, respectively, and for each* $i = 0, \ldots, s$*,* $\pi_{w_i} \cong \mathrm{St}_{2,E_{w_i}} \boxplus \mathrm{St}_{l-1,E_{w_i}}$*.*
– π *is cuspidal and for each* $i = 0, \ldots, s$*,* $\pi_{w_i} \cong \mathrm{St}_{2,E_{w_i}} \boxplus \mathrm{St}_{l-1,E_{w_i}}$*.*
– π *is cuspidal and* π_{w_0} *is an unramified twist of the Steinberg representation, and for each* $i = 1, \ldots, s$*,* $\pi_{w_i} \cong \mathrm{St}_{2,E_{w_i}} \boxplus \mathrm{St}_{l-1,E_{w_i}}$*.*

Proof. Let π be as in the statement of the theorem. By Proposition 3.7, we can write $\pi \cong \pi_1 \boxplus \cdots \boxplus \pi_r$, where the π_i are discrete and conjugate self-dual automorphic representations of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$. Let $\rho = r_\iota(\pi)$. Then $\overline{\rho}$ is a direct sum of two irreducible representations of distinct dimensions. In particular, we must have either $r = 1$ and π is cuspidal, or $r = 2$, $n_1 = 2$, $n_2 = n - 2 = l - 1$ and π_1 and π_2 are both cuspidal. In this case π_1 and π_2 are also regular algebraic. We now apply the following.

Lemma 4.3. *Let π be relevant. Then for each $i = 0, \ldots, s$, π_{w_i} has an Iwahori-fixed vector.*

Proof. We fix i to be one of $0, \ldots, s$ for the duration of the proof. Assume first that π is cuspidal. By the identity at the beginning of §3.9 and [Lab11a, Theorem 4.12] we obtain, after separation of Hecke eigenvalues:

$$\sum_\sigma \langle \mathrm{tr}\, \sigma, f \rangle = \langle \mathrm{tr}\, \pi \times I_c, f_E \rangle.$$

Here f is a function on $G(F_\infty \times \prod_{i=0}^s F_{v_i})$, f_E is the function on $G(E_\infty \times \prod_{i=0}^s E_{w_i})$ associated to f by stable base change, and σ runs over the local components of automorphic representations of $G(\mathbb{A}_F)$ associated to π. We may further assume that $f_\infty = 1$. (If we use Mok's full results then the sum is finite, each term occurring with multiplicity one, since the same identity obtains for G^*, isomorphic to G at the finite places. We do not need this.)

Now fix $v = v_i$, $w = w_i$ for some $i = 0, \ldots, s$. Choosing the functions for $v' \neq v$ suitably we obtain

$$c \sum_\sigma \langle \mathrm{tr}\, \sigma, f_v \rangle = \langle \mathrm{tr}\, \pi_w \times I_c, f_{E_w} \rangle,$$

with $c \neq 0$. The representation of $G(\mathbb{A}_F)$ is admissible, so the left-hand side contains a finite number of semistable representations σ with finite multiplicity. (A semistable representation is, by definition, one that has a non-zero Jacquet module for N_0, composed of unramified characters.)

Consider a function f_v with support in the elements contracting N_0. We may further assume that the constant term $\overline{f}^{P_0}(t)$ (cf. §3.8) is an unramified function. By Casselman's theorem (§3.8), the left-hand side is then a finite sum, over the semistable representations:

$$c \sum_\sigma \langle \mathrm{tr}\, \sigma_{N_0}^{\mathrm{norm}}, \overline{f}^{P_0} \rangle_{T_0}.$$

By assumption, the sum contains a representation σ_0 such that $(\sigma_0)_{N_0}$ is a sum of unramified characters; note that there is no cancellation in the sum. However, the identity of orbital integrals shows that we can take for f_{E_w} a function whose orbital integrals have the same property. The right-hand side of the identity is then equal to

$$\langle \mathrm{tr}\, \pi_{N_0(E)}^{\mathrm{norm}}, (\overline{f_{E_w}})^{P_0} \rangle$$

and this implies that π_w is semistable. (Moeglin [Mœg07] shows that in fact the resulting identity of Jacquet modules extends from the contracting elements to all of T_0 and $T_0(E)$.)

Consider now the case where $\pi = \pi_2 \boxplus \pi_{n-2}$ with π_2, π_{n-2} cuspidal. In this case the relevant equality is given by §3.9. The sum $\sum_\sigma \langle \mathrm{tr}\, \sigma, f \rangle$ is equal to the sum of two terms, one pertaining to G^*:

$$1/2 \langle \mathrm{tr}\, \pi \times I_c, f_E \rangle,$$

where however π is an Eisenstein representation $\pi = \pi_2 \boxplus \pi_{n-2}$. This is the term (4.4.2) in [CHL11]; the proof is identical. The second term is

$$1/2 \langle \mathrm{tr}(\pi_2 \otimes \pi_{n-2}) \times I_c, f_E^H \rangle,$$

where $H = U(2) \times U(n-2)$ is the endoscopic group of our datum. We choose f_v and the $f_{v'}$ for $v' \neq v$ as above, so the previous argument applies to $\sum_\sigma$, non-zero by assumption. If the first term does not vanish, $\pi_2 \boxplus \pi_{n-2}$ and therefore π_2, π_{n-2} are semistable. If the H-term does not vanish, the computation of the transfer in §3.8 shows that we may choose f^H unramified, with regular support, thus also f_E^H, the transfer being obvious on the split torus. Again this implies that π_2 and π_{n-2} are semistable. $\square$

We now return to the proof of Proposition 4.2. Suppose first that $r = 2$, so that $\rho = \rho_a(1 - n/2) \oplus \rho_b(-1)$, where $\rho_a = r_\iota(\pi_a)$ and $\rho_b = r_\iota(\pi_b)$. The hypotheses on the residual representations $\overline{\rho}_a \cong \overline{r}_2$, $\overline{\rho}_b \cong \overline{r}_{l-1}$ now imply that for each $i = 0, \ldots, s$, the representation π_{a,w_i} (resp. π_{b,w_i}) is an unramified twist of $\mathrm{St}_{2,E_{w_i}}$ (resp. $\mathrm{St}_{l-1,E_{w_i}}$). Indeed, it is easy to see that since π_{w_i} has an Iwahori-fixed vector, the same must be true for the representations π_{a,w_i} and π_{b,w_i}. We therefore have, for example, an isomorphism

$$\pi_{b,w_i} \cong \mathrm{St}_{b_1,E_{w_i}}(\psi_1) \boxplus \cdots \boxplus \mathrm{St}_{b_t,E_{w_i}}(\psi_t),$$

where $b_1 + \cdots + b_t = n - 2 = l - 1$ and each $\psi_1, \ldots, \psi_t : E_{w_i}^\times \to \mathbb{C}^\times$ is an unramified character. Let $t_{w_i} \in I_{w_i}$ denote a generator of the l-part of tame inertia. Local-global compatibility in its strong form (cf. [Car12]) now implies that $\rho_b(t_{w_i})$ is a unipotent matrix with Jordan form corresponding to the partition $b_1 + \cdots + b_t = l - 1$. After conjugating and possibly enlarging K, we can assume that ρ_b takes values in $\mathrm{GL}_{l-1}(\mathcal{O})$, and that the composite $G_E \to \mathrm{GL}_{l-1}(\mathcal{O}) \to \mathrm{GL}_{l-1}(k)$ is equal to $\overline{r}_{l-1}$. By hypothesis, $\overline{r}_{l-1}(t_{w_i})$ is a regular unipotent matrix. It follows that we must have $t = 1$, and then π_{b,w_i} is an unramified twist of the Steinberg representation, as claimed. To see that the first bullet point holds in this situation, we must check that these twists are all actually trivial. To do this we look at the Frobenius eigenvalues of $\overline{\rho}_a$ and $\overline{\rho}_b$. Let us treat, for example, $\pi_{b,w_0} \cong \mathrm{St}_{l-1,E_{w_0}}(\psi)$. Since $\psi\psi^c = 1$ (as π_b is of unitary type) and $\psi = \psi^c$ (as ψ is unramified), we see that $\psi^2 = 1$ and we must rule out the case that ψ is the non-trivial unramified character of order 2.

Let ϖ_{w_0} be a uniformizer of E_{w_0}, and let $N = \rho_b(t_{w_0}) - 1 \in M_{l-1}(\mathcal{O})$. Then $N \bmod \lambda$ is a regular nilpotent element, by hypothesis, and the natural map

$$(\ker N) \otimes_{\mathcal{O}} k \to \ker(N \bmod \lambda)$$

is an isomorphism. In particular, $\overline{\rho}_b$ preserves the line $\ker(N \bmod \lambda)$ and Frobenius acts with eigenvalue

$$\iota^{-1}\psi(\varpi_{w_0})\epsilon(\mathrm{Frob}_{w_0})^{-1} q_{w_0}^{l-2} \equiv \iota^{-1}\psi(\varpi_{w_0}) \bmod \lambda.$$

Since π_{l-1,w_0} is, by hypothesis, the untwisted Steinberg representation, performing the same calculation for $\overline{r}_{l-1}$ gives $\iota^{-1}\psi(\varpi_{w_0}) \equiv 1 \bmod \lambda$, and hence $\psi = 1$.

Now suppose that $r = 1$, so that π is cuspidal. Let $0 \leq i \leq s$. Since π_{w_i} has Iwahori-fixed vectors, there is an isomorphism

$$\pi_{w_i} \cong \mathrm{St}_{n_1,E_{w_i}}(\psi_1) \boxplus \cdots \boxplus \mathrm{St}_{n_t,E_{w_i}}(\psi_t)$$

for some $t \geq 1$, where the ψ_i are unramified characters of $E_{w_i}^\times$. The congruence $\overline{\rho} \cong \overline{r_\iota(\sigma_0)}$ implies that the nilpotent conjugacy class of GL_n corresponding to the partition $n = n_1 + \cdots + n_t$ specializes to the class corresponding to the partition $2 + (n-2)$. This rules out all but the possibilities $n = 2 + (n-2)$, $n = 1 + (n-1)$, and $n = n$. We must rule out the case $n = 1 + (n-1)$ and show that in case $n = 2 + (n-2)$ the characters ψ_1, ψ_2 are trivial, and that in case $n = n$ we necessarily have $i = 0$. This will complete the proof of the proposition.

To rule out the case $n = 1 + (n-1)$, we note that no representation $\psi_1 \boxplus \mathrm{St}_{n-1,E_{w_i}}(\psi_2)$ with ψ_1, ψ_2 unramified is in the image of the stable base change map, as the corresponding parameter is not conjguate symplectic, cf. Lemma 2.6. Suppose instead that $\pi_{w_i} \cong \mathrm{St}_{2,E_{w_i}}(\psi_1) \boxplus \mathrm{St}_{l-1,E_{w_i}}(\psi_2)$. After conjugating, we may assume that ρ takes values in $\mathrm{GL}_n(\mathcal{O})$ and that $\rho \bmod \lambda$ is semisimple. Let $N = \rho(t_{w_i}) - 1$. For each $j \geq 0$ the natural map $(\ker N^j) \otimes_{\mathcal{O}} k \to \ker(N^j \bmod \lambda)$ is an isomorphism and comparing the eigenvalues of Frobenius on $\ker(N \bmod \lambda)$ of $\overline{\rho}$ and $\overline{r}_2(1 - n/2) \oplus \overline{r}_{l-1}(-1)$, we get

$$\iota^{-1}\psi_1(\varpi_{w_i}) q_{w_i}^{n/2} \equiv q_{w_i}^{n/2} \bmod \lambda \text{ and } \iota^{-1}\psi_2(\varpi_{w_i}) q_{w_i}^{l-1} \equiv q_{w_i}^{l-1} \bmod \lambda.$$

We again have $\psi_1^2 = \psi_2^2 = 1$. It follows that ψ_1 and ψ_2 are both trivial. Finally, suppose that we have $\pi_{w_i} \cong \mathrm{St}_{n,E_{w_i}}(\psi)$ for some unramified character $\psi : E_{w_i}^\times \to \mathbb{C}^\times$. Comparing the Frobenius eigenvalues at w_i of ρ and $r_\iota(\sigma_0)$ shows that

$$\iota^{-1}\psi(\varpi_{w_i})\{q_{w_i}^l, q_{w_i}^{l-1}, \ldots, q_{w_i}, 1\} \equiv \{q_{w_i}^{n/2}, q_{w_i}^{n/2-1}, q_{w_i}^{l-1}, q_{w_i}^{l-2}, \ldots, q_{w_i}\} \bmod \lambda,$$

where again $\psi(\varpi_{w_i}) = \pm 1$. Suppose for contradiction that $i > 0$. If $\psi(\varpi_{w_i}) = 1$ then the above equality of sets of eigenvalues cannot hold, since $q_{w_i}^{n/2-1} = q_{w_i}^{(l-1)/2} \equiv -1 \bmod l$. If $\psi(\varpi_{w_i}) = -1$, then the injection $(\ker N^3) \otimes_{\mathcal{O}} k \hookrightarrow \ker(N^3 \bmod \lambda)$ shows that

$$\{-q_{w_i}^l, -q_{w_i}^{l-1}, -q_{w_i}^{l-2}\} \bmod \lambda \subset \{q_{w_i}^{n/2}, q_{w_i}^{n/2-1}, q_{w_i}^{l-1}, q_{w_i}^{l-2}, q_{w_i}^{l-3}\} \bmod \lambda,$$

or equivalently

$$\{-q_{w_i}, -1, -1/q_{w_i}\} \bmod \lambda \subset \{-q_{w_i}, -1, 1, 1/q_{w_i}, 1/q_{w_i}^2\} \bmod \lambda.$$

It follows that $-1/q_{w_i} \bmod \lambda \in \{1, 1/q_{w_i}, 1/q_{w_i}^2\} \bmod \lambda$, again giving a contradiction. This completes the proof. ☐

Corollary 4.4. *Let π be relevant. Then each automorphic representation σ of $G(\mathbb{A}_F)$ with base change π and $(\sigma^\infty)^U \neq 0$ occurs with multiplicity one in the space of automorphic forms on G. Moreover, we have the following possibilities:*

1. *If π is not cuspidal then there are exactly 2^s such representations σ. We index them by a choice of element $\mathbf{d} = (d_1, \dots, d_s) \in \prod_{i=1}^s \{X_i, Y_i\}$. The corresponding automorphic representation $\sigma(\mathbf{d})$ is uniquely characterized by the condition*

   $$\sigma(\mathbf{d})_{v_i} \cong d_i, i = 1, \dots, s.$$

   It satisfies the condition

   $$\epsilon_0(\sigma(\mathbf{d})_{v_0}) \cdot \prod_{i=1}^s \epsilon_i(d_i) = 1,$$

   where $\epsilon_i : \{X_i, Y_i\} \to \{\pm 1\}$ is defined by $\epsilon_i(X_i) = -1, \epsilon_i(Y_i) = 1$.
2. *If π is cuspidal and $\pi_{w_0} \cong \mathrm{St}_{2,E_{w_0}} \boxplus \mathrm{St}_{l-1,E_{w_0}}$ then there are exactly 2^{s+1} such representations, corresponding as above to a choice of element of $\prod_{i=0}^s \{X_i, Y_i\}$.*
3. *If π is cuspidal and π_{w_0} is an unramified twist of the Steinberg representation then there are exactly 2^s such representations, corresponding as above to a choice of element of $\prod_{i=1}^s \{X_i, Y_i\}$.*

Proof. This follows from Proposition 4.2 and Theorem 3.11. ☐

This corollary has the following consequence. Let $\mathbf{d} = (X_1, \dots, X_s)$ if s is odd, and $\mathbf{d} = (Y_1, X_2, \dots, X_s)$ is s is even. Let π be relevant, and suppose that π is not cuspidal. Then $\sigma(\mathbf{d})_{v_0} \cong X_0$. We fix this choice of $\mathbf{d}$ for the remainder of this section. (In fact, any choice of $\mathbf{d} \in \prod_{i=1}^s \{X_i, Y_i\}$ with $\prod_{i=1}^s \epsilon_i(d_i) = -1$ would suffice for what follows.)

Now let $V \subset U$ be an open compact subgroup of the kind considered above. There is a perfect pairing

$$\langle \cdot, \cdot \rangle_V : S(V, \mathcal{O}) \times S(V, \mathcal{O}) \to \mathcal{O},$$

which satisfies the formula $\langle [V_v g_v V_v] x, y \rangle = \langle x, [V_v g_v^{-1} V_v] y \rangle$ for any $g_v \in G(F_v)$, $x, y \in S(V, \mathcal{O})$. This pairing need not restrict to a perfect duality on $S(V, \mathcal{O})_{\mathfrak{m}}$. In fact, for any automorphic representation σ of $G(\mathbb{A}_F)$ which contributes to $S(V, \mathcal{O})$, its admissible dual $\sigma^\vee$ also contributes. We write $\mathfrak{m}^\vee \subset \mathbb{T}(U, \mathcal{O})$ for the maximal ideal corresponding to the Hecke eigenvalues of $\sigma_0^\vee$. We have the following result.

Proposition 4.5.

– *The above pairing restricts to a perfect duality*

$$\langle \cdot, \cdot \rangle_{V,\mathfrak{m}} : S(V, \mathcal{O})_{\mathfrak{m}} \times S(V, \mathcal{O})_{\mathfrak{m}^\vee} \to \mathcal{O}.$$

– *The induced pairing*

$$\langle \cdot, \cdot \rangle_{U,\mathfrak{m}} : S(U, k)_{\mathfrak{m}} \times S(U, k)_{\mathfrak{m}^\vee} \to k$$

vanishes on restriction to the subspace $S(U_1, k)_{\mathfrak{m}} \times S(U_1, k)_{\mathfrak{m}^\vee}$. (Note that for any subgroup $V \subset U_1$, there are isomorphisms

$$S(V, \mathcal{O}) \otimes_{\mathcal{O}} k \cong S(V, k)$$

compatible with the action of Hecke operators.)

Proof. For the first part, we decompose

$$S(V,\mathcal{O})\otimes_{\mathcal{O}}\overline{\mathbb{Q}}_l\cong\bigoplus_{\sigma}(\iota^{-1}\sigma^\infty)^V.$$

A separation of eigenvalues argument shows that if $f\in(\iota^{-1}\sigma^\infty)^V, g\in(\iota^{-1}(\sigma')^\infty)^V$, then $\langle f,g\rangle_V=0$ unless $\sigma'\cong\sigma^\vee$. The claim of the proposition easily follows from this statement.

For the second part, let $i:S(U_1,k)\to S(U,k)$ denote the natural inclusion. This can be identified with the trivial Hecke operator for the pair of subgroups $U\subset U_1$, and so for any $f,g\in S(U_1,k)$ we find the formula

$$\langle if,ig\rangle_U=\langle f,i^*ig\rangle_{U_1},$$

where i^* denotes adjoint with respect to the different dualities. An easy calculation (cf. [Tay89, Lemma 2]) shows that i^*i, viewed as endomorphism of $S(U_1,k)$, is multiplication by $[U_1:U]=[\mathfrak{P}_{v_0}:\mathfrak{B}_{v_0}]=q_{v_0}+1\equiv 0 \bmod \lambda$. Restricting to the given subspace gives the desired result. □

We now come to the proof of Theorem 4.1. Suppose for contradiction that there are no relevant automorphic representations π such that π_{v_0} is an unramified twist of the Steinberg representation. The space $\mathcal{M}=S(U,\mathcal{O})_{\mathfrak{m}}$ receives commuting actions of the Iwahori-Hecke algebras $H_{\mathfrak{B}_{v_0},\mathcal{O}},\dots,H_{\mathfrak{B}_{v_s},\mathcal{O}}$. By Corollary 2.10, it admits a direct sum decomposition

$$\mathcal{M}=\bigoplus_{\mathbf{d}'\in\prod_{i=1}^s\{X_i,Y_i\}}\mathcal{M}(\mathbf{d}'),$$

each summand being characterized by the equality (the first sum running over relevant π):

$$\mathcal{M}(\mathbf{d}')\otimes_{\mathcal{O}}\overline{\mathbb{Q}}_l=\bigoplus_{\pi}\bigoplus_{\substack{\sigma\\ \mathrm{BC}(\sigma)=\pi\\ \sigma_{v_i}\cong d_i',i=1,\dots,s}}(\iota^{-1}\sigma^\infty)^{\mathrm{ord},U}.$$

By choice of $\mathbf{d}$, if σ appears in the decomposition of $\mathcal{M}(\mathbf{d})\otimes_{\mathcal{O}}\overline{\mathbb{Q}}_l$ and the base change of σ is not cuspidal, then $\sigma_{v_0}\cong X_0$. In particular, there is an isotypic decomposition of $H_{\mathfrak{B}_{v_0},\mathbb{C}}$-modules

$$\mathcal{M}(\mathbf{d})\otimes_{\mathcal{O},\iota}\mathbb{C}\cong(X_0^{\mathfrak{B}_{v_0}})^a\oplus(Y_0^{\mathfrak{B}_{v_0}})^b,$$

where $a>b$. Indeed, a,b can be calculated as follows. For each relevant automorphic representation π, let $\sigma(\pi)=\iota^{-1}\sigma(\mathbf{d})^{v_0l\infty}\otimes\bigotimes_{v|l}\iota^{-1}\sigma(\mathbf{d})_v^{\mathrm{ord}}$, an admissible representation of $G(\mathbb{A}_F^{v_0l\infty})$, where $\sigma(\mathbf{d})$ is as in Corollary 4.4. We have

$$a=\sum_{\pi}\dim\sigma(\pi)^{U^{v_0l}},\ b=\sum_{\pi\text{ cuspidal}}\dim\sigma(\pi)^{U^{v_0l}}.$$

Let $M=\mathcal{M}(\mathbf{d})$. We define $\mathcal{N}=S(U,\mathcal{O})_{\mathfrak{m}^\vee}$ and $N=\mathcal{N}(\mathbf{d})$ in an analogous manner. Let $M_1=M^{U_1}\subset M$, and $N_1=N^{U_1}\subset N$. The perfect duality $\langle\cdot,\cdot\rangle_{U,\mathfrak{m}}$ restricts to a perfect duality $\langle\cdot,\cdot\rangle:M\times N\to\mathcal{O}$ satisfying the relation $\langle hx,y\rangle=\langle x,\jmath(h)y\rangle$ for all $h\in H_{\mathfrak{B}_{v_0},\mathcal{O}}$, $x\in M$, $y\in N$. By Proposition 4.5, the induced perfect duality $M\otimes_{\mathcal{O}}k\times N\otimes_{\mathcal{O}}k\to k$ vanishes on the subspace $M_1\otimes_{\mathcal{O}}k\times N_1\otimes_{\mathcal{O}}k$.

We recall the abelian subalgebra $\mathcal{O}[\Lambda]\subset H_{\mathfrak{B}_{v_0},\mathcal{O}}$, cf. §2.1. If $\overline{\eta}:\mathcal{O}[\Lambda]\to k$ is a character of this algebra, we write $M(\overline{\eta})$ for its generalized eigenspace, i.e. the localization at $\ker\overline{\eta}$ as $\mathcal{O}[\Lambda]$-module. Given a homomorphism $\overline{\eta}:\mathcal{O}[\Lambda]\to k$ we obtain a new homomorphism $\overline{\eta}\jmath:\jmath\mathcal{O}[\Lambda]\to k$, and the pairing restricts to a perfect pairing $\langle\cdot,\cdot\rangle_{\overline{\eta}}:M(\overline{\eta})\times N(\overline{\eta}\jmath)\to\mathcal{O}$, where we write $N(\overline{\eta}\jmath)$ for the generalized $\overline{\eta}\jmath$-eigenspace of $\jmath\mathcal{O}[\Lambda]$.

By Theorem 2.7, the characters of $K[\Lambda]$ appearing amongst the Jordan-Hölder constituents of $M\otimes_{\mathcal{O}}K$ as $K[\Lambda]$-module are amongst the characters

$$[1,n-3,n-5,\dots,1],[n-3,\dots,n-1-2i,1,n-3-2i,\dots,1],\ i=1,\dots,n/2-2,\text{ and }[n-3,n-5,\dots,1,-1].$$

These all arise from characters $\mathcal{O}[\Lambda]\to\mathcal{O}$, and the last of these, the character $[n-3,n-5,\dots,1,-1]$, has distinct reduction modulo λ from the others. Write $\overline{\eta}_0$ for the character $\mathcal{O}[\Lambda]\to k$ arising from reduction modulo λ of this character. Then $M(\overline{\eta}_0)$ is a direct summand $\mathcal{O}[\Lambda]$-submodule of M and (in the notation of Proposition 2.9, with $X = X_0$ and $\mathfrak{B} = \mathfrak{B}_{v_0}$) $X_{\mathcal{O}}^{\mathfrak{B}} = X_{\mathcal{O}}^{\mathfrak{B}}(\overline{\eta}_0)$.

Let M_X denote the intersection of M with the X_0-isotypic piece of $M \otimes_{\mathcal{O},\iota} \mathbb{C}$. Thus $M_X \subset M$ is a finite free $\mathcal{O}$-module of rank a, and M/M_X is $\mathcal{O}$-torsion free. We have $M_X \subset M_1$, by Theorem 2.7 applied at the place v_0, which shows that $X_0^{\mathfrak{B}_{v_0}} = X_0^{\mathfrak{P}_{v_0}}$. Defining $N_X \subset N$ in the same manner, we have $N_X \subset N_1$ and N_X is a finite free $\mathcal{O}$-module of rank a. Moreover, we have $M_X = M_X(\overline{\eta}_0)$. We also have $N_X = N_X(\overline{\eta}_0 \jmath)$, since $X_{\mathcal{O}}^{\mathfrak{B}} = X_{\mathcal{O}}^{\mathfrak{B}}(\overline{\eta}_0 \jmath)$, by Proposition 2.8.

We now see that the perfect pairing

$$\langle \cdot, \cdot \rangle_{\overline{\eta}_0} : M(\overline{\eta}_0) \otimes_{\mathcal{O}} k \times N(\overline{\eta}_0 \jmath) \otimes_{\mathcal{O}} k \to k$$

vanishes on the subspace $M_X \otimes_{\mathcal{O}} k \times N_X \otimes_{\mathcal{O}} k$. By construction, $M(\overline{\eta}_0) \otimes_{\mathcal{O}} k$ has dimension $a + b$ as a k-vector space, and the subspaces $M_X \otimes_{\mathcal{O}} k$, $N_X \otimes_{\mathcal{O}} k$ have dimension a. Since they annihilate each other, we must therefore have $a \leq b$. This contradicts the assumption above that $a > b$, and this contradiction completes the proof of the theorem.

5 Construction of a special automorphic representation

Let E be an imaginary CM field with totally real subfield F, and let π be a RACSDC automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$ of weight zero. Let $l \geq 5$ be prime, and let $n = l + 1$. Fix an isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$. In order to reduce notation, we now write $\rho = r_\iota(\pi)$. We suppose that the following hypotheses are in effect.

- For each place $v|l$ of F, v is split in E and $[F_v : \mathbb{Q}_l]$ is even. Moreover, π is ι-ordinary.
- The residual representation $\overline{\rho} : G_E \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is irreducible, and its image contains $\mathrm{SL}_2(\mathbb{F}_{l^a})$ up to conjugation for some $a > 1$.
- The $(l-2)^{\text{th}}$ symmetric power of π exists, in the following sense: there exists a RACSDC automorphic representation π_{l-1} of $\mathrm{GL}_{l-1}(\mathbb{A}_E)$ such that $r_\iota(\pi_{l-1}) \cong \mathrm{Sym}^{l-2} \rho$.
- Let $y_0, \ldots, y_s$ denote the places of F ramified in E. Then $s \geq 1$. Let z_i denote the place of E above y_i. For each $i = 0, \ldots, s$, $\pi_{z_i} \cong \mathrm{St}_{2,E_{z_i}}$, $\overline{\rho}$ is ramified at z_i, and q_{z_i} is a primitive element modulo l, and is odd.
- There exists an everywhere unramified totally real quadratic extension F'/F, linearly disjoint over F from the extension of $E(\zeta_l)$ cut out by $\overline{\rho}$, in which each place $y_0, \ldots, y_s$ is inert. We write $\omega_{F'/F}$ for the corresponding quadratic character of G_F.
- If $w \neq z_0, \ldots, z_s$ is a place of E at which π is ramified, then w is split over F.

Let $\chi = \det \rho$, and let φ denote a continuous automorphism of $\overline{\mathbb{Q}}_l$ lifting the arithmetic Frobenius. There is an isomorphism (cf. [CT, §4]):

$$(\mathrm{Sym}^l \overline{\rho})^{\mathrm{ss}} \cong {}^{\varphi}\overline{\rho} \oplus \overline{\chi} \otimes \mathrm{Sym}^{l-2} \overline{\rho}.$$

In this section we will prove the following result.

Theorem 5.1. *There exists a soluble CM extension M/E, linearly disjoint over E from the extension of $E(\zeta_l)$ cut out by $\overline{\rho}$, and a RACSDC automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_M)$ satisfying the following:*

1. *Π is ι-ordinary of weight zero.*
2. *There is an isomorphism $\overline{r_\iota(\Pi)} \cong (\mathrm{Sym}^l \overline{\rho})^{ss}|_{G_M}$.*
3. *There exists a place w of M above z_0 such that Π_w is an unramified twist of the Steinberg representation.*

By Proposition 3.10 and base change for U_2, we can find a RACSDC automorphic representation π_2 of $\mathrm{GL}_2(\mathbb{A}_E)$ satisfying the following:

- π_2 is ι-ordinary. Writing $\rho' = r_\iota(\pi_2)$, we have $\mathrm{HT}_\tau(\rho') = \{(1-l)/2, (1+l)/2\}$ for every embedding $\tau : E \hookrightarrow \overline{\mathbb{Q}}_l$.
- For each $i = 0, \ldots, s, \pi_{2,z_i}$ has an Iwahori-fixed vector.
- $\overline{\rho}' \cong \omega^{(l-1)/2} \otimes \overline{\rho}$, where ω denotes the Teichmüller lift of the mod l cyclotomic character.
- If π_2 is ramified at a place $w \neq z_0, \ldots, z_s$ of E, then w is split over F.

Then the representation $\pi_{l+1} = {}^{\varphi}\pi_2 \boxplus \pi_{l-1} \otimes \iota\epsilon\chi$ is conjugate self-dual and regular algebraic of weight zero, and we have

$$r_\iota(\pi_{l+1}) = \epsilon^{(1-l)/2} \otimes {}^{\varphi}\rho' \oplus \chi \otimes \mathrm{Sym}^{l-2}\rho.$$

In particular, the reduction modulo l of this representation is the same as that of $\mathrm{Sym}^l \rho$. For each $i = 0, \ldots, s$ there is an isomorphism $\pi_{l+1,z_i} \cong \mathrm{St}_{2,E_{z_i}}(\omega_{F'/F}) \boxplus \mathrm{St}_{l-1,E_{z_i}}$.

Now let L denote the extension of $E(\zeta_l)$ cut out by $\overline{\rho}$. We may choose a set S_1 of places of E such that every place of F below a place of S_1 is split in F', and any finite extension E'/E which is S_1-split is linearly disjoint over E from L. We can moreover assume that π and L are unramified above S_1 (see [BLGGT, Lemma A.2.2]). Let S_0 denote the set of places of F below a place of S_1. Let b denote the least positive integer such that $q_{z_0}^b \equiv -1 \bmod l$, and choose a cyclic totally real extension F_1/F of degree b which is S_0-split and in which y_0 is inert and each place $y_1, \ldots, y_s$ splits. (This is possible, by the Grunwald-Wang theorem, since q_{z_0} is odd.) We write v_0 for the place of F_1 above y_0 and $v_1, \ldots, v_{bs}$ for the places of F_1 above $y_1, \ldots, y_s$. Let $E_1 = E \cdot F_1$, and let w_i denote the place of E_1 above v_i. Let π'_{l+1} denote the base change of ${}^{\varphi}\pi_2 \otimes \omega_{F'/F} \boxplus \pi_{l-1} \otimes \iota\epsilon\chi$ to E_1. Then π'_{l+1} is regular algebraic and conjugate self-dual. Moreover, for each $i = 0, \ldots, bs$, we have $\pi'_{l+1,w_i} \cong \mathrm{St}_{2,E_{1,w_i}} \boxplus \mathrm{St}_{l-1,E_{1,w_i}}$. We can therefore apply Theorem 4.1 to π'_{l+1} to deduce the existence of an automorphic representation Π' of $\mathrm{GL}_n(\mathbb{A}_{E_1})$ such that Π' is ι-ordinary of weight zero, Π'_{w_0} is an unramified twist of the Steinberg representation, and

$$\overline{r_\iota(\Pi')} \cong \left(\omega_{F'/F} \otimes {}^{\varphi}\overline{\rho} \oplus \overline{\chi} \otimes \mathrm{Sym}^{l-2}\overline{\rho}\right)\Big|_{G_{E_1}}.$$

Now let $M = E_1 \cdot F'$. Then M/E is a soluble extension, and S_1-split. Let Π denote the base change of Π' to M. Then Π is a RACSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_M)$ which is ι-ordinary of weight zero, such that Π_w is an unramified twist of the Steinberg representation for any place w of M above w_0, and such that

$$\overline{r_\iota(\Pi)} \cong \left({}^{\varphi}\overline{\rho} \oplus \overline{\chi} \otimes \mathrm{Sym}^{l-2}\overline{\rho}\right)\Big|_{G_M}.$$

This completes the proof of Theorem 5.1.

6 Proof of Theorem 1.2

In this section we prove the theorem from the introduction. Fix throughout this section a prime $l \geq 5$ and an isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$, and a finite Galois extension $K/\mathbb{Q}$. We assume throughout this section the following hypothesis:

Conjecture 6.1 ($\mathrm{SP}_{l-1}(K(\zeta_l))$)**.** *Let F be a totally real number field, linearly disjoint over $\mathbb{Q}$ from $K(\zeta_l)$. Let (π, χ) be a RAESDC automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ without CM. Then the $(l-2)^{th}$ symmetric power lifting exists, in the following sense: there exists an RAESDC automorphic representation (π_{l-1}, χ_{l-1}) of $\mathrm{GL}_{l-1}(\mathbb{A}_F)$ and an isomorphism*

$$\mathrm{Sym}^{l-2} r_\iota(\pi) \cong r_\iota(\pi_{l-1}).$$

We must show that $\mathrm{SP}_{l+1}(K(\zeta_l))$ holds. We begin by proving a special case, using the results accumulated above. We will reduce the general case to this one.

Theorem 6.2. *Let F be a totally real number field. Let (π, χ) be a RAESDC automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$, and suppose that the following hypotheses hold:*

- *π is ι-ordinary of weight zero.*
- *Let $\rho = r_\iota(\pi)$. Then the residual representation $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is irreducible, and its image contains $\mathrm{SL}_2(\mathbb{F}_{l^a})$, up to conjugation, for some $a > 1$.*
- *There exists a place v of F such that π_v is an unramified twist of the Steinberg representation and q_v is a primitive root mod l. Moreover, $\overline{\rho}$ is ramified at v.*

Then $\mathrm{Sym}^l \rho$ is automorphic.

Proof. Let F_0/F denote a totally real quadratic extension in which v is inert, and let $\omega_{F_0/F} : G_F \to \overline{\mathbb{Q}}_l^\times$ denote the corresponding quadratic character. Let E/F be a CM imaginary quadratic extension which is ramified at v, and in which every place of F dividing l splits. By [CHT08, Lemma 4.1.4] we can find an algebraic character $\psi : G_E \to \overline{\mathbb{Q}}_l^\times$, unramified above v, such that $\psi\psi^c = r_\iota(\chi)|_{G_E}$. Let F_1/F be a soluble extension satisfying the following:

- The place v splits in F_1.
- Let $E_1 = E \cdot F_1$. If $w \nmid v$ is a place of E_1 at which π_E or ψ_E is ramified, then w is split over F_1.
- The extension E_1/F_1 is unramified away from places dividing v. The extension $F_1 \cdot F_0/F_1$ is everywhere unramified.
- For each place $v|l$ of F_1, the local degree $[F_{1,v} : \mathbb{Q}_l]$ is even.

By choosing these extensions so that certain auxiliary primes split, we can force $F_0 \cdot E_1$ to be disjoint over F from the extension of $F(\zeta_l)$ cut out by $\overline{\rho}$ (see [BLGGT, Lemma A.2.2]). The hypotheses of §5 are now satisfied, either for the automorphic representation $(\pi \otimes \iota\psi^{-1})_{E_1}$ of $\mathrm{GL}_2(\mathbb{A}_{E_1})$, or its twist $(\pi \otimes \iota\psi^{-1}\omega_{F_0/F})_{E_1}$. (Since the representation $(\pi \otimes \iota\psi^{-1})_{E_1}$ is conjugate self-dual, its local component at a prime w dividing v is isomorphic either to $\mathrm{St}_{2,E_{1,w}}$ or its twist by the quadratic unramified character of $E_{1,w}^\times$.) We may assume without loss of generality that it is the former. We deduce by Theorem 5.1 the existence of a soluble CM extension M/E_1 disjoint over E_1 from the extension of $E_1(\zeta_l)$ cut out by $\overline{\rho}$, and an automorphic representation Π of $\mathrm{GL}_{l+1}(\mathbb{A}_M)$ such that Π is ι-ordinary, such that for some place w of M above v, Π_w is an unramified twist of the Steinberg representation, and such that

$$\overline{r_\iota(\Pi)} \cong (\mathrm{Sym}^l(\overline{\rho} \otimes \overline{\psi}^{-1}))|_{G_M}^{\mathrm{ss}}.$$

We claim that the hypotheses of [Tho, Theorem 7.1] now apply, and thus $\mathrm{Sym}^l \rho \otimes \psi^{-1}|_{G_M}$ is automorphic. Indeed, it remains to check the following points:

- The irreducible constituents of $(\mathrm{Sym}^l \overline{\rho})^{\mathrm{ss}}|_{G_{M(\zeta_l)}}$ are adequate, in the sense of [Tho12, §3].
- The extension $M(\zeta_l)$ is not contained in the extension of M cut out by $\mathrm{ad}\,\overline{\rho}$.

The first point follows from our hypothesis on the image of $\overline{\rho}$ and [Gur, Theorem 1.5]. For the second point, we note that by construction we have $M \cap \mathbb{Q}(\zeta_l) = \mathbb{Q}$, while the image of $\mathrm{ad}\,\overline{\rho}$ contains a simple normal subgroup of index at most 2. It follows that $\mathrm{Sym}^l \rho$ is automorphic, and this completes the proof. ∎

We now reduce the general case of $\mathrm{SP}_{l+1}(K(\zeta_l))$ to the above one using a chain of congruences. The arguments are similar to those of [CT, §5], but since the hypotheses of the above theorem are more stringent we work a little harder. We begin by fixing a totally real field F, linearly disjoint over $\mathbb{Q}$ from $K(\zeta_l)$, and a RAESDC automorphic representation (π, χ) of $\mathrm{GL}_2(\mathbb{A}_F)$ without CM. Arguing as in the proof of [CT, Proposition 5.3], we can assume (after replacing F by a soluble extension and passing to a congruent automorphic representation) that there is a finite set T of places of F, a place u of F not in T, and that π satisfies the following:

1. π is unramified outside $T \cup \{u\}$.
2. π is of weight zero.
3. For each place $v|l$, π_v is an unramified twist of the Steinberg representation (and hence π is ι-ordinary).
4. There exists a rational prime $t > l^2$ such that $q_u \equiv -1 \bmod t$ and

$$r_\iota(\pi)|_{I_{F_u}} \cong \begin{pmatrix} \psi_u & 0 \\ 0 & \psi_u^{q_u} \end{pmatrix},$$

   where $\psi_u : I_{F_u} \to \overline{\mathbb{Q}}_l^\times$ is a character of order t. Moreover, the place u is split in the maximal abelian extension of F of exponent 2 which is unramified away from T.

Fix an open compact subgroup $U \subset \mathrm{GL}_2(\mathbb{A}_F^\infty)$ such that $(\pi^\infty)^U \neq 0$, and let $\pi_1, \ldots, \pi_n$ denote the RAESDC automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$ such that $(\pi_i^\infty)^U \neq 0$ and π_i satisfies the above conditions. We can assume after renumbering that $\pi_1 = \pi$.

For each $i = 1, \ldots, n$, the residual representation $\overline{r_\iota(\pi_i)}$ is irreducible, and its image contains $\mathrm{SL}_2(\mathbb{F}_{l^a})$ up to conjugation, for some $a > 1$. This follows from an argument of Khare-Wintenberger, as follows. Let us write $\overline{\rho} = \overline{r_\iota(\pi_i)}$. Since $t > l^2$, the projective image of $\overline{\rho}$ contains an element of order $t > 5$. The classification of finite subgroups of $\mathrm{PGL}_2(\overline{\mathbb{F}}_l)$ implies that the projective image of $\overline{\rho}$ is conjugate either to $\mathrm{PSL}_2(\mathbb{F}_{l^a})$ or $\mathrm{PGL}_2(\mathbb{F}_{l^a})$, or to a dihedral subgroup. In the first case, we must have $a > 1$ since $t > l^2$, by hypothesis. If the projective image is dihedral, then there exists a totally imaginary quadratic extension M/F and a continuous character $\alpha : G_M \to \overline{\mathbb{F}}_l^\times$ such that $\overline{\rho} \cong \mathrm{Ind}_M^F \alpha$.

If the extension M/F is ramified at a place y of F, then $\overline{\rho}$ and hence π is ramified at y, and so $y \in T \cup \{u\}$. In fact, we have $y \in T$, since $\overline{\rho}(I_{F_u})$ has order t, prime to 2. Thus M/F is unramified outside T, and the place u splits in M. This implies that the representation $\overline{\rho}|_{G_{F_u}}$ is reducible, a contradiction.

Proposition 6.3. *There exists a prime $p > 2(l+2)$, an isomorphism $\iota_p : \overline{\mathbb{Q}}_p \cong \mathbb{C}$, and an automorphic representation π' satisfying conditions 2 – 4 above, as well as the following conditions:*

- $\overline{r_{\iota_p}(\pi')} \cong \overline{r_{\iota_p}(\pi)}$.
- *The image of the residual representation $\overline{r_\iota(\pi')}$ contains $\mathrm{SL}_2(\mathbb{F}_{l^a})$ up to conjugation, for some $a > 1$.*
- *There exists a place v of F such that q_v is odd and is a primitive root modulo l and π'_v is an unramified twist of the Steinberg representation, and the restriction of $\overline{r_\iota(\pi')}$ to G_{F_v} is ramified.*
- *The symmetric l^{th} power lifting of π exists if and only if the symmetric l^{th} power lifting of π' exists.*

Proof. We construct π' by raising the level from π, modulo p. To ease notation, let us write $\overline{\rho}_i = \overline{r_\iota(\pi_i)}$ for $i = 1, \ldots, n$. Choose a prime $p > t$ and an isomorphism $\iota_p : \overline{\mathbb{Q}}_p \cong \mathbb{C}$ such that the image of $\overline{r_{\iota_p}(\pi)}$ contains $\mathrm{SL}_2(\mathbb{F}_p)$ up to conjugation, and set $\overline{\rho}_0 = \overline{r_{\iota_p}(\pi)}$. We say that one of the representations $\overline{\rho}_i$ admits level raising at the place v of F if $\overline{\rho}_i$ is unramified at v and the eigenvalues α, β of $\overline{\rho}_i(\mathrm{Frob}_v)$ satisfy $\alpha = q_v^{\pm 1}\beta$. This condition depends only on the image of Frob_v under the projective representation associated to $\overline{\rho}_i$.

We claim that to prove the proposition, it suffices to exhibit a place v of F such that q_v is odd and is a primitive root modulo l, and $\overline{\rho}_0$ admits level-raising at the place v of F, but none of $\overline{\rho}_1, \ldots, \overline{\rho}_n$ admits level raising at v. Indeed, in this case we can construct using e.g. [Gee11, Corollary 3.1.7] an automorphic lift of $\overline{\rho}_0$ which corresponds to a RAESDC automorphic representation π' unramified outside $T \cup \{u, v\}$ and satisfying the desired properties, except possibly for the following:

- $\overline{r_\iota(\pi')}$ is irreducible, and its image contains $\mathrm{SL}_2(\mathbb{F}_{l^a})$, up to conjugation, for some $a > 1$.
- The restriction of $\overline{r_\iota(\pi')}$ to G_{F_v} is ramified.

We check that these conditions also hold. We first note that $\overline{r_\iota(\pi')}$ is, by construction, irreducible even after restriction to G_{F_u}. We claim that it is ramified after restriction to G_{F_v}. If not, then applying [Gee11, Corollary 3.1.7] once more we can find a RAESDC automorphic representation π'' which satisfies conditions 1–4 above and such that $\overline{r_\iota(\pi'')} \cong \overline{r_\iota(\pi')}$. Then there exists $1 \le i \le n$ such that $\pi'' = \pi_i$, and this implies that $\overline{\rho}_i$ admits level-raising at v, a contradiction. In particular, the image of $\overline{r_\iota(\pi')}$ contains an element of order l. Since it also contains an element of order t, the classification of finite subgroups of $\mathrm{GL}_2(\overline{\mathbb{F}}_l)$ now shows that the image must contain a conjugate of $\mathrm{SL}_2(\mathbb{F}_{l^a})$ for some $a > 1$. To complete the proof of the claim, we must show that the symmetric l^{th} power lifting of π exists if and only if the symmetric l^{th} power lifting of π' exists. Since both $r_{\iota_p}(\pi)$ and $r_{\iota_p}(\pi')$ are potentially Barsotti-Tate, hence potentially diagonalizable, and their symmetric l^{th} powers are adequate, this follows from [BLGGT, Theorem 4.2.1] (cf. the proof of [CT, Proposition 5.2]).

We now introduce some more notation. Let F_i denote the extension of F cut out by the projective representation associated to $\overline{\rho}_i$, $i = 0, \ldots, n$. Let L denote the compositum of the extensions $F_1, \ldots, F_n$. Let us write L^{ab} and F_0^{ab} for the maximal subextensions of L and F_0, respectively, which are abelian over F. Let $G = \mathrm{Gal}(L/F)$, $G_i = \mathrm{Gal}(F_i/F)$. For each $i = 1, \ldots, n$ there is a surjective homomorphism $p_i : G \to G_i$. The group G_i contains a simple normal subgroup of index at most 2, isomorphic to $\mathrm{PSL}_2(\mathbb{F}_{l^a})$ for some $a > 1$. By the Chebotarev density theorem, it now suffices to construct an element $\sigma \in \mathrm{Gal}(L \cdot F_0 \cdot F(\zeta_p, \zeta_l)/F)$ satisfying the following conditions:

1. The projection of σ to $\mathrm{Gal}(F(\zeta_l)/F)$ generates this group.
2. The projection of σ to $\mathrm{Gal}(F_0(\zeta_p)/F)$ is trivial.

3. For each $i = 1, \dots, n$, the eigenvalues α, β (which are defined only up to scalars) of $p_i(\sigma)$ satisfy

$$\alpha \neq \epsilon(\sigma)^{\pm 1}\beta.$$

We first note that the extensions $F(\zeta_l)$ and $F_0(\zeta_p)$ are linearly disjoint over F. Indeed, $F_0^{\mathrm{ab}}(\zeta_p)$ is unramified at the primes dividing l. It follows that we can choose an element $\sigma_0 \in \mathrm{Gal}(L \cdot F_0 \cdot F(\zeta_p, \zeta_l)/F)$ satisfying the first two requirements above. We now claim that we can choose $\tau \in \mathrm{Gal}(L \cdot F_0(\zeta_p, \zeta_l)/F_0(\zeta_p, \zeta_l))$ such that $\sigma = \tau \cdot \sigma_0$ satisfies all three requirements. Note that multiplying by such an element τ does not disturb the first two points.

We will in fact choose an element $\tau \in \mathrm{Gal}(L \cdot F_0 \cdot F(\zeta_p, \zeta_l)/L^{\mathrm{ab}} \cdot F_0(\zeta_p, \zeta_l)) = H$, say. The group H is a product of simple groups, each isomorphic to $\mathrm{PSL}_2(\mathbb{F}_{l^a})$ for some $a > 1$, and each map $p_i|_H : H \to G_i$ has image of index at most 2. We show by induction on j that we can choose τ such that the condition on eigenvalues is satisfied for $i = 1, \dots, j$. For the case $j = 1$, we look at the image of σ_0 in G_1. Either the condition is satisfied for $p_1(\sigma_0) \in \mathrm{PGL}_2(\mathbb{F}_{l^a})$ or we can choose $x \in \mathrm{PSL}_2(\mathbb{F}_{l^a})$ such that the condition is satisfied for $p_1(\sigma_0)x$. We now take τ be be an arbitrary lift of x to H.

For the induction step, we look at $p_{j+1}(\tau\sigma_0) \in \mathrm{PGL}_2(\mathbb{F}_{l^a})$. If the condition is satisfied for this element, then we are done. If the condition is not satisfied, then the extensions $F_{j+1} \cdot L^{\mathrm{ab}}(\zeta_p, \zeta_l)$ and $F_1 \cdot \dots \cdot F_j \cdot L^{\mathrm{ab}}(\zeta_p, \zeta_l)$ are linearly disjoint over $L^{\mathrm{ab}}(\zeta_p, \zeta_l)$. For otherwise, there exists $i = 1, \dots, j$ such that $F_i \cdot L^{\mathrm{ab}}(\zeta_p, \zeta_l) = F_{j+1} \cdot L^{\mathrm{ab}}(\zeta_p, \zeta_l)$; on the other hand one knows that every automorphism of $\mathrm{PSL}_2(\mathbb{F}_{l^a})$ is a composite of conjugation by an element of $\mathrm{PGL}_2(\mathbb{F}_{l^a})$ and the automorphism induced by a power of Frobenius, and such an automorphism does not affect the condition $\alpha \neq \epsilon(\sigma)^{\pm 1}\beta$. We can therefore choose an element $\tau' \in H$ such that $p_i(\tau) = p_i(\tau')$ for each $i = 1, \dots, j$ and $p_{j+1}(\tau'\sigma_0)$ satisfies the condition on eigenvalues. $\square$

This proposition implies the result, since Theorem 6.2 now shows that the symmetric l^{th} power lifting of π' exists.

Appendix A : Calculation of Jacquet modules

By Colette Mœglin

6.1 Le cas quasi déployé, introduction

On fixe une extension quadratique E/F de corps p-adiques ; on s'intéresse au groupe $U(n,E/F)$ et $\widetilde{\mathrm{GL}}(n,E)$ est la composante non neutre qui intervient dans l'endoscopie tordue.

On écrit abusivement les induites en oubliant le parabolique mais on considère les paraboliques standard, le Borel étant les triangulaires supérieures de sorte que la représentation de Steinberg de $U(n,E/F)$ a pour module de Jacquet pour le Borel $\bigotimes_{\ell\in[(n-1)/2,1/2]} |\,|^{\ell}$.

Pour π une représentation de $U(n,E/F)$ et pour χ un caractère de $E^{\times}$ (en général une puissance d'une valeur absolue $|\,|^{x}$, avec x demi-entier), on note $Jac_{\chi}\pi$ l'élément du groupe de Grothendieck de $U(n-2,E/F)$ tel que le module de Jacquet de π pour le parabolique maximal de Levi $E^{\times}\times U(n-2,E/F)$ est de la forme $\chi\otimes Jac_{\chi}\pi\oplus\bigoplus_{\chi'\neq\chi,\pi'}\chi'\otimes\pi'$. Pour $\tilde{\pi}$ une représentation de $\widetilde{\mathrm{GL}}(n,E)$, on note $Jac_{\chi}^{GL}\tilde{\pi}$ la même chose sauf que l'on regarde le Levi $E^{\times}\times\widetilde{\mathrm{GL}}(n-2,E)\times E^{\times}$ (on peut avoir $n=2$ mais je ne l'utiliserai pas) et le module de Jacquet est la somme de $\chi\otimes Jac_{\chi}^{\mathrm{GL}}\tilde{\pi}\otimes\overline{\chi}^{-1}$ plus d'autres termes où au moins l'une des composantes $E^{\times}$ agit par un autre caractère. Comme $\tilde{\pi}$ est muni d'une action de θ, de fait $Jac_{\chi}^{\mathrm{GL}}(\pi)$ en a une aussi tout à fait canoniquement. Pour nous, cela n'interviendra pas car on évite cette difficulté.

6.1.1 Le cas de $U(4,E)$

Proposition. *L'induite de la représentation de Steinberg de* $\mathrm{GL}(2,E)$ *à* $U(4,E/F)$ *est réductible.*

Voir [Gol93].

Lemme. *L'induite de la proposition précédente est de longueur deux. L'un de ses sous-modules a un module de Jacquet (pour le Borel) de longueur 3 ; on note cette représentation* $\pi_{4,+}$*. L'autre représentation* $\pi_{4,-}$ *a un module de Jacquet irréductible. Avec des notations intuitives, le semi-simplifié du module de Jacquet de* $\pi_{4,+}$ *contient le terme :*

$$|\,|^{1/2}\otimes|\,|^{1/2} \tag{1}$$

avec multiplicité 2 et le module de Jacquet de $\pi_{4,+}$ *et* $\pi_{4,-}$ *contiennent tous deux avec multiplicité 1 le terme*

$$|\,|^{1/2}\otimes|\,|^{-1/2}. \tag{2}$$

Le module de Jacquet de toute l'induite contient exactement les 2 termes décrits, chacun avec multiplicité 2 et chacune des sous-représentations irréductibles contient au moins avec multiplicité 1 le terme (2) par réciprocité de Frobenius. Fixons π', un des sous-modules irréductible dont le module de Jacquet contient avec multiplicité au moins 1, le terme

$$|\,|^{1/2}\otimes|\,|^{1/2}.$$

On montre qu'il contient ce terme avec multiplicité au moins 2 : en effet on calcule le module de Jacquet de π' par rapport au parabolique, P_2, de Levi $\mathrm{GL}(2,E)$. Par transitivité le calcul du module de Jacquet de π' par rapport au Borel se calcule en prenant d'abord le module de Jacquet par rapport au parabolique P_2 puis en passant de $\mathrm{GL}(2,E)$ au Borel de $\mathrm{GL}(2,E)$. Donc dans la première opération, on a nécessairement une représentation de $\mathrm{GL}(2,E)$ de support cuspidal $|\,|^{1/2},|\,|^{1/2}$. Il n'y a qu'une représentation de $\mathrm{GL}(2,E)$ ayant cette propriété, c'est l'induite de $|\,|^{1/2}\otimes|\,|^{1/2}$ qui est irréductible. Le module de Jacquet de cette induite a bien le terme (1) de l'énoncé avec multiplicité 2. D'où le lemme.

6.2 Le cas de $U(n,E)$, n pair et >4

6.2.1 Nombre de séries discrètes dans le paquet

Proposition. *Il existe exactement 2 représentations elliptiques dans le paquet associé à* $St(2), St(n-2)$ *et ce sont des séries discrètes.*

J'admets essentiellement cette proposition : [Mœg] 7.1 où ici $Jord(\pi)$ est, par définition, l'ensemble à deux éléments $(trivial, 2), (trivial, 4)$.

6.2.2 Rappel d'un petit lemme technique

Lemme. *Soit π une série discrète irréductible pour U(n,E/F) et χ un caractère de $E^\times$ de la forme $| |^x$.*

(i) $Jac_\chi \pi$ est soit nul soit $x > 0$; si $Jac_\chi \pi \neq 0$, alors $Jac_\chi \pi$ est irréductible. De plus si $Jac_\chi \pi$ n'est pas nul alors π est un sous-module irréductible de l'induite de $\chi \otimes Jac_\chi \pi$ pour le parabolique standard de Levi $E^\times \times U(n-2, E/F)$.

(ii) Soit π et π' deux séries discrètes irréductibles et inéquivalentes ; alors on ne peut avoir $Jac_\chi \pi = Jac_\chi \pi'$ sauf si ces deux modules sont nuls.

En fait il y a unicité du sous-module irréductible dans (i) mais on n'en a pas besoin.

La première assertion de (i) est uniquement le critère de Casselman pour les séries discrètes. Pour l'irréductibilité de (i), c'est [Mœg] corollaire de 2.7 (i). Montrons l'inclusion : la non nullité de $Jac_\chi \pi$ entraîne que le module de Jacquet de π pour le parabolique standard de Levi isomorphe à $E^\times \times U(n-2, E/F)$ a un quotient irréductible de la forme $\chi \otimes \sigma$; par irréductibilité de $Jac_\chi \pi$, nécessairement $\sigma = Jac_\chi \pi$. Par réciprocité de Frobenius, on a alors une inclusion de π dans l'induite comme annoncé.

Pour (ii) c'est [Mœg] corollaire 2.7 (ii) avec le fait que toute série discrète est dans un paquet stable ([Mœg] 2.4, où n'importe quelle autre référence)

6.2.3 Calcul des modules de Jacquet

On peut aller plus loin, en utilisant le fait que le module de Jacquet commute au transfert. On a, pour $n \geq 6$, $Jac_\chi^{\mathrm{GL}}(St(2) \times St(n-2)) = 0$ sauf exactement si $\chi = | |^{1/2}$ ou $\chi = | |^{(n-3)/2}$. On a $Jac^{\mathrm{GL}}_{| |^{1/2}} St(2) \times St(n-2) = St(n-2)$ et $Jac^{\mathrm{GL}}_{| |^{(n-3)/2}} St(2) \times St(n-2) = St(2) \times St(n-4)$ et il n'y a pas de multiplicité ; donc l'action de θ est bien déterminée à un signe près, dont on se moque.

Proposition. *On suppose que $n \geq 6$*

(i) On suppose que π est dans le paquet de séries discrètes associées à $St(2), St(n-2)$. Alors $Jac_\chi \pi = 0$ sauf éventuellement si $\chi = | |^{1/2}$ ou $\chi = | |^{(n-3)/2}$.

(ii) $Jac_{| |^{1/2}}(\pi) = 0$ ou est la représentation de Steinberg de $U(n-2, E/F)$,chacun de ces deux cas se produisant pour un bon choix de π dans le paquet ; on note $\pi_{n,+}$ celle des deux représentations du paquet telle que $Jac_{| |^{1/2}} \pi \neq 0$ et $\pi_{n,-}$ l'autre représentation.

(iii) Avec la définition glissée dans (ii) et celle du paragraphe 6.1.1, pour tout $n \geq 6$, on a pour $\zeta = \pm$, $Jac_{| |^{(n-3)/2}} \pi_{n,\zeta} = \pi_{n-2,\zeta}$.

Le (i) est juste la compatibilité des modules de Jacquet au transfert. Pour (ii) et (iii) on introduit les notations suivantes : soit π_i pour $i = 1, 2$ les deux séries discrètes dans le paquet considéré. Soit a_i des nombres complexes non nuls tels que $\sigma := a_1 \pi_1 + a_2 \pi_2$ est stable.

Montrons (ii) $Jac_{| |^{1/2}} \sigma$ est une distribution stable (compatibilité de la stabilité à la restriction) et elle se transfère (à un scalaire près) en la trace tordue de la représentation de Steinberg de $\mathrm{GL}(n-2, E)$; donc $Jac_{| |^{1/2}} \sigma$ est nécessairement (à un scalaire près) la représentation de Steinberg de $U(n-2, E/F)$. Par l'irréductibilité rappelée ci-dessus (6.2.2 (i)) et le fait que $Jac_\chi \pi_1 \neq Jac_\chi \pi_2$ si l'un des deux modules de Jacquet est non nul (6.2.2 (ii)), il existe exactement un i tel que $Jac_{| |^{1/2}} \pi_i \neq 0$ et ce module de Jacquet vaut alors la représentation de Steinberg de $U(n-2, E/F)$.

Pour (iii), $Jac_{| |^{(n-3)/2}}(a_1 \pi_1 + a_2 \pi_2)$ est (via la trace) une distribution stable et elle se transfère à un scalaire près en la trace tordue de $Ind\, St(2) \otimes St(n-4)$. Si $n = 6$, l'induite $Ind\, St(2) \otimes St(n-4)$ n'est pas θ-elliptique c'est une induite à partir d'une représentation θ-stable. Son caractère est le transfert de la représentation de $U(4, E/F)$, $Ind\, St_{\mathrm{GL}(2,E)}(2)$ et on a calculé cette distribution ; c'est le caractère de $\pi_{4,+} + \pi_{4,-}$. On montre par récurrence sur n que $a_1 = a_2$ et l'égalité d'ensembles non ordonnés :

$$(Jac_{| |^{(n-3)/2}} \pi_1, Jac_{| |^{(n-3)/2}} \pi_2) = (\pi_{n-2,+}, \pi_{n-2}, -).$$

Initialiser la récurrence avec $n = 6$ se fait en même temps que le pas général.

En effet, quitte à multiplier a_1 et a_2 par le même nombre complexe non nul $Jac_{|\,|^{(n-3)/2}}(a_1\pi_1+a_2\pi_2) = \pi_{n-2,+} + \pi_{n-2,-}$, car c'est (à un scalaire près) la distribution stable portée par $\pi_{n-2,+}$ et $\pi_{n-2,-}$ que l'on connaît pour $n=6$, on vient de le rappeler, et par l'hypothèse de récurrence pour $n>6$. Le membre de gauche vaut

$$a_1 Jac_{|\,|^{(n-3)/2}}\pi_1 \;+\; a_2 Jac_{|\,|^{(n-3)/2}}\pi_2$$

et chaque terme est soit nul soit une représentation irréductible, les deux ne pouvant être simultanément non nuls et égaux (6.2.2(i) et (ii)). Ainsi $a_1=a_2=1$ et l'égalité d'ensemble non ordonné annoncée.

Supposons que $Jac_{|\,|^{(n-3)/2}}\pi_1 = \pi_{n-2,+}$ donc $Jac_{|\,|^{(n-3)/2}}\pi_2 = \pi_{n-2,-}$. On sait (6.2.2 (i)) que $\pi_2 \hookrightarrow Ind\,|\,|^{(n-3)/2}\otimes\pi_{n-2,-}$.

On traite d'abord le cas de $n=6$ qui est le seul cas où $Jac_{|\,|^{1/2}}\pi_{n-2,-}\neq 0$. Dans ce cas, par les formules générales de calcul de module de Jacquet, on a :

$$Jac_{|\,|^{1/2}}Ind(|\,|^{5/2}\times\pi_{4,-}) = Ind\,|\,|^{5/2}\otimes|\,|^{-1/2},$$

car $Jac_{|\,|^{1/2}}\pi_{4,-} = |\,|^{-1/2}$ d'après la description donnée dans le paragraphe 6.1.1. Par exactitude du module de Jacquet, on a l'inclusion :

$$Jac_{|\,|^{1/2}}\pi_{6,-}\hookrightarrow Ind\,|\,|^{5/2}\otimes|\,|^{-1/2}\simeq Ind\,|\,|^{-1/2}\otimes|\,|^{5/2};$$

l'isomorphisme est, par transitivité, une propriété de $\mathrm{GL}(2,E)$ et dans ce groupe l'induite de $|\,|^{5/2}\otimes|\,|^{-1/2}$ est irréductible. Ainsi si $Jac_{|\,|^{1/2}}\pi_{6,-}$ est non nul c'est un sous-module irréductible de l'induite $Ind\,|\,|^{-1/2}\otimes|\,|^{5/2}$. D'où une inclusion (cf. 6.2.2 (i))

$$\pi_{6,-}\hookrightarrow Ind\,(|\,|^{1/2}\otimes Jac_{|\,|^{1/2}}\pi_{6,-})\hookrightarrow Ind\,|\,|^{1/2}\otimes|\,|^{-1/2}\otimes|\,|^{5/2}.$$

Ceci est impossible pour une série discrète car $1/2+(-1/2)=0$ et cela contredit le critère de Casselman. Ainsi $Jac_{|\,|^{1/2}}\pi_{6,-}=0$.

On suppose $n>6$. Par hypothèse de récurrence $Jac_{|\,|^{1/2}}\pi_{n-2,-}=0$ et $(n-3)/2\neq\pm 1/2$; donc les formules standard de calcul de module de Jacquet donnent

$$Jac_{|\,|^{1/2}}Ind\,|\,|^{(n-3)/2}\otimes\pi_{n-2,-} = Ind\,(|\,|^{(n-3)/2}\otimes Jac_{|\,|^{1/2}}\pi_{n-2,-}) = 0,$$

par l'hypothèse de récurrence puisque $n-2\geq 6$. Par exactitude des modules de Jacquet cela force aussi $Jac_{|\,|^{1/2}}\pi_2 = 0$ donc $\pi_2\neq\pi_{n-2,+}$; d'où $\pi_1=\pi_{n-2,+}$ par (ii). Cela termine la preuve de (iii).

Corollaire. *Ici on suppose $n\geq 4$.*

(i) Le module de Jacquet (pour le Borel) de $\pi_{n,-}$ est de longueur 1 ; il est réduit à

$$\bigotimes_{\ell\in[(n-3)/2,-1/2]}|\,|^{\ell}$$

où on décale de 1 en 1 (et non 1/2)

(ii) Le module de Jacquet (pour le Borel) de $\pi_{n,+}$ est de longueur $(n-2)/2+2$. Il contient avec multiplicité 1 le terme

$$\bigotimes_{\ell\in[(n-3)/2,-1/2]}|\,|^{\ell}$$

avec multiplicité 2 le terme

$$\bigotimes_{\ell\in[(n-3)/2,1/2]}|\,|^{\ell}\otimes|\,|^{1/2}$$

et avec mutliplicité 1 tous les termes

$$\bigotimes_{\ell\in[(n-3)/2,1/2]}|\,|^{\ell}$$

et où on glisse $|\,|^{1/2}$ juste à gauche de l'un des $|\,|^{\ell}$ avec $\ell>1/2$.

Le corollaire est vrai pour $n = 4$ grâce au paragraphe 6.1.1. Pour $n > 4$ on le démontre ainsi.

Le (i) se démontre par récurrence : on sait que $Jac_\chi \pi_{n,-} = 0$ sauf pour $\chi = |\,|^{(n-3)/2}$ et $Jac_{|\,|^{(n-3)/2}} \pi_{n,-} = \pi_{n-2,-}$. Par transitivité, le module de Jacquet pour le Borel de $\pi_{n,-}$ est le produit tensoriel de $|\,|^{(n-3)/2}$ avec le module de Jacquet (pour le Borel) de $\pi_{n-2,-}$.

Pour (ii) le même argument que pour (i) calcule tous les termes du module de Jacquet de $\pi_{n,+}$ qui commence par $|\,|^{(n-3)/2}$ et il faut ajouter les termes qui commencent par $|\,|^{1/2}$. Mais il n'y a en qu'un puisque $Jac_{|\,|^{1/2}} = St_{U(n-2)}$ et c'est la description de l'énoncé.

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