Raising the level for GL_n

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Abstract

We prove a simple level-raising result for regular algebraic, conjugate self-dual automorphic forms on GL_n. This gives a systematic way to construct irreducible Galois representations whose residual representation is reducible.

Contents

1 Introduction

Let N be a positive integer, and let f be an elliptic modular newform of weight 2 and level $\Gamma_0(N)$. If l is a prime and ι is a choice of isomorphism $\overline{\mathbb{Q}}_l \cong \mathbb{C}$, then there is an associated Galois representation

$$r_\iota(f) : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l),$$

unramified outside Nl, uniquely characterized by the requirement that the trace of Frobenius at a prime $p \nmid Nl$ equal the p^{th} Fourier coefficient of f (or rather, its image in $\overline{\mathbb{Q}}_l$ under ι).

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After possibly making a change of basis, we may assume that $r_\iota(f)$ takes its values in $\mathrm{GL}_2(\overline{\mathbb{Z}}_l)$. We may then consider the reduced representation $\overline{r_\iota(f)} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$, which we assume to be irreducible. Let p be a prime not dividing Nl, and let $\alpha_1, \alpha_2 \in \overline{\mathbb{F}}_l^\times$ denote the eigenvalues of Frob_p. If $\alpha_1 = p^{\pm 1}\alpha_2$ then there exists a lift of $\overline{r_\iota(f)}|_{G_{\mathbb{Q}_p}}$ to a representation

$$\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{Z}}_l)$$

such that $\rho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ corresponds, under the local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$, to an unramified twist of the Steinberg representation, which has conductor p. It therefore makes sense to ask if there exists an elliptic modular newform g of weight 2 and level $\Gamma_0(Np)$ such that $\overline{r_\iota(g)} \cong \overline{r_\iota(f)}$, there being in this instance no obstruction from local-global compatibility.

This question was first posed and answered by Ribet [Rib84], and the theme of congruences between algebraic automorphic representations has been developed in many different directions since that work. In particular, an understanding of such congruences plays a fundamental role in the proofs of all known automorphy lifting theorems.

The aim of this work is to prove new level raising theorems for automorphic representations π of $\mathrm{GL}_n(\mathbb{A}_E)$, where E is a CM field satisfying some additional hypotheses. Suppose that π is regular algebraic and conjugate self-dual. In this case, it is known that there exists a Galois representation $r_\iota(\pi) : G_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$, and one can formulate the question of level-raising in much the same way as we have done for elliptic modular forms above. Broadly speaking, there are two main approaches. The first is to try to understand directly the natural integral structures appearing in spaces of algebraic automorphic forms. In this case, one can attempt to generalize Ribet's original argument. For unitary groups this rests on the still unproven 'Ihara's Lemma' of [CHT08]. If the residual representation $\overline{r_\iota(\pi)}$ has large image (and in particular, is irreducible), a second approach is possible. A trick due to Richard Taylor [Tay08] allows one to use automorphy lifting theorems to construct automorphic representations π' congruent to π modulo l, and such that π' has essentially any local behavior away from l not ruled out by the existence of a lifting ρ as above, cf. [Gee11].

In this work we therefore restrict focus to regular algebraic, conjugate self-dual automorphic representations π of the form $\pi = \pi_1 \boxplus \pi_2$, where π_i are cuspidal automorphic representations of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$ and $n_1 + n_2 = n$. By the theory of endoscopy, these representations often admit a descent to discrete automorphic representations of unitary groups. In this paper we exploit this fact to find congruences between representations of this form and cuspidal automorphic representations on $\mathrm{GL}_n(\mathbb{A}_E)$, by studying the integral structure of spaces of algebraic automorphic forms on unitary similitude groups.

As an example of the kind of thing we can prove, suppose that E is a CM imaginary field with totally real subfield F, and let p be a rational prime which is inert in F. Let w_0 denote a place of E above p, and suppose that w_0 is split over F. We assume that $[F : \mathbb{Q}]$ is odd. Let n_1, n_2 be distinct even integers, and let π_1, π_2 be cuspidal, conjugate self-dual automorphic representations of $\mathrm{GL}_{n_1}(\mathbb{A}_E)$ and $\mathrm{GL}_{n_2}(\mathbb{A}_E)$, respectively, such that $\pi_1 \boxplus \pi_2$ is regular algebraic of strictly regular weight (cf. (2.1) below).

Theorem 1.1. *Suppose that π_{1,w_0} and π_{2,w_0} are isomorphic to unramified twists of the Steinberg representation. Then there exists a set $\mathcal{L}$ of rational primes l of Dirichlet density one such that for all $l \in \mathcal{L}$ and for all isomorphisms $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$, there exists a finite order character $\psi : G_E \to \mathbb{C}^\times$ with $\psi\psi^c = 1$, a CM quadratic extension E_1/E and a RACSDC automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_{E_1})$ satisfying the following:*

- $\overline{r_\iota(\Pi)} \cong \overline{r_\iota(\pi_1 \boxplus (\pi_2 \otimes \psi))}|_{G_{E_1}}$.
- *If w_1 is a place of E_1 above w_0, then Π_{w_1} is an unramified twist of the Steinberg representation.*
- *Π has the same infinity type as the base change of $\pi_1 \boxplus \pi_2$ to E_1 and is unramified at the primes dividing l.*

Moreover, if $\pi_1 \boxplus \pi_2$ is ι-ordinary in the sense of [Ger, Definition 5.1.2], then we can assume that Π is also ι-ordinary.

For our main theorem, see Theorem 7.1 below. It is worth noting that at the same time as proving our main result, we also establish the analogue of Ihara's lemma in the simplest possible non-trivial case. This is a new result even when we localize at a non-Eisenstein maximal ideal, and would presumably allow one to establish the first non-minimal $R = \mathbb{T}$ theorems for Galois representations of unitary type, when our hypotheses are satisfied, although we have not pursued this here.

Our main interest in proving such theorems is the applications to automorphy lifting theorems for RACSDC automorphic representations with residually reducible Galois representations. We note that for applications of this type it is essential to be able to find congruences to automorphic representations which have the same l-adic Hodge type at the primes dividing l. By combining the theorems of this paper with the main theorem of [Tho], one can often prove the automorphy of Galois representations $r : G_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ satisfying the following kinds of conditions:

- r is ordinary, and there exists a place w of E at which r looks like it corresponds to the Steinberg representation.
- The residual representation $\overline{r}$ is reducible, and the Jordan-Hölder factors of $\overline{r}$ are residually automorphic.

We now come to a description of the contents and main ideas of this paper. In sections §§2–3 we first set up notation and recall some background results.

Let I be a definite unitary group over $\mathbb{Q}$ associated to the extension E/F as above. Then the $\mathbb{Z}$-arithmetic subgroups of $I(\mathbb{R})$ are essentially trivial, but if p is a prime such that $I(\mathbb{Q}_p)$ is non-compact, then the $\mathbb{Z}[1/p]$-arithmetic subgroups of $I(\mathbb{Q}_p)$ are highly non-trivial. If one knows that the cohomology of these arithmetic subgroups is torsion-free, then one can prove level-raising results for automorphic forms on I. This is essentially the content of §4.

In order to show such torsion vanishing, we compare the cohomology of these arithmetic groups with the cohomology of a PEL-type Shimura variety $S(G, U)$ obtained by 'switching primes', which is associated to an inner form G of I which has the type $U(1, n-1) \times U(n)^{d-1}$ at infinity and looks like a division algebra locally at p. According to a theorem of Rapoport-Zink [RZ96], these varieties admit a p-adic uniformization by the Drinfeld upper half plane. It turns out that the weight spectral sequence (whose definition we recall in §5) describing the cohomology of these Shimura varieties can be written down, at least at the E_1 page, in terms of spaces of algebraic modular forms on the definite group I. We remark that the weight spectral sequence of varieties uniformized by the Drinfeld upper half plane has been studied previously by Ito [Ito05].

Lan and Suh [LS12] have proved torsion vanishing results for the cohomology of local systems on Shimura varieties of sufficiently regular weight, using geometric methods. When the weight spectral sequence of $S(G, U)$ degenerates at E_2, we deduce from their results that the the cohomology of our arithmetic groups with corresponding coefficient systems has trivial torsion subgroup. We can prove this degeneration when l is a banal characteristic for $\mathrm{GL}_n(E_{w_0})$ by using a trick inspired by the use of weights to show that the spectral sequence with rational coefficients degenerates at E_2. The comparison of cohomologies and the study of the weight spectral sequence is made in §6. Finally, we deduce our main theorems in §7.

1.1 Acknowledgements

The idea of using the weight spectral sequence together with the p-adic uniformization of unitary Shimura varieties in order to prove cases of Ihara's lemma is due to Michael Harris, cf. [CHT08, Introduction]. Richard Taylor suggested to me that it might be possible to use these ideas to prove new cases of level-raising. I am grateful to both of them for allowing me to pursue these ideas here. I would also like to thank Laurent Clozel for a number of useful comments.

1.2 Notation

If F is a number field then we write G_F for its absolute Galois group. If v is a finite place of F, then we write G_{F_v} for a choice of decomposition group at v and q_v for the cardinality of the residue field at v.

We fix for every prime l an algebraic closure $\overline{\mathbb{Q}}_l$ of $\mathbb{Q}_l$. If $\rho : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ is a continuous representation then the semisimplification of the reduction modulo l of ρ with respect to some invariant lattice depends only on ρ, up to isomorphism, and we will write $\overline{\rho} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ for this reduced representation.

If p is a prime and K is a finite extension of $\mathbb{Q}_p$, then there is a bijection

$$\mathrm{rec}_K : \mathrm{Adm}_{\mathbb{C}} \mathrm{GL}_n(K) \leftrightarrow \mathrm{WD}^n_{\mathbb{C}} W_K,$$

characterized by a certain equality of epsilon- and L-factors on either side, cf. [HT01], [Hen02]. Here we write (for $\Omega = \mathbb{C}$ or $\overline{\mathbb{Q}}_l$) $\mathrm{Adm}_\Omega \mathrm{GL}_n(K)$ for the set of isomorphism classes of irreducible admissible representations of this group over Ω, and $\mathrm{WD}^n_\Omega W_K$ for the set of Frobenius-semisimple Weil-Deligne representations (r, N) of W_K valued in $\mathrm{GL}_n(\Omega)$. We define $\mathrm{rec}^T_K(\pi) = \mathrm{rec}_K(\pi|\cdot|^{(1-n)/2})$. This is the normalization of the local Langlands correspondence with good rationality properties; in particular, for any $\sigma \in \mathrm{Aut}(\mathbb{C})$ and any $\pi \in \mathrm{Adm}_{\mathbb{C}} \mathrm{GL}_n(K)$ there is an isomorphism

$$\mathrm{rec}^T_K({}^\sigma\pi) \cong {}^\sigma \mathrm{rec}^T_K(\pi).$$

This can be seen using, for example, the characterization of rec_K and the description given in [Tat79, §3] of the action of Galois on local ϵ- and L-factors. As a consequence, rec^T_K gives rise to a well-defined bijection

$$\mathrm{rec}^T_K : \mathrm{Adm}_\Omega \mathrm{GL}_n(K) \leftrightarrow \mathrm{WD}^n_\Omega W_K.$$

Suppose instead that K is a finite extension of $\mathbb{R}$. Then there is a bijection

$$\mathrm{rec}_K : \mathrm{Adm}_{\mathbb{C}} \mathrm{GL}_n(K) \leftrightarrow \mathrm{Rep}^n_{\mathbb{C}} W_K.$$

Here we write $\mathrm{Adm}_{\mathbb{C}} \mathrm{GL}_n(K)$ for the set of infinitesimal equivalence classes of irreducible admissible representations of $\mathrm{GL}_n(K)$ and $\mathrm{Rep}^n_{\mathbb{C}} W_K$ for the set of continuous representations of W_K into $\mathrm{GL}_n(\mathbb{C})$. We define $\mathrm{rec}^T_K(\pi) = \mathrm{rec}_K(\pi|\cdot|^{(1-n)/2})$.

2 Automorphic representations

2.1 GL_n

Let E be an imaginary CM field with totally real subfield F, and let $c \in \mathrm{Gal}(E/F)$ denote the non-trivial element. We say that an automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_E)$ is RACSDC if it satisfies the following conditions:

- It is conjugate self-dual: $\pi^c \cong \pi^\vee$.
- It is cuspidal.
- It is regular algebraic. By definition, this means that for each place $v|\infty$ of E, the representation $\mathrm{rec}^T_{E_v}(\pi_v)$ is a direct sum of pairwise distinct algebraic characters.

If π is a regular algebraic automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$, then for each embedding $\tau : E \hookrightarrow \mathbb{C}$, we are given a representation $r_\tau : \mathbb{C}^\times \to \mathrm{GL}_n(\mathbb{C})$, induced by $\mathrm{rec}_{E_v}(\pi_v)$, where v is the infinite place induced by τ, and the isomorphism $E_v^\times \cong \mathbb{C}^\times$ induced by τ. This representation has the form

$$r_\tau(z) = \left((z/\overline{z})^{a_{\tau,1}}, \ldots, (z/\overline{z})^{a_{\tau,n}}\right),$$

where $a_{\tau,i} \in (n-1)/2 + \mathbb{Z}$. We will refer to the tuple $\mathbf{a} = (a_{\tau,1}, \ldots, a_{\tau,n})_{\tau \in \mathrm{Hom}(E,\mathbb{C})}$, where for each τ we have $a_{\tau,1} > a_{\tau,2} > \cdots > a_{\tau,n}$, as the infinity type of π. More generally if π is any automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ and the parameters $r_\tau(z)$ associated to π are given by the above formula for some real numbers $a_{\tau,i} \in \mathbb{R}$, we use the same formula to define the infinity type $\mathbf{a}$ of π. We will say that the infinity type of π is strictly regular if for each embedding $\tau : E \hookrightarrow \mathbb{C}$, we have

$$a_{\tau,i} > a_{\tau,i+1} + 1 \tag{2.1}$$

for each i.

Suppose that π_1, π_2 are conjugate self-dual cuspidal automorphic representations of $\mathrm{GL}_{n_1}(\mathbb{A}_E)$, $\mathrm{GL}_{n_2}(\mathbb{A}_E)$, respectively, and that $\pi = \pi_1 \boxplus \pi_2$ is regular algebraic. Then the representations $\pi_i |\cdot|^{(n_i - n)/2}$ are regular algebraic. We call a representation π arising in this way a RACSD sum of cuspidal representations. In this case, define $\mathbf{a}^i = (a^i_\tau)_{\tau \in \mathrm{Hom}(E,\mathbb{C})}$ by the requirement that $(a^i_{\tau,1} + (n_i - n)/2, \dots, a^i_{\tau,n_i} + (n_i - n)/2)$ equal the infinity type of $\pi_i |\cdot|^{(n_i-n)/2}$, and define $\mathbf{b} = (b_\tau)_{\tau \in \mathrm{Hom}(E,\mathbb{C})}$ by the formula

$$(b_{\tau,1}, \dots, b_{\tau,n}) = (a^1_{\tau,1}, \dots, a^1_{\tau,n_1}, a^2_{\tau,1}, \dots, a^2_{\tau,n_2}).$$

Then there is a unique tuple $\mathbf{w} = (w_\tau)_{\tau \in \mathrm{Hom}(E,\mathbb{C})} \in S_n^{\mathrm{Hom}(E,\mathbb{C})}$ such that for each $\tau \in \mathrm{Hom}(E,\mathbb{C})$, the infinity type of π is $(b_{\tau,w_\tau(1)}, \dots, b_{\tau,w_\tau(n)})_{\tau \in \mathrm{Hom}(E,\mathbb{C})}$. We will say that $\pi = \pi_1 \boxplus \pi_2$ satisfies the sign condition if the following condition is satisfied. Choose for each place $v|\infty$ of F an embedding $\tau : E \hookrightarrow \mathbb{C}$ inducing v. Then:

$$\prod_v \det w_{\tau(v)} = 1. \tag{2.2}$$

We remark that this condition is always satisfied if, for example, there is an imaginary CM subfield $E' \subset E$ such that $[E : E'] = 2$ and π arises as a base change from E'.

Theorem 2.1. *Suppose that π_1, π_2 are cuspidal conjugate self-dual automorphic representations of $\mathrm{GL}_{n_i}(\mathbb{A}_E)$ and that $\pi = \pi_1 \boxplus \pi_2$ is a regular algebraic automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$. Then for each isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$, there is a continuous semisimple representation*

$$r_\iota(\pi) : G_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l),$$

uniquely characterized by the following local-global compatibility property at all primes w of E not dividing l:

$$WD(r_\iota(\pi)|^{F\text{-}ss}_{G_{E_w}}) \cong \mathrm{rec}^T_{E_w}(\iota^{-1}\pi_w). \tag{2.3}$$

Proof. Arguing as in the proof of [Gue11, Theorem 2.3], we can find continuous characters $\psi_i : \mathbb{A}_E^\times / E^\times \to \mathbb{C}^\times$ such that $\psi\psi^c = 1$ and the restriction of ψ_i to $(E \otimes_{E,\tau} \mathbb{C})^\times$ is given by $\psi_i(z) = (z/z^c)^{\delta_{i,\tau}}$, where $\delta_{i,\tau} = 0$ if $n - n_i$ is even and $\delta_{i,\tau} = 1/2$ if $n - n_i$ is odd. Then each $\pi_i \psi_i$ is RACSDC, and the representations $r_\iota(\pi_i \psi_i)$, characterized by a similar local-global compatibilty condition, exist, cf. [Car12, Theorem 1.1]. We now simply take

$$r_\iota(\pi) = r_\iota(\pi_1\psi_1) \otimes r_\iota(\psi_1^{-1}|\cdot|^{(n_1-n)/2}) \oplus r_\iota(\pi_2\psi_2) \otimes r_\iota(\psi_2^{-1}|\cdot|^{(n_2-n)/2}).$$

□

If π is a regular algebraic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ of infinity type $\mathbf{a}$, we also define a tuple $\boldsymbol{\lambda} = (\lambda_\tau)_{\tau \in \mathrm{Hom}(E,\mathbb{C})} = (\lambda_{\tau,1}, \dots, \lambda_{\tau,n})_{\tau \in \mathrm{Hom}(E,\mathbb{C})}$, which we call the weight of π, by the formula $\lambda_{\tau,i} = -a_{\tau,n+1-i} + (n-1)/2 - (n-i)$. Then for each $\tau : E \hookrightarrow \mathbb{C}$, we have $\lambda_{\tau,1} \geq \cdots \geq \lambda_{\tau,n}$, and the irreducible admissible representation of $\mathrm{GL}_n(\mathbb{C})$ corresponding to r_τ has the same infinitesimal character as the dual of the algebraic representation of $\mathrm{GL}_n(\mathbb{C})$ with highest weight λ_τ. The representation π is strictly regular if and only if for each τ we have $\lambda_{\tau,1} > \cdots > \lambda_{\tau,n}$.

2.2 Algebraic modular forms

Let E be an imaginary CM field with totally real subfield F. We suppose that $E = E_0 \cdot F$, where E_0 is a quadratic imaginary extension of $\mathbb{Q}$, and that E/F is everywhere unramified. Let $\dagger$ denote an involution of the second kind on the matrix algebra $M_n(E)$ corresponding to a Hermitian form on E^n. We define reductive groups I over $\mathbb{Q}$ and I_1 over F by their functors of points:

$$I(R) = \{g \in M_n(E) \otimes_{\mathbb{Q}} R \mid gg^\dagger = c(g) \in R^\times\}$$

and

$$I_1(R) = \{g \in M_n(E) \otimes_F R \mid gg^\dagger = 1\}$$

We suppose that I is quasi-split at every finite place and that $I_1(\mathbb{R})$ is compact. (This can always be achieved. Indeed, there is an obstruction from the Hasse principle only if n is even and $[F:\mathbb{Q}]$ is odd. However, the assumption that E/F is everywhere unramified implies that $[F:\mathbb{Q}]$ is even, by [Gro03, Proposition 3.1].) If $v = ww^c$ is a place of F split in E and dividing the rational prime p, then there are isomorphisms

$$\iota_w : I(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times \prod_{w'|p} \mathrm{GL}_n(E_{w'}),$$

$$\iota_w : I_1(F_v) \cong \mathrm{GL}_n(E_w),$$

the product being over the primes w' of E above p with the same restriction to E_0 as w. We observe that $I(\mathbb{R})$ is not compact, but that the group I nevertheless satisfies the conditions of [Gro99, Proposition 1.4]. In particular, we can define spaces of automorphic forms on the groups I and I_1 with integral coefficients.

Fix a prime l, and let K be a finite extension of $\mathbb{Q}_l$ inside $\overline{\mathbb{Q}}_l$ with ring of integers $\mathcal{O}$ and residue field k. Let $U_l \subset I(\mathbb{Q}_l)$ be an open compact subgroup, and suppose that M is a finite $\mathcal{O}$-module on which U_l acts continuously in the l-adic topology. In this case we define $\mathcal{A}(M)$ to denote the set of locally constant functions $f : I(\mathbb{A}^\infty) \to M$ such that for all $\gamma \in I(\mathbb{Q}), f(\gamma g) = f(g)$. We endow this space with an action of $I(\mathbb{A}^{l,\infty}) \times U_l$ by setting $(g \cdot f)(h) = g_l f(hg)$, where g_l denotes the projection to the l-component. If $U \subset I(\mathbb{A}^{l,\infty}) \times U_l$ is a subgroup, we set $\mathcal{A}(U, M) = \mathcal{A}(M)^U$.

Lemma 2.2. *Let $p \neq l$ be a prime, and suppose that U^p is an open compact subgroup of $I(\mathbb{A}^{p,\infty})$ whose projection to $I(\mathbb{Q}_l)$ is contained in U_l. Then $\mathcal{A}(U^p, M)$ is an admissible representation of $I(\mathbb{Q}_p)$, in the following sense: for any open compact subgroup $U_p \subset I(\mathbb{Q}_p)$, $\mathcal{A}(U^p, M)^{U_p}$ is a finite $\mathcal{O}$-module.*

Proof. Let $U_p \subset I(\mathbb{Q}_p)$ be an open compact subgroup. By [Gro99, Proposition 1.4], $I(\mathbb{Q}) \subset I(\mathbb{A}^\infty)$ is a discrete cocompact subgroup, and the quotient $I(\mathbb{Q})\backslash I(\mathbb{A}^\infty)/U^pU_p$ is finite. Let $g_1, \ldots, g_s \in I(\mathbb{A}^\infty)$ be representatives. There is an isomorphism of $\mathcal{O}$-modules

$$\mathcal{A}(U^pU_p, M) \cong \oplus_{i=1}^s M^{\Gamma_i},\ f \mapsto (f(g_i))_{i=1,\ldots,s},$$

where $\Gamma_i = I(\mathbb{Q}) \cap g_i U^p U_p g_i^{-1}$. □

Lemma 2.3. 1. *Let σ be an automorphic representation of $I(\mathbb{A})$ such that σ_∞ is the restriction of an algebraic representation of $I(\mathbb{R}) \subset I(\mathbb{C})$. Then there exists an automorphic representation σ_1 of $I_1(\mathbb{A}_F)$ satisfying the following:*

- *For each place p of $\mathbb{Q}$ split in E_0, $\sigma_{1,p}$ is isomorphic to the restriction of σ_p to the group $I_1(F \otimes_{\mathbb{Q}} \mathbb{Q}_p) \subset I(\mathbb{Q}_p)$.*
- *$\sigma_{1,\infty}$ is isomorphic to the restriction of σ_∞ to $I_1(\mathbb{R})$.*

2. *Let σ_1 be an automorphic representation of $I_1(\mathbb{A}_F)$. Then there exists an automorphic representation σ of $I(\mathbb{A})$ satisfying the following:*

- *σ_∞ is the restriction of an algebraic representation of $I(\mathbb{R}) \subset I(\mathbb{C})$. The restriction of σ_∞ to $I_1(\mathbb{R})$ is isomorphic to $\sigma_{1,\infty}$.*
- *For each prime p split in E_0, the restriction of σ_p to the group $I_1(F \otimes_{\mathbb{Q}} \mathbb{Q}_p) \subset I(\mathbb{Q}_p)$ is isomorphic to $\sigma_{1,p}$. If $\sigma_{1,p}$ is unramified then σ_p is unramified. If $\sigma_{1,p}$ has an Iwahori-fixed vector, then σ_p has an Iwahori-fixed vector.*

Proof. Let $T = \mathrm{Res}_{\mathbb{Q}}^{E_0} \mathbb{G}_m$, and let $T_1 \subset T$ denote the subtorus of elements of norm 1. Then there is an exact sequence of algebraic groups

$$1 \longrightarrow T_1 \longrightarrow T \times \mathrm{Res}_{\mathbb{Q}}^F I_1 \longrightarrow I \longrightarrow 1,$$

where T_1 is embedded diagonally. Let $\mathcal{A}$ denote the space of automorphic forms on I, an admissible semisimple representation of $I(\mathbb{A})$. Arguing as in the proof of [HT01, Theorem VI.2.1], we see that given an automorphic representation σ of $I(\mathbb{A})$ appearing in $\mathcal{A}$, there is an element $g \in I(\mathbb{A})$ and an irreducible admissible constituent τ of $\sigma|_{T(\mathbb{A})\times I(\mathbb{A}_F)}$ such that $\tau_1^g \cong \psi \otimes \sigma_1$ is automorphic. The representation σ_1 then satisfies the desired properties.

Suppose conversely that σ_1 is as in the second part of the lemma. Arguing as in the proof of [HT01, Lemma VI.2.10], we can find an algebraic Hecke character $\psi : E_0^\times \backslash \mathbb{A}_{E_0}^\times$ such that the central character ω_{σ_1} of σ_1 satisfies the relation $\omega_{\sigma_1}(z) = \psi(z^{-1})$. If p is a prime split in E_0 and $\sigma_{1,p}$ is unramified or has an Iwahori-fixed vector, then ω_{σ_1} is unramified at p, and after multiplying ψ by a character of the form $\chi \circ \mathbb{N}_{E_0/\mathbb{Q}}$, χ a Dirichlet character, we can assume that ψ is unramified at all such primes.

Now $\psi \otimes \sigma_1$ is an automorphic representation of the group $T(\mathbb{A}) \times I_1(\mathbb{A}_F)$, and (cf. the proof of [HT01, Theorem VI.2.9]) it is a subrepresentation of the pullback to $T(\mathbb{A}) \times I_1(\mathbb{A}_F)$ of an automorphic representation σ of $I(\mathbb{A})$, which now satisfies the desired properties. □

Proposition 2.4. *1. Let π_1, π_2 be cuspidal, conjugate self-dual automorphic representations of $\mathrm{GL}_{n_1}(\mathbb{A}_E)$, $\mathrm{GL}_{n_2}(\mathbb{A}_E)$, respectively, such that $\pi = \pi_1 \boxplus \pi_2$ is regular algebraic. Suppose that the following conditions are satisfied:*

- *If π_w is ramified, then w is split over F.*
- *$n_1 n_2$ is even.*
- *$\pi = \pi_1 \boxplus \pi_2$ satisfies the sign condition 2.2.*

Then there exists a cuspidal automorphic representation σ of $I_1(\mathbb{A}_F)$ of which π is the base change in the following sense: at every place of E at which π is unramified, the correspondence is given by the unramified base change. For every place $v = ww^c$ of F split in E, we have $\pi_w \cong \sigma_v \circ \iota_w$. The representation σ_∞ is dual of the algebraic representation of $I_1(F \otimes_{\mathbb{Q}} \mathbb{R})$ of highest weight equal to the weight of π.

2. Suppose conversely that σ is a cuspidal automorphic representation of $I_1(\mathbb{A}_F)$. Then there exists a partition $n = n_1 + \cdots + n_r$ and discrete automorphic representations π_i of $\mathrm{GL}_{n_i}(\mathbb{A}_E)$ such that at finite places, $\pi_1 \boxplus \cdots \boxplus \pi_r$ is the base change of σ in the above sense. If we suppose furthermore that the π_i are cuspidal, then π_∞ is the base change of σ_∞.

Proof. The first part is proved in [CT]. The second part follows immediately from [Lab11, Corollaire 5.3]. □

3 Drinfeld's upper half plane

In this section let F be a finite extension of $\mathbb{Q}_p$, and fix an integer $n \geq 2$. We write ϖ for a choice of uniformizer of F and q for the cardinality of the residue field $\mathcal{O}_F/\varpi$. The Drinfeld p-adic upper half plane over F is a formal scheme over $\mathcal{O}_F$ whose rigid generic fiber can be identified with the open subspace of $\mathbb{P}_F^{n-1}$ obtained by deleting all F-rational hyperplanes. It receives a faithful action of the group $\mathrm{PGL}_n(F)$ and uniformizes certain Shimura varieties.

We first recall the Bruhat-Tits building BT of $\mathrm{PGL}_n(F)$. It is a simplicial complex with vertices the homothety classes of $\mathcal{O}_F$-lattices $M \subset F^n$. A set $\{M_1, \ldots, M_r\}$ of lattices up to homothety represents a simplex if we can choose representatives such that $\varpi M_r \subset M_1 \subset \cdots \subset M_r$. The maximal simplices have dimension $n-1$, and for each k, $\mathrm{PGL}_n(F)$ acts transitively on the set of simplices of dimension k with a marked vertex. We write $\mathrm{BT}(i)$ for the set of simplices of BT of dimension i.

We write $\Omega_{\mathcal{O}_F}$ for the Drinfeld upper half plane over $\mathcal{O}_F$, see [RZ96, §3.71] or [Mus78]. It is a p-adic formal scheme, formally locally of finite type over $\mathrm{Spf}\, \mathcal{O}_F$, which receives a left action of $\mathrm{PGL}_n(F)$. The irreducible components of the special fiber of $\Omega_{\mathcal{O}_F}$ are geometrically irreducible, and in canonical bijection with the vertices of $\mathrm{BT}(0)$. Moreover, they are smooth, and the special fiber is a strict normal crossings divisor. In fact, BT can also be described as follows: it is the simplicial complex whose vertices are in bijection with the set of irreducible components of the special fiber of $\Omega_{\mathcal{O}_F}$. Vertices $v_1, \dots, v_r$ give rise to a simplex if and only if the corresponding irreducible components have non-trivial intersection.

The irreducible component of the special fiber corresponding to the homothety class of the lattice M can be described as follows: let $Y_0 = \mathbb{P}(M) \otimes_{\mathcal{O}_F} (\mathcal{O}_F/\varpi)$. For each i, let Y_i denote the blowing-up of Y_{i-1} along the strict transforms in Y_{i-1} of the codimension i, $\mathcal{O}_F/\varpi$-rational linear subspaces of Y_0. Then (as observed in [Ito05, §6]) the desired variety is Y_{n-1}. In particular:

Proposition 3.1. *Let $\overline{s}$ be a geometric point above the closed point of* $\operatorname{Spec} \mathcal{O}_F$. *For each prime $l \neq p$, the action of Frobenius on the étale cohomology groups $H^{2i}(Y_{n-1,\overline{s}}, \mathbb{Z}_l)$ is by the scalar q^i. These groups are torsion-free. For each odd integer i, $H^i(Y_{n-1,\overline{s}}, \mathbb{Z}_l)$ is zero.*

Proof. This follows from the calculation of the cohomology of the blow-up of a smooth variety along a smooth center, cf. [Ito05, §3]. □

For global applications, we will need to introduce a simple enlargement of $\Omega_{\mathcal{O}_F}$. We write $\mathcal{M}^{\mathrm{split}}$ for the p-adic formal scheme formally locally of finite type over $\mathcal{O}_F$ given by the formula

$$\mathcal{M}^{\mathrm{split}} = \Omega_{\mathcal{O}_F} \times \mathbb{Q}_p^\times/\mathbb{Z}_p^\times \times \mathrm{GL}_n(F)/\mathrm{GL}_n(F)^0,$$

where $\mathrm{GL}_n(F)^0 \subset \mathrm{GL}_n(F)$ is the open subgroup consisting of matrices with determinant a p-adic unit. Here we identify the sets on the right hand side with the corresponding constant $\mathcal{O}_F$-formal schemes. We define $\mathcal{M} = \mathcal{M}^{\mathrm{split}} \widehat{\otimes}_{\mathcal{O}_F} \mathcal{O}_{\mathbb{F}}$, where $\mathbb{F}$ denotes the completion of a maximal unramified extension of $\mathcal{O}_F$. The group $\mathbb{Q}_p^\times \times \mathrm{GL}_n(F)$ acts on both of these formal schemes on the left.

The set of irreducible components in the special fiber of $\mathcal{M}$ is in bijection with the set $BT(0) \times \mathbb{Z} \times \mathbb{Z}$. We define a coloring map $\kappa : BT(0) \times \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ by sending (M, a, b) to $\kappa(M, a, b) = \log_q [M : \mathcal{O}_F^n] + b$. We observe that κ is equivariant for the action of the group $\mathbb{Q}_p^\times \times \mathrm{GL}_n(F)$, and its fibers are precisely the orbits of this group.

If we make some more choices, then we can get an even more concrete realization of this set. Let $B \subset U_0 = \mathrm{GL}_n(\mathcal{O}_F)$ denote the standard Iwahori subgroup inside the standard maximal compact subgroup. Let

$$\zeta = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ \varpi & 0 & \dots & 0 & 0 \end{pmatrix}.$$

For $i = 0, \dots, n-1$, let $U_i = \zeta^{-i} U_0 \zeta^i$. These maximal compact subgroups stabilize the $n-1$ distinct vertices of the closure of the unique chamber of BT fixed by B, and their intersection is exactly equal to B. Let $x_0, \dots, x_{n-1}$ denote these vertices. Then we have $\kappa(x_i, 1, 1) = i$, and therefore an isomorphism of $\mathbb{Q}_p^\times \times \mathrm{GL}_n(F)$-sets

$$BT(0) \times \mathbb{Z} \times \mathbb{Z} \cong \mathbb{Q}_p^\times/\mathbb{Z}_p^\times \times \coprod_{i=0}^{n-1} \mathrm{GL}_n(F)/U_i.$$

For each $i = 0, \dots, n-1$ there is then a bijection between the set of non-empty $(i+1)$-fold intersections of irreducible components of the special fiber of $\mathcal{M}$ and the set

$$BT(i) \times \mathbb{Z} \times \mathbb{Z} \cong \mathbb{Q}_p^\times/\mathbb{Z}_p^\times \times \coprod_{E \subset \{0, \dots, n-1\}} \mathrm{GL}_n(F)/U_E.$$

Here the disjoint union runs over subsets E of order $i+1$, and by definition we have $U_E = \cap_{i \in E} U_i$. Finally, we have the following.

Lemma 3.2. *1. Let $\Gamma \subset \mathrm{GL}_n(F)^0$ denote a discrete cocompact subgroup, and suppose that for all $x \in$ $\mathrm{BT}(0)$, the stabilizer $Z_\Gamma(x)$ is trivial. Then for all $x \in \mathrm{BT}(0)$ and for all $\gamma \in \Gamma$, $\gamma \neq 1$, we have $d(x, \gamma \cdot x) \geq 2$.*

The quotient $\Gamma\backslash\Omega_{\mathcal{O}_F}$ exists, and has a canonical algebraization, which is a projective algebraic variety, strictly semistable over $\mathcal{O}_F$. The irreducible components of its special fiber are geometrically irreducible and globally smooth.

2. *Let $\Gamma \subset \mathbb{Q}_p^\times \times \mathrm{GL}_n(F)$ denote a discrete cocompact subgroup, and suppose that for all $x \in \mathrm{BT}(0) \times \mathbb{Z} \times \mathbb{Z}$, the stabilizer $Z_\Gamma(x)$ is trivial. Then the quotient $\Gamma\backslash\mathcal{M}^{split}$ exists, and has a canonical algebraization, which is a projective algebraic variety, strictly semistable over $\mathcal{O}_F$. The irreducible components of its special fiber are geometrically irreducible and globally smooth.*

Proof. For the first part, we note that if $d(x, y) = 1$ then there exists a chamber in BT whose closure contains x, y. Then x, y are represented by $\mathcal{O}_F$-lattices $M_x \subset M_y$. If $\gamma \in \Gamma$ and $\gamma x = y$ then we must therefore have $x = y$ and hence $\gamma = 1$. The formal scheme $\Omega_{\mathcal{O}_F}$ has a covering by Zariski open subsets, formally of finite type over $\mathcal{O}_F$, which are in bijective correspondence with the set BT(0). Two Zariski opens intersect if and only if the corresponding vertices are connected by an edge. Thus Γ acts discontinuously with respect to this covering, and the quotient formal scheme can be obtained by simply gluing these Zariski opens. The ample line bundle which defines the algebraization is the relative dualizing sheaf over Spf $\mathcal{O}_F$, cf. [Mus78, Theorem 4.1].

For the second part, let $\Gamma^0 = \Gamma \cap (\mathbb{Z}_p^\times \times \mathrm{GL}_n(F)^0)$. The quotient $\Gamma\backslash\mathcal{M}^{\mathrm{split}}$ is a finite union of quotients of the form $\Gamma^0\backslash\Omega_{\mathcal{O}_F}$. □

4 A level raising formalism in banal characteristic

Let $p \neq l$ be distinct prime numbers. Let K be a finite extension of $\mathbb{Q}_l$ inside $\overline{\mathbb{Q}}_l$with ring of integers $\mathcal{O}$ and residue field k, and let F be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_F$ and uniformizer ϖ. We write q for the cardinality of the residue field $\mathcal{O}_F/\varpi$. We fix throughout a choice of square-root of q in K. Throughout this section we make the following assumption:

- l is a banal characteristic for $\mathrm{GL}_n(F)$. By definition, this means that l is coprime to the pro-order of $\mathrm{GL}_n(F)$.

In this section we show how one can prove level-raising results for $\mathrm{GL}_n(F)$-modules under the assumption that suitable cohomology groups are torsion free. Let $G = \mathrm{GL}_n(F)$. Let $T \subset P \subset G$ denote the standard maximal torus and Borel subgroup, and $R \subset \Phi^+ \subset \Phi$ the corresponding subsets of simple roots, positive roots, and roots of GL_n. Let $T_0 \subset T$ denote the unique maximal compact subgroup, and $B \subset G$ for the Iwahori subgroup. In this section, all admissible representations of G will be considered as being defined over $\overline{\mathbb{Q}}_l$.

If $\chi : T \to \overline{\mathbb{Q}}_l^\times$ is a continuous character, we define

$$\mathrm{Ind}_P^G \chi = \{f : G \to \overline{\mathbb{Q}}_l \mid f(bg) = \chi(b)f(g) \forall b \in P\},$$

the un-normalized induction. The normalized induction is defined as

$$\text{n-Ind}_P^G \chi = \mathrm{Ind}_P^G \delta_P^{1/2} \chi,$$

where $\delta_P : P \to \overline{\mathbb{Q}}_l^\times$ is the modulus character sending tn to $|t_1^{n-1} t_2^{n-3} \cdots t_n^{1-n}|$. In particular, $\text{n-Ind}_P^G \delta_P^{-1/2} = \mathrm{Ind}_P^G 1 = C^\infty(G/P)$ may be identified with the space of locally constant functions $G/P \to \overline{\mathbb{Q}}_l$. If π is an admissible representation of G, then we define the normalized restriction

$$r_P^G \pi = \delta_P^{-1/2} \otimes \pi_N.$$

This is an admissible representation of T, and the functor r_P^G is left adjoint to n-Ind_P^G. If π is an admissible representation of G and $\alpha \in \overline{\mathbb{Q}}_l^\times$, then we write $\pi(\alpha) = \pi \otimes (\det \circ \lambda_\alpha)$, where λ_α is the unramified character satisfying $\lambda_\alpha(\varpi) = \alpha$.

We describe the decomposition of $\text{n-Ind}_P^G \delta_P^{-1/2} = C^\infty(G/P, \overline{\mathbb{Q}}_l)$. Let $I \subset R$. We write P_I for the group generated by P and the subgroups $U_{-\alpha}$ for $\alpha \in I$. Thus $P_\emptyset = P$ and $P_R = G$. For each $I \subset J$ there is an injection $C^\infty(G/P_J, \overline{\mathbb{Q}}_l) \hookrightarrow C^\infty(G/P_I, \overline{\mathbb{Q}}_l)$. We define

$$\pi_I = C^\infty(G/P_I) / \sum_{I \subsetneq J} C^\infty(G/P_J).$$

Proposition 4.1. *The π_I are irreducible and pairwise non-isomorphic, and exhaust the composition factors of* $\text{n-Ind}_P^G \delta_P^{-1/2}$.

Proof. See [BW00, Chapter X]. A convenient reference for this and for some facts below is [Orl05]. □

The π_I may be described in terms of the Zelevinsky classification [Zel80] as follows. The irreducible constituents $\pi(\overrightarrow{\Gamma})$ of $\text{n-Ind}_P^G \delta_P^{-1/2}$ are in bijection with the orientations $\overrightarrow{\Gamma}$ of the graph Γ with vertices corresponding to the characters $|\cdot|^{(1-n)/2}, \ldots, |\cdot|^{(n-1)/2}$, and edges joining two characters whose quotient is $|\cdot|^{\pm 1}$. The elements of R are $\alpha_i = \epsilon_i - \epsilon_{i+1}$, for $i = 1, \ldots, n-1$. Given an orientation $\overrightarrow{\Gamma}$, we write $I(\overrightarrow{\Gamma}) \subset R$ for the subset of roots α_i such that the edge connecting $|\cdot|^{(1-n)/2+i-1}$ and $|\cdot|^{(1-n)/2+i}$ starts at the former and ends at the latter.

Proposition 4.2. *We have $\pi_{I(\overrightarrow{\Gamma})} \cong \pi(\overrightarrow{\Gamma})$. In particular, $\pi_\emptyset = \mathrm{St}_n$ is the Steinberg representation and π_R is the trivial representation of G.*

We now introduce part of the theory of the Bernstein center. If π is any admissible representation of G over $\overline{\mathbb{Q}}_l$, then we can endow the Iwahori invariants π^B with an action of the algebra $\overline{\mathbb{Q}}_l[T/T_0] \cong \overline{\mathbb{Q}}_l[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ as follows. Let $U \subset T/T_0$ denote the submonoid consisting of those elements

$$(\varpi^{a_1}, \ldots, \varpi^{a_n}) T_0 \in T/T_0$$

where $a_1 \geq a_2 \geq \cdots \geq a_n$ are integers. We let an element uT_0 act on π^B by the Hecke operator $[BuB]$. This induces an action of the algebra $\overline{\mathbb{Q}}_l[U]$, which extends uniquely to an action of the algebra $\overline{\mathbb{Q}}_l[T/T_0]$. We write $t_i = e_i(X_1, \ldots, X_n) \in \overline{\mathbb{Q}}_l[T/T_0]^W$, where e_i is the symmetric polynomial of degree i in n variables. As the notation indicates, these elements are fixed under the natural action of the Weyl group on $\overline{\mathbb{Q}}_l[T/T_0]$.

Proposition 4.3.
1. *For any admissible representation V of G over $\overline{\mathbb{Q}}_l$, there is a functorial isomorphism $V^B \cong (r_P^G V)^{T_0}$ of $\overline{\mathbb{Q}}_l[T/T_0]$-modules.*
2. *If π is an irreducible admissible representation of G over $\overline{\mathbb{Q}}_l$ and $\pi^B \neq 0$, then π is a subquotient of $\text{n-Ind}_P^G \chi$ for some unramified character $\chi = \chi_1 \otimes \cdots \otimes \chi_n$. The operator t_i has the unique eigenvalue $e_i(\chi_1(\varpi), \ldots, \chi_n(\varpi))$ on π^B.*

We introduce reduction modulo l, cf. [Vig94, §1.5]. Let V be an admissible G-module over $\overline{\mathbb{Q}}_l$ of finite length. We say that V admits an integral structure if there exists a G-invariant $\overline{\mathbb{Z}}_l$-lattice $\Lambda \subset V$ such that $\Lambda \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \cong V$. If V admits an integral structure, then the reduction modulo the maximal ideal of $\overline{\mathbb{Z}}_l$ of Λ is a finite length admissible representation of G over $\overline{\mathbb{F}}_l$. Its Jordan-Hölder factors are independent of the choice of integral structure.

If π is an irreducible admissible representation of G over $\overline{\mathbb{Q}}_l$, then it admits an integral structure if and only its cuspidal support does. In particular, if π is a subquotient of a principal series representation $\text{n-Ind}_P^G \chi$, then π admits an integral structure if and only if χ takes is values in $\overline{\mathbb{Z}}_l^\times \subset \overline{\mathbb{Q}}_l^\times$.

Proposition 4.4.
1. *Each representation π_I admits an integral structure, and its reduction modulo l is irreducible. We write $\pi_{I, \overline{\mathbb{F}}_l}$ for this reduced representation.*
2. *Let $\pi = \text{n-Ind}_Q^G \mathrm{St}_a(\alpha) \otimes \mathrm{St}_b(\beta)$ be an irreducible representation of G over $\overline{\mathbb{Q}}_l$ admitting an integral structure, where $a + b = n$, and Q is the standard parabolic subgroup corresponding to this partition. Then $\alpha, \beta \in \overline{\mathbb{Z}}_l$. Suppose that $\beta \equiv q^a \alpha \mod \mathfrak{m}_{\overline{\mathbb{Z}}_l}$. Then the reduction modulo l of π has exactly two Jordan-Hölder factors, which are both absolutely irreducible. The first is the reduction modulo l of $\pi_\emptyset(\alpha)$. The second is the reduction modulo l of $\pi_I(\alpha)$, where $I \subset R$ is such that $P_{R \setminus I} = Q$.*

Proof. For the first part, the existence of the integral structure is immediate from the remarks above. The irreducibility of the representations $\pi_{I,\overline{\mathbb{F}}_l}$ in banal characteristic seems to have first been noted by Lazarus [Laz00, Theorem 4.7.2]. Here we refer again to the work of Orlik [Orl05]. The second part follows from the corresponding fact in characteristic zero, cf. [HT01, Lemma I.3.2], and by reduction modulo l. $\square$

Suppose that M is a smooth $\mathcal{O}[G]$-module. We define cohomology groups $H^*(M)$ as follows. Let $U_0 = \mathrm{GL}_n(\mathcal{O}_F)$ denote the standard maximal compact subgroup, and let $U_1, \dots, U_{n-1}$ denote the conjugates of U_0 containing B, as defined in the previous section. Similarly, if $E \subset \{0, \dots, n-1\}$ is a subset then we write $U_E = \cap_{i \in E} U_i$. We define a complex $C^\bullet(M)$ by the formula

$$C^i(M) = \bigoplus_{E \subset \{0,\dots,n-1\}} M^{U_E},$$

the direct sum being over subsets E of cardinality $i+1$. The differential $d_i : C^i(M) \to C^{i+1}(M)$ is given by the sum of the restriction maps $r_{E,E'} : M^{U_E} \to M^{U_{E'}}$ for $E \subset E'$, each multiplied by the sign $\epsilon(E, E')$, where if $E' = \{i_1, \dots, i_r\}$, $i_1 < \cdots < i_r$, and $E = E' \setminus \{i_s\}$ then $\epsilon(E, E') = (-1)^s$. We then define $H^*(M)$ to be the cohomology of this complex.

Proposition 4.5. *1. Suppose that $M = \pi$ is an irreducible admissible representation of G over $\overline{\mathbb{Q}}_l$. Then $H^*(M)$ is non-zero if and only if π is an unramified twist of one of the representations π_I, $I \subset R$.*

2. If $M = \pi_I(\alpha)$ for some $\alpha \in \overline{\mathbb{Q}}_l$, then $H^i(M)$ is non-zero if and only if $i = \#(R \setminus I)$.

3. If $M = \pi_{I,\overline{\mathbb{F}}_l}(\overline{\alpha})$ for some $\alpha \in \overline{\mathbb{F}}_l$ then $H^i(M)$ is non-zero if and only if $i = \#(R \setminus I)$.

Proof. If $M = \pi$ is an irreducible admissible representation and $H^*(M) \neq 0$, then $\pi^B \neq 0$. In particular π is a subquotient of an unramified principal series representation, and its central character is unramified. After twisting, we can suppose that the center of G acts trivially on π. Then there is a canonical isomorphism $H^*(M) \cong H^*_{\mathrm{e}}(\mathrm{PGL}_n(F), M)$, these latter groups taken in the sense of [BW00, Ch. X, §5]. The first and second parts therefore follow from [BW00, Ch. X, Theorem 4.12]. The third part follows in a similar manner from [Orl05, Theorem 1]. $\square$

We now come to the main result of this section. Suppose that M, N are $\mathcal{O}$-flat admissible $\mathcal{O}[G]$-modules, in the sense that for each open compact subgroup $U \subset G$, M^U and N^U are finite free $\mathcal{O}$-modules. Suppose further that $M \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_l$ and $N \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_l$ are semisimple and that all of their irreducible constituents are generic, and that there is a perfect G-equivariant pairing $M \times N \to \mathcal{O}$.

Theorem 4.6. *Suppose that $M^B \neq 0$, and that if $\pi \subset M \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_l$ is an irreducible admissible representation of G sastisfying $\pi^B \neq 0$, then $\mathrm{rec}^T_F(\pi)$ has at most two irreducible constituents. Suppose that $H^{n-2}(N \otimes_{\mathcal{O}} k)$ and $H^{n-2}(M \otimes_{\mathcal{O}} k)$ are both zero. Finally, suppose that there exists $\overline{\alpha} \in \overline{\mathbb{F}}_l^\times$ such for any maximal ideal $(t_1 - \alpha_1, \dots, t_n - \alpha_n) \subset \overline{\mathbb{Q}}_l[T/T_0]^W$ in the support of M^B, we have $\alpha_i \equiv \overline{\alpha} e_i(q^{(n-1)/2}, \dots, q^{(1-n)/2})$ mod $\mathfrak{m}_{\overline{\mathbb{Z}}_l}$ for each $i = 1, \dots, n$. (Note that we necessarily have $\alpha_i \in \overline{\mathbb{Z}}_l$.) Then there exists $\alpha \in \overline{\mathbb{Z}}_l^\times$ lifting $\overline{\alpha}$ such that $\mathrm{St}_n(\alpha) \subset M \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_l$.*

Proof. After twisting by an unramified character, we can assume that $\overline{\alpha} = 1$. Decompose $N \otimes_{\mathcal{O}} k = N_0 \oplus N_1$, where N_0 is generated by $N^B \otimes_{\mathcal{O}} k$ and $N_1^B = 0$. (This is possible since the representations of G with non-zero Iwahori-fixed vectors form a block in the category of admissible representations of G over $\overline{\mathbb{F}}_l$.) Then the irreducible constituents of $N_0 \otimes_k \overline{\mathbb{F}}_l$ are of the form $\pi_\emptyset$ or $\pi_{\{\alpha\}}$ for some $\alpha \in R$, by Proposition 4.4. If $\pi_{\{\alpha\}} \hookrightarrow N \otimes_{\mathcal{O}} \overline{\mathbb{F}}_l$, then $H^{n-2}(N_0) \neq 0$. Thus $\pi_\emptyset \hookrightarrow N \otimes_{\mathcal{O}} \overline{\mathbb{F}}_l$. By duality, there is a surjection $M \otimes_{\mathcal{O}} \overline{\mathbb{F}}_l \twoheadrightarrow \pi_\emptyset$, hence $H^{n-1}(M \otimes_{\mathcal{O}} \overline{\mathbb{F}}_l) \neq 0$. Using that $H^{n-2}(M \otimes_{\mathcal{O}} \overline{\mathbb{F}}_l) = 0$, we deduce that $H^{n-1}(M \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_l) \neq 0$, and hence M that contains a twist of the Steinberg representation. $\square$

If π is an irreducible admissible representation of G over $\overline{\mathbb{Q}}_l$ which admits an integral structure, and $\pi^B \neq 0$, then we will say that π satisfies the level-raising congruence if there exists $\overline{\alpha} \in \overline{\mathbb{F}}_l^\times$ such that the eigenvalue α_i of t_i on π^B satisfies the congruence

$$\alpha_i \equiv \overline{\alpha} e_i(q^{(n-1)/2}, \dots, q^{(1-n)/2}) \bmod \mathfrak{m}_{\overline{\mathbb{Z}}_l}. \tag{4.1}$$

5 The weight spectral sequence

Let $\mathcal{O}_F$ be a complete discrete valuation ring, and $S = \operatorname{Spec} \mathcal{O}_F$. Write s for the closed point of S, η for the generic point. Let $F = \operatorname{Frac} \mathcal{O}_F$, and let $\overline{F}$ denote a fixed algebraic closure. We write $\overline{s}, \overline{\eta}$ for the induced geometric points of S above s and η, respectively. Suppose that $f : X \to S$ is a proper, strictly semistable (in the sense of [Sai03, §1.1]) morphism of relative dimension n. Then X_s is a strict normal crossings divisor on X; write $X_1, \dots, X_h$ for its irreducible components. We suppose moreover that each X_i is globally smooth over $\kappa(s)$. For $E \subset \{1, \dots, h\}$ we write X_E for the intersection $\cap_{i \in E} X_E$, and $X^{(m)} = \coprod_{\#E = m+1} X_E$ (disjoint union). Let K be a finite extension of $\mathbb{Q}_l$ with ring of integers $\mathcal{O}$, uniformizer λ, and residue field k, where l is coprime to the residue characteristic of $\mathcal{O}_F$. Let $\Lambda = K$, $\mathcal{O}$, or k, and let V be a local system of flat Λ-modules on X. The weight spectral sequence of Rapoport-Zink is a spectral sequence

$$E_1^{p,q} = \bigoplus_{i \geq \max(0,-p)} H^{q-2i}(X_{\overline{s}}^{(p+2i)}, V(-i)) \Rightarrow H^{p+q}(X_{\overline{K}}, V). \tag{5.1}$$

It is equivariant for the natural action of G_F on both sides, and the differentials commute with the action of the group G_F. Note that the groups $E_1^{p,q}$ vanish for $q < 0$ and $q > 2n$. Let us briefly recall the construction of this spectral sequence, following Saito [Sai03]. Consider the following diagram:

$$\begin{tikzcd}
X_{\overline{s}} \arrow[r, "\overline{i}"] \arrow[d] & X_{\mathcal{O}_{\overline{F}}} \arrow[d] & X_{\overline{\eta}} \arrow[l, "\overline{j}"'] \arrow[d] \\
X_s \arrow[r, "i"] & X & X_\eta. \arrow[l, "j"']
\end{tikzcd}$$

The complex $R\Psi V = \overline{i}^* R\overline{j}_* V$ in $D_c^b(X_{\overline{s}}, V)$ of nearby cycles receives an action of the inertia group $I_F \subset G_F = \operatorname{Gal}(\overline{F}/F)$. Let $T \in I_F$ denote an element that maps to a generator of $\mathbb{Z}_l(1)$ under the canonical homomorphism $t_l : I_F \to \mathbb{Z}_l(1)$. Let ν denote the endomorphism of $R\Psi V$ induced by the element $T - 1$. We then have (cf. [Sai03, §2]):

Proposition 5.1. 1. *$R\Psi V$ lies in the abelian subcategory $\operatorname{Perv}(X_{\overline{s}}, \Lambda)[-n]$ of $-n$-shifted perverse sheaves with Λ-coefficients.*

2. *Let $M_\bullet$ denote the increasing monodromy filtration of the nilpotent endomorphism ν of $R\Psi\Lambda$. For each positive integer $p \geq 0$, let $a_p : X_{\overline{s}}^{(p)} \to X_{\overline{s}}$ denote the canonical map. Then for each integer $r \geq 0$ there is a canonical isomorphism*

$$\bigoplus_{p-q=r} a_{p+q,*} V(-p)[-(p+q)] \cong \operatorname{Gr}_r^M R\Psi V,$$

compatible with the action of G_F on either side.

The weight spectral sequence is now the spectral sequence associated to the filtered object $R\Psi V$. Note that [Sai03] treats only the case of constant coefficients, but the case of twisted coefficients can be reduced to this one by working étale locally on X.

We compute the first row of the spectral sequence of the pair (X, V). By definition, we have $E_1^{p,0} = H^0(X_{\overline{s}}^{(p)}, V)$. Define a simplicial complex $\mathcal{K}$ as follows: the vertices of $\mathcal{K}$ are in bijection with the X_i, and the set $\{X_{i_1}, \dots, X_{i_r}\}$ corresponds to a simplex σ_E if and only if the intersection X_E is non-empty, $E = \{i_1, \dots, i_r\}$. We define a coefficient system $\mathcal{V}$ on $\mathcal{K}$ by the assignment $\sigma_E \mapsto H^i(X_{E,\overline{s}}, V)$. Let $C^\bullet(\mathcal{K}, \mathcal{V})$ denote the complex calculating the simplicial cohomology of $\mathcal{K}$ with coefficients in $\mathcal{V}$. Thus, by definition we have

$$C^r(\mathcal{K}, \mathcal{V}) = \bigoplus_{E \subset \{1,\dots,h\}} H^i(X_{E,\overline{s}}, V),$$

the sum being over subsets E of cardinality $r+1$. The differential $d_r = C^r(\mathcal{K}, \mathcal{V}) \to C^{r+1}(\mathcal{K}, \mathcal{V})$ is given by the direct sum of the restriction maps

$$\mathrm{res}_{E,E'} : H^0(X_{E,\bar{s}}, V) \to H^0(X_{E',\bar{s}}, V),$$

each multiplied by the sign $\epsilon(E, E')$ of the previous section.

Proposition 5.2. *There is a canonical isomorphism of complexes $E_1^{\bullet,0} \cong C^\bullet(\mathcal{K}, \mathcal{V})$.*

Proof. In the case $V = \Lambda$, this follows immediately from [Sai03, Proposition 2.10]. Again, the case of general V can be reduced to this one by working étale locally. □

6 Shimura varieties and uniformization

Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, and let E be a CM imaginary field with totally real subfield F. We fix a rational prime p, and suppose that p is totally inert in F. We suppose that the unique prime v of F above p is split in E as $v = ww^c$. We let d denote the degree of F over $\mathbb{Q}$. We fix embeddings ϕ_∞, ϕ_p of $\overline{\mathbb{Q}}$ into $\mathbb{C}, \overline{\mathbb{Q}}_p$, respectively. The composite $\phi_\infty \circ \phi_p^{-1}$ induces a bijection of sets

$$\mathrm{Hom}(E, \mathbb{C}) \leftrightarrow \mathrm{Hom}(E, \overline{\mathbb{Q}}_p).$$

Let $n \geq 2$ be an integer, and let D be a central division algebra over E of dimension n^2, whose invariants at the places w and w^c are given respectively by $1/n$ and $-1/n$. We suppose that at every other place of F, D is split. Let $*$ be a positive involution on D. Let $V = D$, viewed as a D-module, and let $\psi : V \times V \to \mathbb{Q}$ be an alternating pairing satisfying the condition $\psi(dv, w) = \psi(v, d^* w)$ for all $d \in D$, $v, w \in V$. Fix a CM-type $\Phi \subset \mathrm{Hom}(E, \mathbb{C})$. Then we can choose an isomorphism $D \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{\tau \in \Phi} D \otimes_{E,\tau} \mathbb{C} \cong \prod_{\tau \in \Phi} M_n(\mathbb{C})$, such that $*$ corresponds to the operation $X \mapsto {}^t\overline{X}$.

Similarly we may decompose $V \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{\tau \in \Phi} V \otimes_{E,\tau} \mathbb{C}$. We can find isomorphisms $V \otimes_{E,\tau} \mathbb{C} = \mathbb{C}^n \otimes_{\mathbb{C}} W_\tau$, where $M_n(\mathbb{C})$ acts on the first factor. The form ψ_τ then admits a decomposition

$$\psi_\tau(z_1 \otimes w_1, z_2 \otimes w_2) = \mathrm{tr}_{\mathbb{C}/\mathbb{R}}({}^t\overline{z}_1 \cdot z_2 h_\tau(w_1, w_2)),$$

where h_τ is a skew-hermitian form on W_τ. We can find a basis $\{e_1, \ldots, e_n\}$ of W_τ such that h_τ is given by the matrix

$$\mathrm{diag}(\underbrace{-i, \ldots, -i}_{r_\tau}, \underbrace{i, \ldots, i}_{r_{\tau^c}}),$$

where $r_\tau + r_{\tau^c} = n$. We define algebraic groups over $\mathbb{Q}$ by their functors of R-points:

$$G(R) = \{g \in \mathrm{GL}_D(V \otimes R) \mid \psi(gv, gw) = c(g)\psi(v, w), c(g) \in R^\times\}.$$

$$G_1(R) = \{g \in \mathrm{GL}_D(V \otimes R) \mid \psi(v, w) = \psi(v, w)\}.$$

The choices above give rise to an embedding $G_{\mathbb{R}} \hookrightarrow \prod_{\tau \in \Phi} GU(r_\tau, r_{\tau^c})$. We write $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_{\mathbb{R}}$ for the homomorphism which corresponds under this identification to the map

$$h : z \in \mathbb{C}^\times \mapsto (\mathrm{diag}(\underbrace{z, \ldots, z}_{r_\tau}), \mathrm{diag}(\underbrace{\overline{z}, \ldots, \overline{z}}_{r_{\tau^c}}))_{\tau \in \Phi}.$$

Let X denote the $G(\mathbb{R})$-conjugacy class of h inside the set of homomorphisms $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_{\mathbb{R}}$.

We now suppose that Φ corresponds under the identification $\mathrm{Hom}(E, \mathbb{C}) \leftrightarrow \mathrm{Hom}(E, \overline{\mathbb{Q}}_p)$ to the set of embeddings inducing the p-adic place w of E. Write $\tau_1, \ldots, \tau_d$ for the elements of Φ; we suppose that $r_{\tau_1} = 1$ and $r_{\tau_i} = 0, i = 2, \ldots, d$. We will also assume that the group G is quasi-split at every finite place not dividing p. PEL data $(D, E, *, F, V, \psi)$ satisfying these assumptions exist provided that $[F : \mathbb{Q}]$ is even, which we always assume in the applications below, cf. [HT01, Lemma I.7.1].

Proposition 6.1. *The pair (G, X) is a Shimura datum. For $U \subset G(\mathbb{A}^\infty)$ a neat open compact subgroup, the Shimura varieties $S(G, U)$ with $S(G, U)(\mathbb{C}) = G(\mathbb{Q})\backslash G(\mathbb{A}^f) \times X/U$ are smooth projective algebraic varieties over $\mathbb{C}$, and admit canonical models over the reflex field $\tau_1(F) \subset \mathbb{C}$.*

The varieties $S(G, U)$ admit p-adic uniformizations. Let $\nu = \phi_p \phi_\infty^{-1}$ denote the induced embedding of $\tau_1(F)$ into $\overline{\mathbb{Q}}_p$. According to [RZ96, §6], there exists an inner form I of G over $\mathbb{Q}$ and isomorphisms $I(\mathbb{A}^{p,\infty}) \cong G(\mathbb{A}^{p,\infty}), I(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times \mathrm{GL}_n(F_v)$, and satisfying the following. Let $\mathbb{F}$ denote the completion of the maximal unramified extension of F_v. The group $I(\mathbb{Q})$ acts on $\Omega_{\mathcal{O}_{F_v}} \widehat{\otimes}_{\mathcal{O}_{F_v}} \mathcal{O}_\mathbb{F}$ via the scalar extension of its action on $\Omega_{\mathcal{O}_{F_v}}$ through the map $I(\mathbb{Q}) \subset I(\mathbb{Q}_p) \to \mathrm{PGL}_n(F_v)$. It also acts on $G(\mathbb{A}^\infty)/U_p$, where $U_p \subset G(\mathbb{Q}_p)$ is the unique maximal compact subgroup, as follows. There is an isomorphism $G(\mathbb{A}^\infty)/U_p = G(\mathbb{A}^{p,\infty}) \times G(\mathbb{Q}_p)/U_p \cong I(\mathbb{A}^{p,\infty}) \times G(\mathbb{Q}_p)/U_p$. $I(\mathbb{Q})$ acts diagonally under this identification via the natural action on $I(\mathbb{A}^{p,\infty})$ and as follows on $G(\mathbb{Q}_p)/U_p$. The choice of place w of E induces a canonical isomorphism $G(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times D_w^\times$. Let $\Pi \in D_w^\times$ denote a uniformizer. Then an element $(c, a) \in I(\mathbb{Q}_p)$ acts by the formula (cf. [RZ96, Lemma 6.45])

$$(c, a) \cdot (c', a') \mod U_p = (cc', \Pi^{\mathrm{val}_{F_v} \det a} a') \mod U_p,$$

where val_{F_v} is normalized so that $\mathrm{val}_{F_v}(F_v^\times) = \mathbb{Z}$. The following theorem is now [RZ96, Corollary 6.51]. In what follows, we say that an open compact subgroup of $G(\mathbb{A}^{p,\infty}) \cong I(\mathbb{A}^{p,\infty})$ is sufficiently small if there exists a prime $q \neq p$ such that the projection of U to $G(\mathbb{Q}_q)$ contains no non-trivial elements of finite order.

Theorem 6.2. *With notations as above, for each sufficiently small open compact subgroup $U^p \subset G(\mathbb{A}^{p,\infty})$, there is an integral model of $S(G, U^p U_p) \otimes_{\tau_1(F),\nu} F_v$ over $\mathcal{O}_{F_v}$ and a canonical isomorphism of formal schemes over* $\mathrm{Spf}\, \mathcal{O}_\mathbb{F}$

$$I(\mathbb{Q})\backslash [\mathcal{M} \times G(\mathbb{A}^{p,\infty})/U^p] \cong \left(S(G, U^p U_p) \otimes_{\mathcal{O}_{F_v}} \mathcal{O}_\mathbb{F}\right)\widehat{}.$$

This isomorphism is equivariant with respect to the action of the prime-to-p Hecke algebra $\mathcal{H}(G(\mathbb{A}^{p,\infty}) /\!\!/ U^p)$ on either side.

From now on, we shall write $S(G, U^p U_p)$ to mean this integral model over $\mathcal{O}_{F_v}$. We will only consider open compact subgroups $U = U^p U_p$, with U_p maximal compact, so that this will always be defined. As is well-known, the left hand side in the above equation can be rewritten as a finite union of quotients of $\Omega_{\mathcal{O}_{F_v}} \widehat{\otimes}_{\mathcal{O}_{F_v}} \mathcal{O}_\mathbb{F}$. Indeed, the double quotient $I(\mathbb{Q})\backslash G(\mathbb{A}^{p,\infty})/U^p$ is finite. Let $g_1, \ldots, g_s$ be representatives, and let $\Gamma_i = I(\mathbb{Q}) \cap (g_i U^p g_i^{-1} \times \widetilde{U}_p)$, the intersection taken inside $I(\mathbb{A}^\infty)$. Here $\widetilde{U}_p \subset I(\mathbb{Q}_p) = \mathbb{Q}_p^\times \times \mathrm{GL}_n(F_v)$ is the subgroup $\mathbb{Z}_p^\times \times (\mathrm{val}_{F_v} \circ \det)^{-1}(0)$. Each $\Gamma_i \subset \mathbb{Q}_p^\times \times \mathrm{GL}_n(F_v)$ is a discrete cocompact subgroup, and there is an isomorphism (cf. Lemma 3.2):

$$(S(G, U) \otimes_{\mathcal{O}_{F_v}} \mathcal{O}_\mathbb{F})\widehat{} \cong \coprod_{i=1}^{s} \Gamma_i \backslash \mathcal{M}.$$

6.1 Automorphic local systems

From now on we consider only sufficiently small open compact subgroups $U = U^p U_p$ as in Theorem 6.2. We now introduce some local systems on the varieties $S(G, U)$ corresponding to algebraic representations of G. Corresponding to the infinity type Φ, there is an isomorphism

$$G(\mathbb{C}) \cong \mathbb{C}^\times \times \prod_{\tau \in \Phi} \mathrm{GL}_n(\mathbb{C}).$$

We write $T \subset G \otimes_\mathbb{Q} \mathbb{C}$ for the product of the diagonal maximal tori:

$$T(\mathbb{C}) \cong \mathbb{C}^\times \times \prod_{\tau \in \Phi} \underbrace{\mathbb{C}^\times \times \cdots \times \mathbb{C}^\times}_{n}.$$

Then there is a canonical isomorphism $X^*(T) \cong \mathbb{Z} \times (\mathbb{Z}^n)^\Phi$, and we write $X^*(T)_+$ for the subset of dominant weights $\boldsymbol{\mu} = (c, (\mu_\tau)_{\tau\in\Phi})$, namely those satisfying the condition

$$\mu_{\tau,1} \geq \mu_{\tau,2} \geq \cdots \geq \mu_{\tau,n}$$

for each embedding $\tau : E \hookrightarrow \mathbb{C}$ in Φ. If l is a rational prime, we say that $\boldsymbol{\mu}$ is l-small if for each $\tau \in \Phi$, we have

$$0 \leq \mu_{\tau,i} - \mu_{\tau,j} < l \tag{6.1}$$

for all $0 \leq i < j \leq n$. If l is unramified in E and $\boldsymbol{\mu}$ is l-small, we associate to $\boldsymbol{\mu}$ an l-adic local system on $S(G,U)$ as follows, cf. [HT01, §III.2], [Har, §7.1]. Fix a choice of isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$, and let K be a finite extension of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$ with ring of integers $\mathcal{O}$, maximal ideal λ, and residue field k. Let $U_l \subset G(\mathbb{Q}_l)$ be a hyperspecial maximal compact subgroup. We suppose that the algebraic representation of $G \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_l$ of highest weight $\iota^{-1}\boldsymbol{\mu}$ can be defined over K. Let $W_{\boldsymbol{\mu},K}$ denote this representation. There is, up to homothety, a unique U_l-invariant $\mathcal{O}$-lattice of $W_{\boldsymbol{\mu},K}$. Choose one and write it as $W_{\boldsymbol{\mu},\mathcal{O}}$. It is unique since, by the l-small hypothesis, the reduced lattice $W_{\boldsymbol{\mu},k} = W_{\boldsymbol{\mu},\mathcal{O}} \otimes_{\mathcal{O}} k$ is an irreducible representation of U_l, and up to isomorphism does not depend on the choice of invariant lattice.

Given an integer $m \geq 1$, let $U(m) = U^p(m)U_p \subset U$ denote a normal open compact subgroup which acts trivially on $W_{\boldsymbol{\mu},\mathcal{O}/\lambda^m} = W_{\boldsymbol{\mu},\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{O}/\lambda^m$. Then U acts on the constant sheaf $\underline{W_{\boldsymbol{\mu},\mathcal{O}/\lambda^m}}$ on $S(G,U(m))$ in a way covering its action on $S(G,U)$, and the quotient defines an étale local system on $S(G,U)$, which we write as $V_{\boldsymbol{\mu},\mathcal{O}/\lambda^m}$. The sections of $V_{\boldsymbol{\mu},\mathcal{O}/\lambda^m}$ over an étale open $T \to S(G,U)$ can be identified with the set of functions $f : \pi_0(S(G,U(m)) \times_{S(G,U)} T) \to W_{\boldsymbol{\mu}}$ such that for all $\sigma \in U, C \in \pi_0(S(G,U(m)) \times_{S(G,U)} T)$, we have the relation $f(C\sigma) = \sigma^{-1}f(C)$. We then take $V_{\boldsymbol{\mu},\mathcal{O}} = \varprojlim_m V_{\boldsymbol{\mu},\mathcal{O}/\lambda^m}$ and $V_{\boldsymbol{\mu},K} = V_{\boldsymbol{\mu},\mathcal{O}} \otimes_{\mathcal{O}} K$. These local systems are isomorphic to the local systems ${}_{\text{ét}}\underline{V}_{[\mu]}$ constructed in [LS12, §4.3] using geometric means.

6.2 A split descent

The scheme $S(G,U) \otimes_{\mathcal{O}_{F_v}} \mathcal{O}_{\mathbb{F}}$ has another descent $S(G,U)^{\text{split}}$ to $\mathcal{O}_{F_v}$ whose p-adic formal completion is given by

$$S(G,U)^{\text{split}} = I(\mathbb{Q})\backslash \left[\mathcal{M}^{\text{split}} \times G(\mathbb{A}^{p,\infty})/U^p\right] \cong \coprod_i \Gamma_i \backslash \mathcal{M}^{\text{split}}.$$

This is *not* the descent defined by $S(G,U)$. However, the local systems $V_{\boldsymbol{\mu},\Lambda}$, where $\Lambda = K$, $\mathcal{O}$ or $\mathcal{O}/\lambda^m$, also admit descents to $S(G,U)^{\text{split}}$, using exactly the same recipe as before. We write $V^{\text{split}}_{\boldsymbol{\mu},\Lambda}$ for the local systems defined this way.

Lemma 6.3. *The pullback of $V^{split}_{\boldsymbol{\mu},k}$ to any irreducible (hence geometrically irreducible) component Y of the special fiber of $S(G,U)^{split}$ is a constant sheaf. If $Y_1, \dots, Y_s$ are irreducible components of the special fiber of $S(G,U)^{split}$, then Frobenius acts as the scalar $q_v^{i/2}$ on the group $H^i((Y_1 \cap \cdots \cap Y_s)_{\overline{s}}, V^{split}_{\boldsymbol{\mu},k})$. (We recall that this group is zero if i is odd.)*

Proof. Let $Y \subset S(G,U(1))^{\text{split}}$ denote an irreducible component of the special fiber of this scheme. Let $\pi : S(G,U)^{\text{split}} \to S(G,U)^{\text{split}}$ denote the natural projection. Then the restriction $\pi|_Y$ induces an isomorphism from Y to its image in $S(G,U)^{\text{split}}$. Pulling back $V^{\text{split}}_{\boldsymbol{\mu},k}$ by the inverse of this isomorphism now gives the first assertion. The second assertion now follows from the first and Proposition 3.1. □

6.3 Hecke actions and weight spectral sequence

We now compute the complex of abelian groups $C^\bullet(\mathcal{K}, \mathcal{V})$ of Proposition 5.2 for the local system $V_{\boldsymbol{\mu},k}$ on the Shimura variety $S(G,U)$ in terms of the p-adic uniformization

$$I(\mathbb{Q})\backslash [\mathcal{M} \times G(\mathbb{A}^{p,\infty})/U^p] \cong (S(G,U) \otimes_{\mathcal{O}_{F_v}} \mathcal{O}_{\mathbb{F}})\widehat{\ }.$$

Since U is sufficiently small, the irreducible components of the special fiber are in bijection with the set

$$I(\mathbb{Q})\backslash\left[\mathrm{BT}(0)\times\mathbb{Q}_p^\times/\mathbb{Z}_p^\times\times\mathrm{GL}_n(F_v)/\mathrm{GL}_n(F_v)^0\times I(\mathbb{A}^{p,\infty})/U^p\right]\cong\coprod_{i=0}^{n-1}I(\mathbb{Q})\backslash I(\mathbb{A}^\infty)/\mathbb{Z}_p^\times U_iU^p,$$

where the subgroup $U_i\subset\mathrm{GL}_n(F_v)$ is as in §3. For each $i=0,\ldots,n-1$, there is now a bijection

$$\pi_0(S(G,U)^{(i)}_{\overline{s}})\cong\coprod_{E\subset\{0,\ldots,n-1\}}I(\mathbb{Q})\backslash I(\mathbb{A}^\infty)/\mathbb{Z}_p^\times U_EU^p,$$

the union running over subsets E of cardinality $i+1$. If $x\in I(\mathbb{Q})\backslash I(\mathbb{A}^\infty)/\mathbb{Z}_p^\times U_EU^p$, then the images of x under the natural maps $I(\mathbb{Q})\backslash I(\mathbb{A}^\infty)/\mathbb{Z}_p^\times U_EU^p\to I(\mathbb{Q})\backslash I(\mathbb{A}^\infty)/\mathbb{Z}_p^\times U_{E\backslash\{i\}}U^p$, $i\in E$, correspond exactly to those i-fold intersections of irreducible components which contain the $(i+1)$-fold intersection corresponding to x.

For the weight spectral sequence to be defined for $S(G,U)$, we must first choose a partial ordering of the set of irreducible components of the special fiber which restricts to a total ordering on all subsets of irreducible components which have non-trivial intersection. We choose the partial ordering on $\mathbb{Z}/n\mathbb{Z}$ given by $0\leq\cdots\leq n-1$, and pull this back to the set $I(\mathbb{Q})\backslash\left[\mathrm{BT}(0)\times\mathbb{Q}_p^\times/\mathbb{Z}_p^\times\times\mathrm{GL}_n(F_v)/\mathrm{GL}_n(F_v)^0\times I(\mathbb{A}^{p,\infty})/U^p\right]$ via the function κ defined in §3. Let $E_1^{p,q}\Rightarrow H^{p+q}(S(G,U)_{\overline{\eta}},V_{\boldsymbol{\mu},k})$ denote the weight spectral sequence of §5. We observe that the groups $E_1^{p,q}$ are zero if q is odd, and if $q=2k$ is even then the groups $E_1^{p,2k}$ are non-zero only if $-k\leq p\leq n-1-k$.

Proposition 6.4. *1. For each $i=0,\ldots,n-1$, there is a canonical isomorphism*

$$E_1^{i,0}\cong\bigoplus_{E\subset\{0,\ldots,n-1\}}\mathcal{A}(\mathbb{Z}_p^\times U^pU_E,W_{\boldsymbol{\mu},k}),$$

the direct sum running over the set of all subsets E of order $i+1$.

2. There is a canonical isomorphism of complexes

$$E_1^{\bullet,0}\cong C^\bullet(\mathcal{A}(\mathbb{Z}_p^\times U^p,W_{\boldsymbol{\mu},k})),$$

and hence for each $i=0,\ldots,n-1$,

$$E_2^{i,0}\cong H^i(\mathcal{A}(\mathbb{Z}_p^\times U^p,W_{\boldsymbol{\mu},k})).$$

Proof. By definition we have $E_1^{i,0}=H^0(S(G,U)^{(i)}_{\overline{s}},V_{\boldsymbol{\mu},k})$, and this space can be identified with the set of all functions $f:\pi_0(S(G,U(1))^{(i)}_{\overline{s}})\to W_{\boldsymbol{\mu},k}$ satisfying the relation $f(C\sigma)=\sigma^{-1}f(C)$ for all $C\in\pi_0(S(G,U(1))^{(i)}_{\overline{s}})$, $\sigma\in U$. We have identified the set $\pi_0(S(G,U(1))^{(i)}_{\overline{s}})$ with $\coprod_E I(\mathbb{Q})\backslash I(\mathbb{A}^\infty)/\mathbb{Z}_p^\times U_EU^p$, compatibly as U varies. The isomorphism of the first part of the proposition now follows from the very definition of the spaces $\mathcal{A}(\mathbb{Z}_p^\times U^pU_E,W_{\boldsymbol{\mu},k})$.

The the remainder of the proposition, it remains to show that the differentials in the two complexes correspond under the isomorphism of the first part. This follows after noting that the restriction maps of sections under this isomorphism to the natural inclusions $\mathcal{A}(\mathbb{Z}_p^\times U_EU^p,W_{\boldsymbol{\mu},k})\to\mathcal{A}(\mathbb{Z}_p^\times U_{E'}U^p,W_{\boldsymbol{\mu},k})$, and that the signs that must be inserted in either complex agree because of the choices we have made. □

6.4 Degeneration

Proposition 6.5. *Let $r=2s+1$. With notation as above, the differentials*

$$d_r^{p,q}:E_r^{p,q}\to E_r^{p+r,q+1-r}$$

are all zero as long as $q_v^s\not\equiv 1$ modulo l.

Proof. We recall that the differentials in the weight spectral sequence are Galois equivariant. The proposition would therefore follow if the action of Frobenius on $E_1^{p,q}$ was given by the scalar $q_v^{q/2}$. (We recall that these groups are zero if q is odd.) This is not the case. However, this is the case for the weight spectral sequence of the pair $(S(G,U)^{\text{split}}, V_{\boldsymbol{\mu},k}^{\text{split}})$, by Lemma 6.3. The weight spectral sequence of a pair (X, V), where X is a strictly semistable scheme over $\mathcal{O}_{F_v}$ and V is a local system on X, viewed as a spectral sequence of abelian groups (forgetting the Galois action), depends only on $(X \otimes_{\mathcal{O}_{F_v}} \mathcal{O}_{\mathbb{F}}, V)$, i.e. the pullback of X to the maximal unramified extension of $\mathcal{O}_{F_v}$. Since the pairs $(S(G,U)^{\text{split}}, V_{\boldsymbol{\mu},k}^{\text{split}})$ and $(S(G,U), V_{\boldsymbol{\mu},k})$ become canonically isomorphic over $\mathcal{O}_{\mathbb{F}}$, we are done. $\square$

Corollary 6.6. *Suppose that l is a banal characteristic for $\mathrm{GL}_n(F_v)$. Then the weight spectral sequence for the pair $(S(G,U), V_{\boldsymbol{\mu},k})$ degenerates at E_2, and there is for each $i \geq 0$ an injection, equivariant for the prime-to-p Hecke algebra $\mathcal{H}(G(\mathbb{A}^{p,\infty}) /\!\!/ U^p)$:*

$$H^i(\mathcal{A}(\mathbb{Z}_p^\times U^p, W_{\boldsymbol{\mu},k})) \hookrightarrow H^i(S(G, U_p U^p)_{\overline{F_v}}, V_{\boldsymbol{\mu},k}).$$

6.5 Raising the level

We now suppose in addition that $E = F \cdot E_0$, where E_0 is a quadratic imaginary extension of $\mathbb{Q}$ and that E/F is everywhere unramified. We now change notation slightly and write v_0 for the place of F above the rational prime p, and w_0 for one of the places of E above it. Let $l \neq p$ be another prime, and fix an isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$. We assume that l is unramified in E.

Let $\boldsymbol{\mu}$ be a choice of l-small dominant weight, and let $U = \prod_q U_q \subset I(\mathbb{A}^\infty)$ denote a open compact subgroup. Then there is defined a finite free $\mathcal{O}$-module $W_{\boldsymbol{\mu},\mathcal{O}}$ on which U_l acts, and a space of automorphic forms $\mathcal{A}(U, W_{\boldsymbol{\mu},\mathcal{O}})$. It is a finite free $\mathcal{O}$-module. We recall that this space has the following interpretation. Let $\mathcal{A}$ denote the space of automorphic forms on I, a semisimple admissible representation of $I(\mathbb{A})$. Let $W_{\boldsymbol{\mu},\mathbb{C}}$ denote the representation of $I(\mathbb{R}) \subset I(\mathbb{C}) \cong \mathbb{C}^\times \times \prod_{\tau \in \Phi} \mathrm{GL}_n(\mathbb{C})$ which is the restriction of the algebraic representation of highest weight $\boldsymbol{\mu}$. Then there is an isomorphism

$$\mathcal{A}(U, W_{\boldsymbol{\mu},\mathcal{O}}) \otimes_{\mathcal{O},\iota} \mathbb{C} \cong \mathrm{Hom}_{I(\mathbb{R})}(W_{\boldsymbol{\mu},\mathbb{C}}^\vee, \mathcal{A}).$$

If T is a finite set of rational primes containing l, and such that U_q is a hyperspecial maximal compact subgroup for all $q \notin T$, let $\mathbb{T}_T^{\text{univ}} = \mathcal{O}[\{T_1^v, \ldots, T_n^v, (T_n^v)^{-1}\}]$ denote the the polynomial ring in infinitely many indeterminates corresponding to the unramified Hecke operators at places v of F which split in E and are not contained in T. Then $\mathbb{T}_T^{\text{univ}}$ acts on $\mathcal{A}(U, W_{\boldsymbol{\mu},\mathcal{O}})$ by $\mathcal{O}$-algebra endomorphisms, and on the spaces $H^i(S(G,U)_{\overline{E}}, V_{\boldsymbol{\mu},k})$, via the fixed isomorphism $I(\mathbb{A}^{p,\infty}) \cong G(\mathbb{A}^{p,\infty})$. If σ is an automorphic representation of $I(\mathbb{A})$ such that $(\sigma^\infty)^U \neq 0$ and $\sigma_\infty \cong W_{\boldsymbol{\mu},\mathbb{C}}^\vee$, then we can associate to it a maximal ideal $\mathfrak{m}_\sigma \subset \mathbb{T}_T^{\text{univ}}$ by assigning to each Hecke operator the reduction modulo l of its eigenvalue on $\iota^{-1}(\sigma^\infty)^U \subset \mathcal{A}(U, W_{\boldsymbol{\mu},\mathcal{O}}) \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_l$. If σ' is another automorphic representation of $I(\mathbb{A})$, we say that σ' contributes to $\mathcal{A}(U, W_{\boldsymbol{\mu},\mathcal{O}})_{\mathfrak{m}_\sigma}$ if $\sigma'_\infty \cong W_{\boldsymbol{\mu},\mathbb{C}}^\vee$, $(\sigma'^\infty)^U \neq 0$, and the intersection of $(\iota^{-1}\sigma'^\infty)^U$ and $\mathcal{A}(U, W_{\boldsymbol{\mu},\mathcal{O}})_{\mathfrak{m}_\sigma}$ inside $\mathcal{A}(U, W_{\boldsymbol{\mu},\mathcal{O}}) \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_l$ is non-trivial.

There is an isomorphism $\iota_{w_0} : I(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times \mathrm{GL}_n(E_{w_0})$, and if σ_p is an irreducible admissible representation of $I(\mathbb{Q}_p)$, then $(\sigma_p \circ \iota_{w_0})|_{\mathrm{GL}_n(E_{w_0})}$ remains irreducible. We assume that $\iota_{w_0}(U_p) = \mathbb{Z}_p^\times \times B$, where $B \subset \mathrm{GL}_n(E_{w_0})$ is the standard Iwahori subgroup. We write $U_p' \subset G(\mathbb{Q}_p)$ for the unique maximal compact subgroup.

Theorem 6.7. *Suppose that σ is as above, and let $\mathfrak{m}_\sigma \subset \mathbb{T}_T^{univ}$ denote the associated maximal ideal. Suppose that the following hypotheses hold:*

1. *The group U^p is a sufficiently small open compact subgroup of $I(\mathbb{A}^{p,\infty})$.*
2. *If σ' is another automorphic representation which contributes to $\mathcal{A}(U, W_{\boldsymbol{\mu},\mathcal{O}})_{\mathfrak{m}_\sigma}$, then $(\sigma'_p \circ \iota_{w_0})|_{\mathrm{GL}_n(E_{w_0})}$ is a subquotient of a parabolic induction $\text{n-Ind}_Q^G \mathrm{St}_a(\alpha) \otimes \mathrm{St}_b(\beta)$ for some $a + b = n$.*
3. *$\iota^{-1}\sigma_{1,w_0}$ satisfies the level-raising congruence (4.1).*

4. *$\boldsymbol{\mu}$ is l-small (6.1) and l is a banal characteristic for $\mathrm{GL}_n(E_{w_0})$.*

5. *The groups $H^{n-2}(S(G, U^p U_p')_{\overline{F_v}}, V^\vee_{\boldsymbol{\mu},k})$ and $H^{n-2}(S(G, U^p U_p')_{\overline{F_v}}, V_{\boldsymbol{\mu},k})$ are zero.*

Then we can raise the level: there exists another irreducible constituent σ' contributing to $\mathcal{A}(U, W_{\boldsymbol{\mu},\mathcal{O}})_{\mathfrak{m}_\sigma}$, and such that σ' is an unramified twist of the Steinberg representation.

We remark that [LS12, Theorem 8.12] implies that hypothesis 5 above is satisfied provided that U_l is a hyperspecial maximal compact subgroup, $\boldsymbol{\mu}$ is strictly regular, and the following inequalities hold:

$$2n + \sum_{\tau \in \Phi} \sum_{j=1}^{n} (2\lceil \mu_{\tau,1}/2 \rceil - \mu_{\tau,n+1-j}) \leq l \text{ and } 2n + \sum_{\tau \in \Phi} \sum_{j=1}^{n} (\mu_{\tau,j} - 2\lfloor \mu_{\tau,n}/2 \rfloor) \leq l.$$

By adding some further local hypotheses at a prime $q \neq p$, we could also appeal to the main result of [Shi].

Proof. Combining hypothesis 5 and Corollary 6.6, we see that the groups $H^i(\mathcal{A}(U^p \mathbb{Z}_p^\times, W^\vee_{\boldsymbol{\mu},k}))$ and $H^i(\mathcal{A}(U^p \mathbb{Z}_p^\times, W_{\boldsymbol{\mu},k}))$ vanish when $i = n-2$. On the other hand, there is a perfect pairing

$$\mathcal{A}(U^p \mathbb{Z}_p^\times, W^\vee_{\boldsymbol{\mu},\mathcal{O}}) \times \mathcal{A}(U^p \mathbb{Z}_p^\times, W_{\boldsymbol{\mu},\mathcal{O}}) \to \mathcal{O}.$$

Indeed, given an open compact subgroup $V \subset B$ and $f_1 \in \mathcal{A}(U^p \mathbb{Z}_p^\times V, W^\vee_{\boldsymbol{\mu},\mathcal{O}})$, $f_2 \in \mathcal{A}(U^p \mathbb{Z}_p^\times V, W_{\boldsymbol{\mu},\mathcal{O}})$, we define $\langle f_1, f_2 \rangle$ by the formula

$$\langle f_1, f_2 \rangle = \frac{1}{[B : V]} \sum_{x \in I(\mathbb{Q}) \backslash I(\mathbb{A}^\infty)/U^p \mathbb{Z}_p^\times V} (f_1(x), f_2(x)).$$

This is independent of the choice of V, and for every such V restricts to a perfect pairing $\mathcal{A}(U^p \mathbb{Z}_p^\times V, W^\vee_{\boldsymbol{\mu},\mathcal{O}}) \times \mathcal{A}(U^p \mathbb{Z}_p^\times V, W_{\boldsymbol{\mu},\mathcal{O}}) \to \mathcal{O}$. For any $g \in \mathrm{GL}_n(E_{w_0})$, we have the formula $\langle g f_1, g f_2 \rangle = \langle f_1, f_2 \rangle$. The action of $\mathbb{T}_T^{\mathrm{univ}}$ on $\mathcal{A}(U^p \mathbb{Z}_p^\times, W_{\boldsymbol{\mu},\mathcal{O}})$ gives a canonical direct sum decomposition of $\mathcal{O}[\mathrm{GL}_n(E_{w_0})]$-modules:

$$\mathcal{A}(U^p \mathbb{Z}_p^\times, W_{\boldsymbol{\mu},\mathcal{O}}) = \mathcal{A}(U^p \mathbb{Z}_p^\times, W_{\boldsymbol{\mu},\mathcal{O}})_{\mathfrak{m}_\sigma} \oplus C,$$

for some C. The hypotheses of Theorem 4.6 are now satisfied with $M = \mathcal{A}(U^p \mathbb{Z}_p^\times, W_{\boldsymbol{\mu},\mathcal{O}})_{\mathfrak{m}_\sigma}$ and N taken to be the annihilator of C under the pairing $\langle \cdot, \cdot \rangle$. The result follows from this. ☐

7 Consequences for GL_n

In this section we deduce our main theorem. We suppose that E is an imaginary CM field of the form $E = E_0 \cdot F$, where F is a totally real number field and E_0 is an imaginary quadratic field. We suppose that E/F is everywhere unramified. Suppose that there exists a prime p which is totally inert in F and split in E_0. Let $v_0 = w_0 w_0^c$ denote the unique place of F above p. Let $n \geq 3$ be an integer, and $l \neq p$ a prime. We fix an isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$.

Let n_1, n_2 be positive integers with $n = n_1 + n_2$. Suppose that π_1, π_2 are conjugate self-dual cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_E)$ such that $\pi = \pi_1 \boxplus \pi_2$ is regular algebraic. We recall that in Theorem 2.1 we have associated to π a continuous semisimple representation $r_\iota(\pi) : G_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$.

Theorem 7.1. *With π as above, suppose that $\iota^{-1}\pi_{w_0}$ satisfies the level-raising congruence 4.1. Suppose further that:*

1. *If $t_l \in G_{E_{w_0}}$ is a generator the l-part of the tame inertia group at w_0, then $\overline{r_\iota(\pi)}(t_l)$ is a unipotent matrix with exactly two Jordan blocks.*

2. *l is a banal characteristic for $\mathrm{GL}_n(E_{w_0})$.*

3. *The weight $\boldsymbol{\lambda} = (\lambda_\tau)_{\tau : E \hookrightarrow \mathbb{C}}$ of π satisfies the following:*

- *For each τ and for each $0 \leq i < j \leq n$, we have $0 < \lambda_{\tau,i} - \lambda_{\tau,j} < l$.*
- *There exists an isomorphism $\iota_p : \overline{\mathbb{Q}}_p \cong \mathbb{C}$ such that the following inequalities hold:*

$$2n + \sum_{\tau: E \hookrightarrow \mathbb{C}} \sum_{j=1}^{n} (\lambda_{\tau,j} - 2\lfloor \lambda_{\tau,n}/2 \rfloor) \leq l \text{ and } 2n + \sum_{\tau: E \hookrightarrow \mathbb{C}} \sum_{j=1}^{n} (2\lceil \lambda_{\tau,1}/2 \rceil - \lambda_{\tau,n+1-j}) \leq l,$$

the first sum in each case being over embeddings τ such that the place of E_0 induced by $\iota_p^{-1}\tau$ is the same as the restriction of the place w_0 to E_0.

4. *If π is ramified at a place w of E, then w is split over F.*

5. *π is unramified at the primes of E dividing l, and the prime l is unramified in E and split in E_0.*

6. *$\pi = \pi_1 \boxplus \pi_2$ satisfies the sign condition 2.2, $n_1 \neq n_2$, and $n_1 n_2$ is even.*

Then there exists a RACSDC automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_E)$ of weight $\boldsymbol{\lambda}$ such that $\overline{r_\iota(\pi)} \cong \overline{r_\iota(\Pi)}$ and Π_{w_0} is an unramified twist of the Steinberg representation. If the places of F above l are split in E, and π is ι-ordinary in the sense of [Ger, Definition 5.1.2], then we can even assume that Π is also ι-ordinary.

Proof. Let I_1 denote the definite unitary group associated to the extension E/F in §2.2. By Proposition 2.4 there exists an automorphic representation σ_1 of $I_1(\mathbb{A}_F)$ such that π is the base change of σ_1. Let I denote the corresponding unitary similitude group. By Lemma 2.3, σ_1 extends to an automorphic representation σ of $I(\mathbb{A})$. We apply Theorem 6.7 to σ. Let $U^p = \prod_{q \neq p} U_q$ be a sufficiently small open compact subgroup of $I(\mathbb{A}^{p,\infty})$ with $\sigma^U \neq 0$, where $U = U^p U_p$ and $U_p \subset I(\mathbb{Q}_p)$ corresponds under the isomorphism $\iota_{w_0} : I(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times \mathrm{GL}_n(E_{w_0})$ to the product $\mathbb{Z}_p^\times \times B$, where $B \subset \mathrm{GL}_n(E_{w_0})$ is the standard Iwahori subgroup. Suppose in addition that U_l is a hyperspecial maximal compact subgroup.

In the notation of Theorem 6.7, let $\boldsymbol{\mu}$ be the weight such that σ contributes to the space $\mathcal{A}(U, W_{\boldsymbol{\mu},\mathcal{O}})$. If σ' is an automorphic representation which contributes to the space $\mathcal{A}(U, W_{\boldsymbol{\mu},\mathcal{O}})_{\mathfrak{m}_\sigma}$, then let σ'_1 and π' be the automorphic representations of the groups $I_1(\mathbb{A}_F)$ and $\mathrm{GL}_n(\mathbb{A}_E)$ associated to σ' by Lemma 2.3 and Proposition 2.4. Then $\overline{r_\iota(\pi')}|_{G_{E_{w_0}}} \cong \overline{r_\iota(\pi)}|_{G_{E_{w_0}}}$, and hence the former representation maps t_l to a unipotent matrix with exactly two Jordan blocks. If σ' is such a representation, then the representation $\sigma'_{w_0} \circ \iota_{w_0}$ of $\mathrm{GL}_n(E_{w_0})$ has an Iwahori-fixed vector and is therefore isomorphic to $\mathrm{St}_{n_1}(\alpha_1) \boxplus \cdots \boxplus \mathrm{St}_{n_s}(\alpha_s)$ for some constants $\alpha_1, \ldots, \alpha_s$ and integers with $n_1 + \cdots + n_s = n$. The nilpotent operator N in the associated Weil-Deligne representation then has a Jordan decomposition corresponding to this partition of n. By hypothesis, the conjugacy class of N specializes to the conjugacy class of a nilpotent matrix with exactly two Jordan blocks. This implies that $s \leq 2$, and hence the second hypothesis of Theorem 6.7 is satisfied. Let σ' be the representation whose existence is guaranteed by that theorem. Applying Proposition 2.4 and Lemma 2.3 to σ', we obtain a representation Π satisfying the conclusion of the present theorem. It must be cuspidal since Π_{w_0} is an unramified twist of the Steinberg representation.

To obtain the last sentence of the theorem, we can enlarge the Hecke algebra $\mathbb{T}_T^{\mathrm{univ}}$ appearing in the proof of Theorem 6.7 to contain the analogues of the U_l operators at the places dividing l, and further localize at a maximal ideal not containing them. We omit the details. □

We remark that when the characteristic is not banal, but n is nevertheless small compared to the order of q_{v_0} modulo l, one can still obtain some information using Proposition 6.5 instead of Corollary 6.6. For example, one can prove an analogue of the above theorem in the case $n = 3$, with no hypothesis on the order of q_{v_0} modulo l. In general it seems an interesting question to decide whether the spectral sequence of §6.3 degenerates at E_2.

7.1 Proof of Theorem 1.1

We now give the proof of the theorem of the introduction. We first note the following.

Proposition 7.2. *Let E be an imaginary CM field with totally real subfield F, and let π be a RACSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$. Suppose that w_0 is a place of E and that π_{w_0} is an unramified twist of the Steinberg representation. Let $\mathcal{L}$ denote the set of rational primes l such that for all isomorphisms $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$, the residual representation $\overline{r_\iota(\pi)}$ is irreducible and, if t_l denotes a generator of the pro-l part of the tame inertia group at w_0, then $\overline{r_\iota(\pi)}(t_l)$ is a regular unipotent element. Then $\mathcal{L}$ has Dirichlet density 1.*

Proof. We sketch the proof, by exhibiting for every $\delta \in (0,1)$ a set $\mathcal{L}_\delta \subset \mathcal{L}$ of lower density at least $1-\delta$. Replacing E by a soluble extension, we can assume without loss of generality that for any prime w at which π is ramified, w is split over F.

Suppose that $E_1, \ldots, E_s$ are quadratic imaginary fields such that for each i, E_i is disjoint over $\mathbb{Q}$ from the compositum of the fields E_j, $j \neq i$. Let E_0 denote the compositum of the fields $E, E_1, \ldots, E_s$. Let F_0 denote the totally real subfield of E_0. If a prime l splits in any E_i, then the primes of F_0 above l all split in E_0. Let Π denote the base change of π to E_0. By [TY07, Corollary B] and [BLGGT, Proposition 5.2.2], there exists a set $\mathcal{M}$ of rational primes l of Dirichlet density 1 such that for all $l \in \mathcal{M}$ and all isomorphisms $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$, the residual representation $\overline{r_\iota(\Pi)}|_{G_{E_0(\zeta_l)}}$ is irreducible and $l > 2(n+1)$. This implies *a fortiori* that $\overline{r_\iota(\pi)}$ is irreducible. After casting out finitely many elements of $\mathcal{M}$, we can suppose further that for all $l \in \mathcal{M}$, E_0 and Π are unramified above l and, if $\boldsymbol{\lambda}$ denotes the weight of Π, then for all embeddings $\tau : E_0 \hookrightarrow \mathbb{C}$, we have $\lambda_{\tau,1} - \lambda_{\tau,n} \leq l - n - 1$ (this means that the Hodge-Tate weights of $r_\iota(\Pi)$ lie in the Fontaine-Lafaille range).

Choose a place x_0 of E_0 above w_0. It follows from [BLGGT, Theorem 4.4.1] that if $l \in \mathcal{M}$ is a prime split in one of $E_1, \ldots, E_s$, $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$ is an isomorphism and $\overline{r_\iota(\pi)}(t_l)$ is not a regular unipotent element, then we can find a RACSDC automorphic representation Π' of $\mathrm{GL}_n(\mathbb{A}_{E_0})$ satisfying the following:

- $\overline{r_\iota(\Pi)} \cong \overline{r_\iota(\Pi')}$,
- If w is a place of E_0 and $U_w \subset \mathrm{GL}_n(E_{0,w})$ is an open compact subgroup such that $\Pi_w^{U_w} \neq 0$, then $(\Pi'_w)^{U_w} \neq 0$.
- Π' has weight $\boldsymbol{\lambda}$.
- There exists an open compact subgroup U_{x_0} of $\mathrm{GL}_n(E_{0,x_0})$ strictly containing the Iwahori subgroup, such that $(\Pi'_{x_0})^{U_{x_0}} \neq 0$.

We claim that there can be only finitely many such primes. Indeed, if there are infinitely many then, by the pigeonhole principle, there exists an automorphic representation Π' of $\mathrm{GL}_{\mathbb{A}_{E_0}}$ satisfying the last three points, and infinitely many primes $l_1, l_2, \dots \in \mathcal{M}$ with isomorphisms $\iota_i : \overline{\mathbb{Q}}_{l_i} \cong \mathbb{C}$ such that $\overline{r_{\iota_i}(\Pi)} \cong \overline{r_{\iota_i}(\Pi')}$. As $\Pi^\infty, (\Pi')^\infty$ are defined over number fields, this implies that we must have $\Pi \cong \Pi'$, a contradiction (cf. [BG06, Lemme 5.1.7]).

Let $\mathcal{L}_s$ denote the set of primes $l \in \mathcal{M}$ which are split in one of $E_1, \ldots, E_s$. This set has Dirichlet density $1 - 2^{-s}$. The above argument shows that after casting out finitely many elements, we have $\mathcal{L}_s \subset \mathcal{L}$. This concludes the proof. □

Proof of Theorem 1.1. We take up the notation of the introduction. Thus E/F is a CM imaginary extension of a totally real field, and π_1, π_2 are RACSDC automorphic representations of $\mathrm{GL}_{n_1}(\mathbb{A}_E), \mathrm{GL}_{n_2}(\mathbb{A}_E)$, respectively. Let $\mathcal{L}$ denote the intersection of the sets $\mathcal{L}_1, \mathcal{L}_2$ of primes associated to the representations π_1, π_2 by Proposition 7.2. After removing finitely many elements from $\mathcal{L}$, we can assume that for all $l \in \mathcal{L}$ and all isomorphisms $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$, $\pi = \pi_1 \boxplus \pi_2$ is unramified at every prime of E above l, l is unramified in E, the order of q_{w_0} in $\mathbb{F}_l^\times$ is greater than $2n$, and the weight $\boldsymbol{\lambda}$ of π satisfies the inequalities

$$([E:\mathbb{Q}]+2)n + \sum_{\tau : E \hookrightarrow \mathbb{C}} \sum_{j=1}^{n} (\lambda_{\tau,j} - \lambda_{\tau,n}) \leq l/2.$$

Fix a prime $l \in \mathcal{L}$ and an isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$.

There exist $\alpha, \beta \in \overline{\mathbb{Z}}_l^\times$ such that the Frobenius eigenvalues of $r_\iota(\pi_1)$ and $r_\iota(\pi_2)$ are given by respectively

$$\alpha, q_{w_0}\alpha, \ldots, q_{w_0}^{n_1-1} \text{ and } \beta, q_{w_0}\beta, \ldots, q_{w_0}^{n_2-1}.$$

Let γ denote the image of $\beta/(\alpha q_{w_0}^{n_1})$ in $\overline{\mathbb{F}}_l^\times$, and let $m \geq 1$ denote the order of γ in this group. By the Grunwald-Wang theorem, there exists a cyclic extension K of E of degree m such that w_0 is inert in K and w_0^c splits in K, and K is unramified above the primes of E dividing l. Let $\varphi : G_E \to \overline{\mathbb{F}}_l^\times$ be the character factoring through $\mathrm{Gal}(K/E)$ such that $\varphi(\mathrm{Frob}_{w_0}) = \gamma$, and let ψ be the Teichmüller lift of φ/φ^c. Then $\psi\psi^c = 1$ and $\iota^{-1}(\pi_1 \boxplus (\pi_2 \otimes \iota\psi))_{w_0}$ satisfies the level-raising congruence.

Let E_0 be a quadratic imaginary extension of $\mathbb{Q}$ in which p is inert, and which is split at l and every prime $q \neq p$ of $\mathbb{Q}$ below a place of E at which $\pi_1 \boxplus (\pi_2 \otimes \iota\psi)$ or the extension E/F is ramified. Let $E_1 = E \cdot E_0$. The hypotheses of Theorem 7.1 now apply to the base change of $\pi_1 \boxplus (\pi_2 \otimes \iota\psi)$ to E_1. This completes the proof. □

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