CONGRUENCES BETWEEN MODULAR FORMS

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Abstract. We survey the connections between modular forms and representations of Galois groups that are predicted by the Langlands programme. We focus in particular on the applications of congruences between modular forms (through automorphy lifting theorems) to an improved understanding of these connections, including the author's recent joint work with James Newton on the existence of the symmetric power liftings of Hilbert modular forms.

Contents

Let k be an integer, and let $\mathfrak{h}$ denote the complex upper half-plane. A modular form of weight k and level $\Gamma(1) = \mathrm{SL}_2(\mathbf{Z})$ is a holomorphic function $f : \mathfrak{h} \to \mathbf{C}$ satisfying the following conditions:

- For every $\tau \in \mathfrak{h}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$, we have
$$f((a\tau + b)/(c\tau + d))(c\tau + d)^{-k} = f(\tau).$$
- The function $|f(\tau)|$ is bounded as $\mathrm{Im}(\tau) \to \infty$.

Modular forms are the most classical incarnation of *automorphic representations*, which are the fundamental objects of the Langlands programme. Congruences between modular forms, to be defined below, are at the heart of many applications of the Langlands programme to questions in number theory, thanks to their interpretation in terms of deformations (in the sense of Mazur [Maz89]) of representations of Galois groups.

In this article, we will discuss congruences from this point of view and sketch some significant applications, including the modularity of elliptic curves over $\mathbf{Q}$ [Wil95, TW95, BCDT01], the modularity of (Serre type) representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ [KW09b, KW09c], and functoriality for holomorphic modular forms [NT21a, NT21b]. These applications can all interpreted as saying that certain Galois representations are modular (or, synonymously, automorphic). We will not address here some equally significant applications of congruences (especially, congruences with Eisenstein series) to arithmetic, such as the proofs of the Iwasawa main conjecture for GL_1 over totally real fields [Wil90] and the Iwasawa main conjecture for many elliptic curves over $\mathbf{Q}$ [SU14]. For a recent survey addressing these themes in the context of the Birch–Swinnerton-Dyer conjecture, see [BST21].

1. Congruences between holomorphic modular forms of level 1

What is a congruence? Congruences can be seen most directly through q-expansions of modular forms. If $f(\tau)$ is a modular form (of weight k and level $\mathrm{SL}_2(\mathbf{Z})$, as in the first paragraph above), then it satisfies the relation $f(\tau+1) = f(\tau)$, hence has a q-expansion

$$f(\tau) = \sum_{n \geq 0} a_n(f) q^n,$$

where $q = e^{2\pi i \tau}$, and the q-expansion coefficients $a_n(f)$ are complex numbers. If f, g are modular forms which happen to have rational q-expansion coefficients, we say provisionally that they are congruent modulo a prime p (and write $f \equiv g \bmod p$) if for every $n \geq 0$, the rational numbers $a_n(f), a_n(g)$ are p-integral (i.e. have denominator prime to p) and satisfy the congruence

$$a_n(f) \equiv a_n(g) \bmod p. \tag{1.1}$$

A famous example is Ramanujan's congruence $E_{12} \equiv \Delta \bmod 691$, where E_{12} is the weight 12 Eisenstein series

$$E_{12}(\tau) = \frac{\zeta(-11)}{2} + \sum_{n \geq 1} \sigma_{11}(n) q^n, \text{ where } \sigma_{11}(n) = \sum_{d|n} d^{11},$$

and Δ is Ramanujan's modular form

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

(also a modular form of weight 12 – in fact, E_{12}, Δ form a basis for the $\mathbf{C}$-vector space $M_{12}(\mathrm{SL}_2(\mathbf{Z}))$ of modular forms of weight 12). Ramanujan showed that we have

$$\sigma_{11}(n) \equiv a_n(\Delta) \bmod 691$$

for all $n \geq 1$, while we can compute

$$\frac{\zeta(-11)}{2} = \frac{691}{65520} \equiv 0 = a_0(\Delta) \bmod 691.$$

Faced with this example, we have a number of questions, including:

(1) Why is congruence modulo p an interesting concept? What are the number-theoretic consequences of the existence of a congruence?
(2) For which pairs of modular forms is the existence of a congruence arithmetically significant? Clearly, the existence of a congruence modulo p between f and $(1+p)f$ is not interesting.
(3) How do we describe and/or measure congruences?
(4) How do we define the notion of congruence for automorphic representations, which may not admit such a concrete interpretation as modular forms (for example, they may not have analogues of the q-expansion)?

Our answers to these questions will be based on the point of view that we are interested in modular forms primarily for their connection with Galois representations.[1] A Galois representation is a continuous representation

$$\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_n(K),$$

where K is a topological ring. The most significant case for us will be when K is a finite extension of $\mathbf{Q}_p$ (in which case ρ is said to be a p-adic Galois representation) or of $\mathbf{F}_p$ (in which case ρ is said to be a mod p Galois representation). The modular forms E_{12} and Δ both have attached p-adic Galois representations (for any prime number p – these representations then collectively form a *compatible system*). In the case of E_{12}, we can describe the attached p-adic Galois representation $\rho_{E_{12},p}$ explicitly: it is a direct sum

$$\rho_{E_{12},p} = \mathbf{1} \oplus \epsilon_p^{-11},$$

where $\mathbf{1}$ is the trivial character of $G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ and

$$\epsilon_p : G_{\mathbf{Q}} \to \mathrm{Aut}(\mu_{p^\infty}(\overline{\mathbf{Q}})) = \mathbf{Z}_p^\times$$

is the p-adic cyclotomic character (that describes the action of the Galois group on the subgroup $\mu_{p^\infty}(\overline{\mathbf{Q}}) \subset \overline{\mathbf{Q}}^\times$ of p-power roots of unity).

To explain the sense in which $\rho_{E_{12},p}$ is 'attached' to E_{12}, we need to describe some of the additional structures carried by the group $G_{\mathbf{Q}}$, including the notion of Frobenius element. If l is any prime number, we can fix an algebraic closure $\overline{\mathbf{Q}}_l$ of $\mathbf{Q}_l$ and an embedding $\overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_l$. This in turn gives an embedding $G_{\mathbf{Q}_l} \to G_{\mathbf{Q}}$ (restriction of automorphisms of $\overline{\mathbf{Q}}_l/\mathbf{Q}_l$ to $\overline{\mathbf{Q}}$), which is well-defined up to conjugacy. The group $G_{\mathbf{Q}_l}$ sits in a short exact sequence

$$1 \to I_{\mathbf{Q}_l} \to G_{\mathbf{Q}_l} \to G_{\mathbf{F}_l} \to 1,$$

where $I_{\mathbf{Q}_l}$ is the inertia subgroup of $G_{\mathbf{Q}_l}$, i.e. the automorphisms which act as the identity on the residue field $\overline{\mathbf{F}}_l$ of $\overline{\mathbf{Q}}_l$. The group $G_{\mathbf{F}_l}$ has a canonical topological generator, namely the Frobenius element $\mathrm{Frob}_l \in G_{\mathbf{F}_l}$.[2]

A representation $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_n(K)$ is said to be unramified at l if $I_{\mathbf{Q}_l}$ is contained in its kernel. In this case, it makes sense to evaluate ρ on Frob_l and the conjugacy class of $\rho(\mathrm{Frob}_l)$ is independent of the choice of embedding $G_{\mathbf{Q}_l} \to G_{\mathbf{Q}}$. In particular, the characteristic polynomial $\det(X - \rho(\mathrm{Frob}_l))$ is well-defined. We can now explain the sense in which $\rho_{E_{12},p}$ is attached to E_{12}: for any prime number $p \neq l$, $\rho_{E_{12},p}$ is unramified at l, and we have the relation

$$\det(X - \rho_{E_{12},p}(\mathrm{Frob}_l)) = X^2 - a_l(E_{12})X + l^{11} = X^2 - \sigma_{11}(l)X + l^{11}.$$

This duality between character values of Galois representations on the one hand and q-expansion coefficients (to be re-interpreted below as the eigenvalues of Hecke operators) is one of the defining features of the Langlands correspondence.

The Galois representation $\rho_{\Delta,p} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Q}_p)$ attached to Δ is also unramified at any prime number $l \neq p$, and satisfies the relation

$$\det(X - \rho_{\Delta,p}(\mathrm{Frob}_l)) = X^2 - a_l(\Delta)X + l^{11}.$$

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[1] Note, however, that even if one is interested only in number theory, modular forms have many other applications – see e.g. [Sar90] for examples.

[2] More precisely, we take Frob_l to be the geometric Frobenius, inverse of the usual 'arithmetic' Frobenius $x \mapsto x^l$, as our preferred normalisation.

It is relatively easy to show (cf. [Ser68]) that this relation determines $\rho_{\Delta,p}$ up to isomorphism, if it exists. Indeed, the Chebotarev density theorem implies that if, for a finite set S of primes we write $G_{\mathbf{Q},S}$ for the Galois group of the maximal subextension of $\overline{\mathbf{Q}}$ which is unramified outside S, then the conjugacy classes of Frobenius elements at primes $l \notin S$ are dense in $G_{\mathbf{Q},S}$. Since $\rho_{\Delta,p}$ factors through the quotient $G_{\mathbf{Q}} \to G_{\mathbf{Q},\{p\}}$, the character of the representation $\rho_{\Delta,p}$ is determined by its values on Frobenius elements at primes $l \neq p$. Constructing $\rho_{\Delta,p}$ is another matter. Its existence was conjectured (in a very precise formulation, with a view to understanding the congruences satisfied by Δ) by Serre [Ser69] and proved by Deligne [Del71b], who constructed it in the étale cohomology of a p-adic local system on a modular curve over $\mathbf{Q}$.

What, then, is the interpretation of the congruence $E_{12} \equiv \Delta \bmod 691$? For this we need to introduce the (modulo p) residual representation of a p-adic Galois representation. Let $E/\mathbf{Q}_p$ be a finite extension of valuation ring $\mathcal{O}$ and residue field $k = \mathcal{O}/(\varpi)$. The group $\mathrm{GL}_n(E)$ has a unique conjugacy class of maximal compact subgroups, namely the class of $\mathrm{GL}_n(\mathcal{O})$. It follows that if $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_n(E)$ is a continuous representation, then it can be conjugated to take values in $\mathrm{GL}_n(\mathcal{O})$. We write $\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_n(k)$ for the semisimplification of the composite

$$G_{\mathbf{Q}} \to \mathrm{GL}_n(\mathcal{O}_E) \to \mathrm{GL}_n(k)$$

(where the second arrow is reduction modulo (ϖ) – we take semisimplification in order that the result is independent, up to isomorphism, of the choice of conjugate of ρ defined over $\mathcal{O}$). We can equivalently characterise $\overline{\rho}$ as follows: it is the unique (up to isomorphism) semisimple representation such that for any $\sigma \in G_{\mathbf{Q}}$, we have

$$\det(X - \rho(\sigma)) \bmod (\varpi) = \det(X - \overline{\rho}(\sigma)).$$

If ρ is known to be (say) unramified at primes $l \neq p$, then the Chebotarev density theorem implies that is equivalent to have

$$\det(X - \rho(\mathrm{Frob}_l)) \bmod (\varpi) = \det(X - \overline{\rho}(\mathrm{Frob}_l))$$

for every prime $l \neq p$.

We can now interpret the existence of the congruence between E_{12} and Δ in Galois-theoretic terms: it implies that the residual representations $\overline{\rho}_{E_{12},691}$ and $\overline{\rho}_{\Delta,691}$ are isomorphic. The representation $\overline{\rho}_{E_{12},691}$ is easy to compute (it is the direct sum of $\mathbf{1}$ and $\overline{\epsilon}_{691}^{-11}$) so we see in particular that the *irreducible* representation $\rho_{\Delta,691}$ becomes *reducible* after reduction modulo 691. This has consequences for concrete questions in number theory! An argument of Ribet [Rib76] shows that the existence of this congruence directly implies:

Proposition 1.1. *There exists a non-zero $\omega \in \mathrm{Pic}(\mathbf{Z}[\zeta_{691}]) \otimes_{\mathbf{Z}} \mathbf{F}_{691}$ such that for every $\sigma \in \mathrm{Gal}(\mathbf{Q}(\zeta_{691})/\mathbf{Q})$, we have $\sigma(\omega) = \overline{\epsilon}_{691}(\sigma)^{-11}\omega$.*

We can now anticipate the answer to the second question above – the modular forms for which the notion of congruence is interesting are those that have attached Galois representations. This class of modular forms can be described in purely modular terms, using the theory of Hecke operators.

For any weight k, there is a family $(T_n)_{n \in \mathbf{N}}$ of linear endomorphisms of $M_k(\mathrm{SL}_2(\mathbf{Z}))$, called Hecke operators. These have the following key properties (see e.g. [Ser73]):

- They commute, and are simultaneously diagonalisable.

- The simultaneous eigenspaces are 1-dimensional, and the eigenvalues of each Hecke operator T_n are algebraic integers.

An element $f \in M_k(\mathrm{SL}_2(\mathbf{Z}))$ which is an eigenvector for the Hecke operators T_n is called an eigenform. It is a fact that if f is an eigenform, then $a_1(f) \neq 0$, so we can pick out a canonical generator for each eigenspace by specifying $a_1(f) = 1$ (in which case we say that f is normalised). Moreover, if f is normalised then the eigenvalue of T_n on f is the q-expansion coefficient $a_n(f)$ – so we can now reinterpret these coefficients as Hecke eigenvalues.

We then have the following theorem, the second part being the main theorem of [Del71a]. We state it just for the subspace $S_k(\mathrm{SL}_2(\mathbf{Z})) \leq M_k(\mathrm{SL}_2(\mathbf{Z}))$ of cuspidal modular forms, i.e. modular forms f with $a_0(f) = 0$. (This has the effect of excluding Eisenstein series and restricting to those eigenforms which have irreducible Galois representations.)

Theorem 1.2. *Let $k \geq 0$.*

(1) *The vector space $S_k(\mathrm{SL}_2(\mathbf{Z}))$ has a unique (up to re-ordering) basis of normalised eigenforms.*
(2) *Let $f \in S_k(\mathrm{SL}_2(\mathbf{Z}))$ be a normalised eigenform. Then the subfield $K_f \subset \mathbf{C}$ generated by the eigenvalues of T_n on f is a number field. For any finite place λ of K_f, there is a continuous irreducible representation*
$$\rho_{f,\lambda} : G_{\mathbf{Q}} \to \mathrm{GL}_2(K_{f,\lambda}),$$
uniquely characterized up to conjugation by the requirement that for any prime $l \nmid \lambda$ of K_f, $\rho_{f,\lambda}$ is unramified at l and we have
$$\det(X - \rho_{f,\lambda}(\mathrm{Frob}_l)) = X^2 - a_l(f)X + l^{k-1}.$$

To compare the Galois representations associated to different cuspidal eigenforms, it is convenient to introduce the following notational device. If f is an eigenform, p is a prime number, and $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ is an isomorphism, then ι^{-1} induces a p-adic place λ of the coefficient field K_f, and we define
$$\rho_{f,\iota} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$$
to be the composite of $\rho_{f,\lambda}$ with the embedding $\mathrm{GL}_2(K_{f,\lambda}) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ induced by ι.

Definition 1.3. *Let f, g be normalised eigenforms. Fix a prime p and an isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$. We say that they are congruent modulo p if one of the following equivalent conditions is satisfied:*

(1) *There is an isomorphism $\overline{\rho}_{f,\iota} \cong \overline{\rho}_{g,\iota}$ of representations $G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$.*
(2) *For each prime number $l \neq p$, we have*
$$\iota^{-1}(a_l(f)) \equiv \iota^{-1}(a_l(g)) \text{ mod } \mathfrak{m}_{\overline{\mathbf{Z}}_p}.$$
(3) *For all but finitely many prime numbers l, we have*
$$\iota^{-1}(a_l(f)) \equiv \iota^{-1}(a_l(g)) \text{ mod } \mathfrak{m}_{\overline{\mathbf{Z}}_p}.$$

Here $\mathfrak{m}_{\overline{\mathbf{Z}}_p}$ is the maximal ideal of the valuation ring $\overline{\mathbf{Z}}_p$ of $\overline{\mathbf{Q}}_p$. The definition depends on the fixed choice of ι, although in a relatively mild way. Indeed, if $\jmath : \overline{\mathbf{Q}}_p \to \mathbf{C}$ is another choice, then $\jmath = \sigma \circ \iota$ for some $\sigma \in \mathrm{Aut}(\mathbf{C})$. The group $\mathrm{Aut}(\mathbf{C})$ acts on the set of normalised eigenforms (with $a_n({}^\sigma g) = \sigma(a_n(g))$), and if

f is congruent g with respect to ι, then ${}^\sigma f$ will be congruent to ${}^\sigma g$ with respect to $\jmath$.

We caution the reader that Definition 1.3 is different to the provisional definition adopted in (1.1). In particular, we are now only asking for the coefficients $a_l(f), a_l(g)$ at primes $l \neq p$ to be congruent modulo p, and not placing a condition on $a_p(f), a_p(g)$. The reason for this is that the reductions modulo p of the a_p's are not determined by the associated Galois representations (although they are when $k \leq p+1$, see e.g. [BG09]).

Relatedly, we are focusing now on the coefficients a_l for primes l and neglecting the a_n for general integers n. If f is an eigenform then the coefficients $a_n(f)$ are determined by the coefficients $a_l(f)$ for primes $l|n$, so in a sense we are not losing much information. However, when we consider more general automorphic representations, it is the eigenvalues of Hecke operators at unramified primes l that we are interested in, and these no longer coincide with Fourier coefficients of automorphic forms. We adopt this point of view now in preparation for the more general discussion beginning in §3.

We have thus decided it is of primary interest to look for congruences between the systems of eigenvalues associated to eigenforms. An effective tool to study these is the notion of Hecke algebra.

Definition 1.4. *Let $k \geq 0$ an integer. Then the weight k Hecke algebra is $\mathbf{T}_k = \mathbf{T}(S_k(\mathrm{SL}_2(\mathbf{Z})))$, the $\mathbf{Z}$-subalgebra of $\mathrm{End}(S_k(\mathrm{SL}_2(\mathbf{Z})))$ generated by Hecke operators T_l ($l \nmid p$).*

This ring has reasonable properties:

Proposition 1.5.
(1) *$\mathbf{T}_k$ is a finite free $\mathbf{Z}$-algebra.*
(2) *The normalised eigenforms $f \in S_k(\mathrm{SL}_2(\mathbf{Z}))$ are naturally in bijection with the homomorphisms $\alpha_f : \mathbf{T}_k \to \mathbf{C}$.*
(3) *Let p be a prime, and let $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ be an isomorphism, and let $f, g \in S_k(\mathrm{SL}_2(\mathbf{Z}))$ be normalised eigenforms. Then there is an isomorphism $\overline{\rho}_{f,\iota} \cong \overline{\rho}_{g,\iota}$ if and only if the associated homomorphisms $\iota^{-1}\alpha_f : \mathbf{T}_k \to \overline{\mathbf{Z}}_p$ have the same reduction modulo $\mathfrak{m}_{\overline{\mathbf{Z}}_p}$.*

The first point follows from the fact that there is a natural $\mathbf{Z}$-structure on $S_k(\mathrm{SL}_2(\mathbf{Z}))$ preserved by the Hecke operators, given by the submodule $S_k(\mathrm{SL}_2(\mathbf{Z}), \mathbf{Z})$ of cuspidal modular forms with integer q-expansion coefficients. The remainder is essentially linear algebra (see [Bel21, Ch. 1]).

The introduction of the Hecke algebra means that we can study the set of eigenforms which are congruent to a given one by studying the localization and completion of $\mathbf{T}_{k,\mathfrak{m}}$ at a maximal ideal $\mathfrak{m} \leq \mathbf{T}_k$. If $\mathfrak{m}$ has residue characteristic p, this will be a finite free $\mathbf{Z}_p$-algebra. In fact, if we are interested in congruences to a fixed eigenform f, then it is convenient to enlarge the coefficients. We therefore fix a finite extension $E/\mathbf{Q}_p$ inside $\overline{\mathbf{Q}}_p$ such that the image of $\iota^{-1}\alpha_f$ is contained in the ring of integers $\mathcal{O} \subset E$, and form $\mathbf{T}_k(\mathcal{O}) = \mathbf{T}_k \otimes_{\mathbf{Z}} \mathcal{O}$. Then $\mathbf{T}_k(\mathcal{O})$ is a finite free $\mathcal{O}$-algebra, and if we write now $\mathfrak{m} \leq \mathbf{T}_k(\mathcal{O})$ for the kernel of the $\mathcal{O}$-algebra homomorphism

$$\iota^{-1}\alpha_f \otimes 1 : \mathbf{T}_k(\mathcal{O}) \to \mathcal{O},$$

then we find that there is a bijection between the set of normalised eigenforms $g \in S_k(\mathrm{SL}_2(\mathbf{Z}))$ congruent to f modulo p, and the set of $\mathcal{O}$-algebra homomorphisms $\mathrm{Hom}_{\mathcal{O}}(\mathbf{T}_k(\mathcal{O})_{\mathfrak{m}}, \overline{\mathbf{Q}}_p)$.

Various devices from commutative algebra can now be introduced to study and measure congruences. With notation as in the previous paragraph, the following objects are all of interest:

- The ring $\mathbf{T}_k(\mathcal{O})_\mathfrak{m}$ itself. The question of the degree of singularity of $\mathbf{T}_k(\mathcal{O})_\mathfrak{m}$ (for example, whether it is a complete intersection ring, Cohen-Macaulay, Gorenstein, etc.) is (anticipating the discussion below) related to the degree of degeneracy of the residual representation $\overline{\rho}_{f,\iota}$.
- The relative cotangent space $\Phi_f = \mathfrak{p}_f/\mathfrak{p}_f^2$, where $\mathfrak{p}_f$ is the kernel of the homomorphism $\mathbf{T}_k(\mathcal{O})_\mathfrak{m} \to \mathcal{O}$ associated to f.
- The congruence module $\Psi_f = \mathbf{T}_k(\mathcal{O})_\mathfrak{m}/(\mathfrak{p}_f, \operatorname{Ann}_{\mathbf{T}_k(\mathcal{O})_\mathfrak{m}} \mathfrak{p}_f)$.

The $\mathcal{O}$-modules Φ_f and Ψ_f have the property that they are non-zero if and only if f is indeed congruent to another eigenform. Various relations between them exist. For example, a famous component of Wiles's proof of Fermat's Last Theorem [Wil95] is his numerical criterion, namely that there is an inequality

$$\#\Phi_f \geq \#\Psi_f,$$

with equality holding if and only if $\mathbf{T}_k(\mathcal{O})_\mathfrak{m}$ is a complete intersection ring. (This statement incorporates an improvement by Lenstra [Len95], who removed the need to assume that $\mathbf{T}_k(\mathcal{O})_\mathfrak{m}$ is Gorenstein.)

A very powerful device for studying these Hecke algebras is Galois deformation theory. Let k denote the residue field of $\mathcal{O}$. Introduced by Mazur [Maz89], Galois deformation theory gives a mechanism to study all of the deformations of a given absolutely irreducible residual representation $\overline{\rho} : G_\mathbf{Q} \to \mathrm{GL}_2(k)$ to representations $\mathrm{GL}_2(A)$ with coefficients in a complete Noetherian local $\mathcal{O}$-algebra A. We can make this precise as follows. Let $\mathcal{C}_\mathcal{O}$ denote the category of complete Noetherian local $\mathcal{O}$-algebras with residue field k – the objects of this category may all be represented as quotients of rings $\mathcal{O}[\![X_1, \ldots, X_g]\!]$ (for some $g \geq 0$). Suppose that $\overline{\rho}$ is unramified outside S, for a finite set of primes S containing p. If $A \in \mathcal{C}_\mathcal{O}$, then a lifting of $\overline{\rho}$, unramified outside S, is a continuous homomorphism

$$\rho_A : G_{\mathbf{Q},S} \to \mathrm{GL}_2(A)$$

such that the composite

$$G_{\mathbf{Q},S} \to \mathrm{GL}_2(A) \to \mathrm{GL}_2(k)$$

equals $\overline{\rho}$. A *deformation* of $\overline{\rho}$ is a conjugacy class of liftings under the action of the group $\ker(\mathrm{GL}_2(A) \to \mathrm{GL}_2(k))$. If f is a normalised eigenform and $\overline{\rho}_{f,\iota}$ is defined over k and absolutely irreducible, then any choice of eigenform g which is congruent to f gives rise to a deformation of $\overline{\rho}_{f,\iota}$. Indeed, we can choose a model $\rho_{g,\iota}$ which takes values in $\mathrm{GL}_2(\overline{\mathbf{Z}}_p)$. We know that there is an isomorphism between the composite

$$G_{\mathbf{Q},\{p\}} \to \mathrm{GL}_2(\overline{\mathbf{Z}}_p) \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$$

and $\overline{\rho}_{f,\iota}$. After replacing $\rho_{g,\iota}$ by a conjugate, we can assume that these are in fact equal (and that $\rho_{g,\iota}$ can be defined with coefficients in a subring of $\overline{\mathbf{Z}}_p$ which is an object of $\mathcal{C}_\mathcal{O}$), making $\rho_{g,\iota}$ into a lifting, whose associated deformation can be checked to be independent of any choices.

The universal deformation ring is an object $R_{S,\overline{\rho}} \in \mathcal{C}_\mathcal{O}$ with the following properties:

- There is a deformation $\rho_S^{univ} : G_{\mathbf{Q},S} \to \mathrm{GL}_2(R_{S,\overline{\rho}})$.

- For any $A \in \mathcal{C}_{\mathcal{O}}$, the map $\phi \mapsto \phi_* \rho_S^{univ}$ gives a bijection from $\mathrm{Hom}_{\mathcal{C}_{\mathcal{O}}}(R_{S,\overline{\rho}}, A)$ to the set of deformations of $\overline{\rho}$ to A that are unramified outside S.

In other words, $R_{S,\overline{\rho}}$ represents the functor $\mathcal{C}_{\mathcal{O}} \to$ Sets of deformations unramified outside S. Yoneda's lemma implies that this representing object is defined up to unique isomorphism.

It is now relatively formal to prove that if f is an eigenform such that $\overline{\rho}_{f,\iota}$ is absolutely irreducible, then there is a unique surjective homomorphism

$$R_{\{p\},\overline{\rho}_{f,\iota}} \to \mathbf{T}_k(\mathcal{O})_{\mathfrak{m}}$$

with the property that for each prime $l \neq p$, $\mathrm{tr}\, \rho_{\{p\}}^{univ}(\mathrm{Frob}_l)$ is mapped to the Hecke operator T_l. Indeed, there is a natural embedding

$$\mathbf{T}_k(\mathcal{O})_{\mathfrak{m}} \to \prod_g \overline{\mathbf{Z}}_p,$$

$$T_l \mapsto (\iota^{-1} a_l(g))_g$$

where the product is over the set of normalised eigenforms g which are congruent to f. The existence of the deformations associated to each such eigenform implies that there is also a map

$$R_{\{p\},\overline{\rho}_{f,\iota}} \to \prod_g \overline{\mathbf{Z}}_p,$$

and we just need to show that the image is exactly $\mathbf{T}_k(\mathcal{O})_{\mathfrak{m}}$. This is a consequence of Carayol's lemma [Car94], which implies that $R_{\{p\},\overline{\rho}_{f,\iota}}$ is topologically generated by the elements $\mathrm{tr}\, \rho_{\{p\}}^{univ}(\mathrm{Frob}_l)$, therefore that its image under the above map is generated, as an $\mathcal{O}$-algebra, by the Hecke operators T_l.

Homomorphisms such as $R_{\{p\},\overline{\rho}_{f,\iota}} \to \mathbf{T}_k(\mathcal{O})_{\mathfrak{m}}$ are one of the main mechanisms for studying congruences between modular forms, at least in the residually irreducible case. We see in particular that $R_{\{p\},\overline{\rho}_{f,\iota}}$ gives an upper bound on the size of $\mathbf{T}_k(\mathcal{O})_{\mathfrak{m}}$, completely independent of the world of modular forms. In actual fact, to obtain a useful upper bound we need to refine the definition of $R_{\{p\},\overline{\rho}_{f,\iota}}$ by imposing deformation conditions. To see the necessity of this refinement, we note that the ring $\mathbf{T}_k(\mathcal{O})_{\mathfrak{m}}$, being a finite free $\mathcal{O}$-algebra, has Krull dimension 1, while Mazur's tangent-obstruction theory [Maz89, Proposition 5] can be used, together with Tate's global Euler characteristic formula, to show that the Krull dimension of $R_{\{p\},\overline{\rho}_{f,\iota}}$ is at least 4.

We define a deformation condition $\mathcal{D} = (S, \psi, \{\mathcal{D}_l\}_{l \in S})$ to be a tuple consisting of the following data:

- S, a finite set of primes containing p.
- ψ, a continuous character $G_{\mathbf{Q},S} \to \mathcal{O}^\times$.
- For each $l \in S$, a subfunctor $\mathcal{D}_l \subset \mathcal{D}_l^\square$ of the functor $\mathcal{D}_l^\square$ of all liftings of $\overline{\rho}_{f,\iota}|_{G_{\mathbf{Q}_l}}$ to objects of $\mathcal{C}_{\mathcal{O}}$ which is invariant under the conjugation action of $\ker(\mathrm{GL}_2(A) \to \mathrm{GL}_2(k))$. We further require that $\mathcal{D}_l$ is representable.

Under these assumptions, we say that a lifting $\rho_A : G_{\mathbf{Q}} \to \mathrm{GL}_2(A)$ of $\overline{\rho}_{f,\iota}$ is of type $\mathcal{D}$ if it satisfies the following conditions:

- ρ_A is unramified outside S.
- $\det \rho_A$ coincides with the pushforward of ψ to a character $G_{\mathbf{Q}} \to A^\times$.
- For each $l \in S$, $\rho_A|_{G_{\mathbf{Q}_l}} \in \mathcal{D}_l(A)$.

The hypothesis that $\mathcal{D}_l$ is invariant under conjugation shows that the notion of 'type $\mathcal{D}$ deformation' makes sense; the hypothesis that $\mathcal{D}_l$ is representable implies that there is a quotient $R_{S,\overline{\rho}_{f,\iota}} \to R_{\mathcal{D}}$ with the property that for any $A \in \mathcal{C}_{\mathcal{O}}$ the homomorphism $R_{S,\overline{\rho}_{f,\iota}} \to A$ corresponding to a deformation $[\rho_A]$ factors through $R_{\mathcal{D}}$ if and only if $[\rho_A]$ is of type $\mathcal{D}$. We say that $R_{\mathcal{D}}$ is the universal type $\mathcal{D}$ deformation ring.

What is the appropriate choice of deformation condition to study the Hecke algebra $\mathbf{T}_k(\mathcal{O})_{\mathfrak{m}}$? The first two pieces of data are easy to specify:

- We already know that we should take $S = \{p\}$.
- The representations $\rho_{g,\iota}$ corresponding to modular forms of weight k have determinant ϵ_p^{1-k}, so we should take $\psi = \epsilon_p^{1-k}$.

It remains to define $\mathcal{D}_p$. This is hard! The correct conditions to impose on the restrictions $\rho_A|_{G_{\mathbf{Q}_p}}$ come from p-adic Hodge theory. In particular, Fontaine defined [Fon84] what it means for a representation of $G_{\mathbf{Q}_p}$ on a $\mathbf{Q}_p$-vector space to be 'crystalline'. Such representations have associated numerical invariants, the Hodge–Tate weights. It follows from work of Faltings and Scholl [Fal89, Sch90] that if g is a normalised eigenform of weight k, then $\rho_{g,\iota}|_{G_{\mathbf{Q}_p}}$ is crystalline, with Hodge–Tate weights $\{0, k-1\}$. However, Fontaine's theory works primarily with rational coefficients, and can be promoted to an integral theory only when k is small relative to p (see e.g. [FL82]). Kisin [Kis08] showed that one can nevertheless define a deformation problem $\mathcal{D}_p^{cr,(0,k-1)} \subset \mathcal{D}_p^{\square}$ capturing crystalline representations in the generic fibre and with good properties. He showed that $\mathcal{D}_p^{cr,(0,k-1)}$ is represented by a quotient $R_p^{cr,(0,k-1)} \in \mathcal{C}_{\mathcal{O}}$ of $R_p^{\square}$ with the following properties:

- $R_p^{cr,(0,k-1)}$ is reduced and $\mathcal{O}$-flat.
- $R_p^{cr,(0,k-1)}[1/p]$ is regular.
- A homomorphism $R_p^{\square} \to \overline{\mathbf{Q}}_p$ factors through $R_p^{cr,(0,k-1)}$ if and only if the pushforward $\rho_p : G_{\mathbf{Q}_p} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ of the universal lifting is crystalline with Hodge–Tate weights $\{0, k-1\}$.

With this choice, we obtain a homomorphism

$$R_{\mathcal{D}} \to \mathbf{T}_k(\mathcal{O})_{\mathfrak{m}}$$

that has a chance of being an isomorphism. This homomorphism is a central motif in the study of congruences between modular forms. We see that the ring $R_{\mathcal{D}}$ gives a purely Galois-theoretic upper bound for the Hecke algebra, and as such gives an upper bound for the congruences that can exist. Some of the measures of congruences introduced above have Galois theoretic analogues. For example, we recall that $\mathfrak{p}_f$ denotes the kernel of the homomorphism $\mathbf{T}_k(\mathcal{O})_{\mathfrak{m}} \to \mathcal{O}$ associated to f. Let $\mathfrak{q}_f$ denote its pullback to $R_{\mathcal{D}}$. Then there is an inequality

$$\#\mathfrak{p}_f/\mathfrak{p}_f^2 \leq \#\mathfrak{q}_f/\mathfrak{q}_f^2,$$

and the quotient $\mathfrak{q}_f/\mathfrak{q}_f^2$ can be interpreted as the Pontryagin dual of the Selmer group

$$H^1_{\mathcal{D}}(\mathbf{Q}, \mathrm{ad}^0\, \rho_{f,\iota} \otimes_{\mathcal{O}} E/\mathcal{O})$$

(where the local conditions defining the Selmer group may be computed in terms of the deformation condition $\mathcal{D}$, and $\mathrm{ad}^0 \subset \mathrm{ad}$ denotes the adjoint action on the Lie subalgebra of trace 0 matrices).

In §2 below, we will broaden our discussion to include modular forms of varying level (or in other words, Galois representations which may be ramified at more than one prime). In this context, the point of view afforded by the connection to Galois representations and their deformation theory becomes indispensable. We conclude this section by mentioning two other generalisations of the ideas considered here.

First, we want to mention that the notion of Hecke algebra is a flexible one and need not be confined just to a discussion of congruences between modular forms of fixed weight. For example, we could fix a bound N and define $\mathbf{T}_{\leq N}$ to be the Hecke algebra which acts faithfully on $\oplus_{k \leq N} M_k(\mathrm{SL}_2(\mathbf{Z}))$, or even p-adically complete and let $N \to \infty$, in which case we enter the world of p-adic modular forms. A readable exposition of this situation, which goes on to discuss the more subtle theories of Hida and Coleman–Mazur and their connections to Galois representations, can be found in [Eme11].

Second, we need to mention that one can also consider the question of deforming Galois representations without assuming that $\overline{\rho}_{f,\iota}$ is irreducible. This residually reducible case presents additional technical difficulties. However, it should not be ignored since theorems proved in this context often have powerful applications. A rule of thumb is that it is difficult because there are more congruences, and also interesting for the very same reason! A typical case is when $p = 2$. There is (up to conjugacy) a unique semisimple representation $\overline{\rho}_2 : G_{\mathbf{Q},\{2\}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_2)$, namely the trivial one, corresponding to the fact that there is a unique modulo 2 congruence class of normalised eigenforms of level $\mathrm{SL}_2(\mathbf{Z})$. The neatest way to surmount these difficulties is to use the notion of pseudodeformation. A nice exposition of this theory that works in small characteristic is given by Chenevier [Che14]. This reference includes a computation of the universal pseudodeformation ring of $\overline{\rho}_2$.

2. Varying the level and the weight

We now discuss modular forms of varying level, sticking to the case of cuspidal modular forms. In the classical theory, one fixes an integer $N \geq 1$ and considers the congruence subgroup

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \mid a \equiv 1 \bmod N, c \equiv 0 \bmod N \right\}$$

of $\mathrm{SL}_2(\mathbf{Z})$, defining a vector space $S_k(\Gamma_1(N))$ of cuspidal modular forms of weight k and level $\Gamma_1(N)$. One can again define Hecke operators, and the Atkin-Lehner theory of newforms uses these Hecke operators to single out those modular forms which should (and do) have associated Galois representations.

We prefer to use the language of automorphic representations of the adele group $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$ (i.e. the group of invertible 2×2 matrices with coefficients in the adele ring $\mathbf{A}_\mathbf{Q} = \prod'_p \mathbf{Q}_p \times \mathbf{R}$ of $\mathbf{Q}$). The holomorphic cuspidal newforms may be naturally identified with a subset of the set of 'algebraic' cuspidal automorphic representations of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$. Conjecturally, the algebraic cuspidal automorphic representations of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$ are in one-to-one correspondence with those compatible systems of irreducible 2-dimensional representations of $G_\mathbf{Q}$ which contribute to the cohomology of algebraic varieties. This correspondence is characterized through the (known) local Langlands correspondence, which similarly relates irreducible representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ with 2-dimensional Weil–Deligne representations of $W_{\mathbf{Q}_p}$ (to be defined below). This enlarged point of view is useful even for those automorphic representations which can be described in more classical terms, since certain phenomena (especially related to the presence of ramification) become much more transparent.

In this section, we will therefore recall some definitions in the theory of automorphic representations on GL_2, before continuing our discussion of congruences in this context. We first give the definition of a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$. Let $\chi : \mathbf{Q}^\times \backslash \mathbf{A}_\mathbf{Q}^\times \to \mathbf{C}^\times$ be a continuous (Hecke) character, and let $\mathcal{A}_{0,\chi}$ denote the $\mathbf{C}$-vector space of functions

$$f : \mathrm{GL}_2(\mathbf{Q}) \backslash \mathrm{GL}_2(\mathbf{A}_\mathbf{Q}) \to \mathbf{C}$$

satisfying the following conditions:

(1) For all $z \in \mathbf{A}_\mathbf{Q}^\times$, $g \in \mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$, $f(zg) = \chi(z) f(g)$.
(2) The span of the right translates of f under the group $\prod_p \mathrm{GL}_2(\mathbf{Z}_p) \times \mathrm{O}_2(\mathbf{R})$ is finite-dimensional.
(3) Writing $\mathbf{A}_\mathbf{Q}^\infty = \prod_p' \mathbf{Q}_p$ for the ring of finite adeles, for all $g^\infty \in \mathrm{GL}_2(\mathbf{A}_\mathbf{Q}^\infty)$, the function $\phi_{g^\infty} : \mathrm{GL}_2(\mathbf{R}) \to \mathbf{C}$, $\phi_{g^\infty}(g_\infty) = f(g^\infty, g_\infty)$ is smooth, and the span of its images under the centre of the universal enveloping algebra $U(\mathfrak{gl}_{2,\mathbf{C}})$ is finite-dimensional.
(4) Writing $\| \cdot \| : \mathbf{A}_\mathbf{Q}^\times \to \mathbf{R}_{>0}$ for the adele norm, for every $c > 0$ and every compact subset $\omega \subset \mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$, there exist $C, N > 0$ such that for all $g \in \omega$, $a \in \mathbf{A}_\mathbf{Q}^\times$ with $\|a\| > c$, we have $|f(\mathrm{diag}(a,1)g)| \leq C \|a\|^N$.
(5) For all $g \in \mathbf{A}_\mathbf{Q}^\times$, we have
$$\int_{x \in \mathbf{Q} \backslash \mathbf{A}} f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx = 0$$
(where the integration is with respect to some choice of Haar measure on $\mathbf{Q} \backslash \mathbf{A}_\mathbf{Q}$).

This is the space of cuspidal automorphic forms of central character χ. It receives actions as follows:

- The group $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q}^\infty) \times \mathrm{O}_2(\mathbf{R})$ acts by right translation.
- The Lie algebra $\mathfrak{gl}_{2,\mathbf{C}}$ acts by the formula ($X \in \mathfrak{gl}_2, f \in \mathcal{A}_{0,\chi}$):

$$(Xf)(g^\infty, g_\infty) = \frac{d}{dt} f(g^\infty, g_\infty \exp(tX))|_{t=0}.$$

The actions of $\mathrm{O}_2(\mathbf{R})$ and $\mathfrak{gl}_{2,\mathbf{C}}$ are related by the formula

$$k \cdot X \cdot f = (\mathrm{Ad}(k)X) \cdot (k \cdot f)$$

for $k \in \mathrm{O}_2(\mathbf{R})$, $X \in \mathfrak{gl}_2$, and $f \in \mathcal{A}_{0,\chi}$. These actions give $\mathcal{A}_{0,\chi}$ the structure of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q}^\infty) \times (\mathfrak{gl}_{2,\mathbf{C}}, \mathrm{O}_2(\mathbf{R}))$-module (see [Car79], [Wal79] for further discussion of the relevant local definitions). By definition, a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$ of central character χ is an irreducible $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q}^\infty) \times (\mathfrak{gl}_{2,\mathbf{C}}, \mathrm{O}_2(\mathbf{R}))$-submodule of $\mathcal{A}_{0,\chi}$.

If π is a cuspidal automorphic representation, then [Fla79] it is isomorphic to a restricted tensor product

$$\pi \cong \left(\bigotimes_l{}' \pi_l \right) \otimes \pi_\infty,$$

where:

- For each prime l, π_l is an irreducible admissible $\mathrm{GL}_2(\mathbf{Q}_l)$-module.
- π_∞ is an irreducible admissible $(\mathfrak{gl}_{2,\mathbf{C}}, \mathrm{O}_2(\mathbf{R}))$-module.

The Langlands conjectures (in a form made precise by Clozel [Clo90]) predict that only those cuspidal automorphic representations which are *algebraic* should have associated Galois representations. To define the word ‘algebraic’, and to make precise the word ‘associated’, we need to have in hand the local Langlands correspondence for GL_2. We state this precisely now, treating separately the archimedean and non-archimedean cases.

The local Langlands correspondence for $\mathrm{GL}_2(\mathbf{R})$ (established for a general reductive group over $\mathbf{R}$ by Langlands [Lan89]) is a bijection $\mathrm{rec}_{\mathbf{R}}$ between the following two sets:

- The set of (isomorphism classes of) irreducible admissible $(\mathfrak{gl}_{2,\mathbf{C}}, \mathrm{O}_2(\mathbf{R}))$-modules.
- The set of (conjugacy classes of) continuous semisimple representations $W_{\mathbf{R}} \to \mathrm{GL}_2(\mathbf{C})$.

Here $W_{\mathbf{R}}$ denotes the Weil group of $\mathbf{R}$, which may be represented as $W_{\mathbf{R}} = \mathbf{C}^\times \sqcup \mathbf{C}^\times j$, where $\mathbf{C}^\times$ is a normal subgroup, $j^2 = -1 \in \mathbf{C}$, and we have $jzj = \overline{z}$ for $z \in \mathbf{C}^\times$.

Definition 2.1. *A cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$ is said to be algebraic if $\mathrm{rec}_{\mathbf{R}}(\pi_\infty \otimes |\det|^{-1/2})|_{\mathbf{C}^\times}$ is conjugate to a representation of the form $z \mapsto \mathrm{diag}(z^{a_1}\overline{z}^{b_1}, z^{a_2}\overline{z}^{b_2})$, where a_i, b_i are integers. It is said to be regular algebraic if further $a_1 \neq a_2$ (which implies $b_1 \neq b_2$).*

The cuspidal automorphic representations π of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$ can be split into classes depending on the behaviour of π_∞ as follows:

- If $\mathrm{rec}_{\mathbf{R}}(\pi_\infty)$ is irreducible, then π may be generated by a vector corresponding to a holomorphic newform of weight $k \geq 2$. In this case, there is a character twist of π that is algebraic (even regular algebraic).
- If $\mathrm{rec}_{\mathbf{R}}(\pi_\infty)$ is reducible, it is a sum of characters of the abelianized Weil group $W_{\mathbf{R}}^{\mathrm{ab}}$, which may be identified with $\mathbf{R}^\times$, therefore of the form $x \mapsto \mathrm{sgn}(x)^{a_1}|x|^{s_1} \oplus \mathrm{sgn}(x)^{a_2}|x|^{s_2}$. If $a_1 \neq a_2$ and $s_1 = s_2$, then π may be generated by a vector corresponding to a holomorphic newform of weight $k = 1$. In this case, there is a character twist of π that is algebraic, but not a character twist that is regular algebraic.
- If $\mathrm{rec}_{\mathbf{R}}(\pi_\infty)$ is reducible and $a_1 = a_2$, then π may be generated by a vector corresponding in classical terms to a Maass form (see e.g. [Bum97]). In this case π may or may not admit a character twist which is algebraic.

To define the local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_l)$, we first need to recall the concept of Weil–Deligne representation of $W_{\mathbf{Q}_l}$. The Weil group $W_{\mathbf{Q}_l}$ is, as an abstract group, the subgroup of $G_{\mathbf{Q}_l}$ of automorphisms inducing an integer power of Frobenius on the residue field $\overline{\mathbf{F}}_l$ of $\overline{\mathbf{Q}}_l$. We give it the topology making the inertia subgroup $I_{\mathbf{Q}_l}$ an open subgroup with its usual topology. By definition, a Weil–Deligne representation is a pair (r, N), consisting of a continuous representation $r : W_{\mathbf{Q}_l} \to \mathrm{GL}_n(\mathbf{C})$ and a nilpotent $n \times n$ matrix N such that for every $\sigma \in W_{\mathbf{Q}_l}$ with image Frob_l^a in $G_{\mathbf{F}_l}$, we have the relation $r(\sigma)Nr(\sigma)^{-1} = l^{-a}N$.

The utility of Weil–Deligne representations is that they are insensitive to the topology in the field of coefficients, and allow comparison of different representations in a single compatible system. (Note that the definition of the topology on $W_{\mathbf{Q}_l}$

means a continuous representation is simply one where the profinite group $I_{\mathbf{Q}_l}$ acts through a discrete (and therefore finite) quotient.) This is the content of the following proposition (see [Tat79, §4.2]):

Proposition 2.2. *Let $p \neq l$ be a prime and let $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$. Then there is a canonical bijection* WD *between the following two classes of objects:*

(1) *The set of (conjugacy classes of) continuous representations $\rho : G_{\mathbf{Q}_l} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$.*
(2) *The set of (conjugacy classes of) Weil–Deligne representations (r, N) of rank n.*

The first example of a Weil–Deligne representation with non-trivial N is the one associated to a Tate elliptic curve. Let $q \in \mathbf{Q}_l^\times$ with $|q| < 1$. Then the Tate curve of parameter q is an elliptic curve E_q over $\mathbf{Q}_l$ with split multiplicative reduction whose Weierstrass equation may be given by power series in q with integer coefficients (see e.g. [Sil94, Ch. V]). There is a $G_{\mathbf{Q}_l}$-equivariant isomorphism

$$\overline{\mathbf{Q}}_l^\times / \langle q \rangle \cong E_q(\overline{\mathbf{Q}}_l).$$

Using this, it is easy to show that for each prime p there is an isomorphism of p-adic representations associated to $H^1_{\text{ét}}(E_{q,\overline{\mathbf{Q}}_l}, \mathbf{Q}_p)$:

$$\rho_{E_q,p} \cong \begin{pmatrix} 1 & * \\ 0 & \epsilon_p^{-1} \end{pmatrix},$$

this representation cutting out the extension $\mathbf{Q}_l(\{\zeta_{p^n}, q^{1/p^n}\}_{n \geq 1})/\mathbf{Q}_l$. This extension depends on p, but the associated Weil–Deligne representation $\mathrm{WD}(\rho_{E_q,p}) = (r, N)$ is given by

$$r = \mathbf{1} \oplus |\cdot|^{-1}, N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.1}$$

(where $|\cdot| : W_{\mathbf{Q}_l} \to \mathbf{C}^\times$ is the unramified character with $|\mathrm{Frob}_l| = l^{-1}$), which is independent of the choice of prime p.

The local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_l)$ (established for a general l-adic local field in [Kut80], and for higher rank general linear groups in [HT01]) is a bijection $\mathrm{rec}_{\mathbf{Q}_l}$ between the following two sets:

- The set of (isomorphism classes of) irreducible smooth $\mathrm{GL}_2(\mathbf{Q}_l)$-modules (over $\mathbf{C}$).
- The set of (conjugacy classes of) continuous Frobenius-semisimple Weil–Deligne representations (r, N).[3]

We are now ready to begin our discussion of the Galois representations attached to regular algebraic cuspidal automorphic representations of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$, starting with:

Theorem 2.3. *Let p be a prime and let $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ be an isomorphism. Let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$. Then there exists a continuous irreducible representation*

$$r_\iota(\pi) : G_\mathbf{Q} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$$

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[3]A Weil–Deligne representation is Frobenius-semisimple if r is semisimple. Conjecturally every Weil–Deligne representation coming from the cohomology of a smooth, projective algebraic variety satisfies this condition, but this is unknown outside of a few cases. This is why the local-global compatibility results to follow are generally stated only up to 'Frobenius semi-simplification'.

such that for every prime $l \neq p$, we have

$$\mathrm{WD}(r_\iota(\pi)|_{G_{\mathbf{Q}_l}})^{F-ss} \cong \mathrm{rec}_{\mathbf{Q}_l}(\pi_l|\det|^{-1/2}).$$

If $l \neq p$ is a prime such that π_l is unramified, then the above relation simply asserts that $r_\iota(\pi)|_{G_{\mathbf{Q}_l}}$ is unramified and that the characteristic polynomial of $r_\iota(\pi)(\mathrm{Frob}_l)$ can be written down in terms of the eigenvalues of the unramified Hecke operators at the prime l on π. In particular, suppose that $f \in S_k(\mathrm{SL}_2(\mathbf{Z}))$ is a normalised eigenform (as appearing in the statement of Theorem 1.2), and define a function $\phi : \mathrm{GL}_2(\mathbf{A_Q}) \to \mathbf{C}$ by the formula

$$\phi(\gamma g^\infty g_\infty) = f(g_\infty \cdot i) j(g_\infty, i)^{-k} \det(g_\infty),$$

for any $\gamma \in \mathrm{GL}_2(\mathbf{Q})$, $g^\infty \in \prod_p \mathrm{GL}_2(\mathbf{Z}_p)$, $g_\infty \in \mathrm{GL}_2(\mathbf{R})^{\det>0}$ (any element of $\mathrm{GL}_2(\mathbf{A_Q})$ admits an expression as a product of such elements). Then one can show that $\phi \in \mathcal{A}_{0,|\cdot|^{2-k}}$ generates a regular algebraic cuspidal automorphic representation π_f, and we can take $r_\iota(\pi_f) = \rho_{f,\iota}$.

Definition 2.4. *Let π, π' be regular algebraic cuspidal automorphic representations of $\mathrm{GL}_2(\mathbf{A_Q})$, and let $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ be an isomorphism. We say that π, π' are congruent modulo p (with respect to ι) if there is an isomorphism $\overline{r}_\iota(\pi) \cong \overline{r}_\iota(\pi')$.*

Once again, the Chebotarev density theorem shows that this is equivalent either to:

- For every prime $l \neq p$ at which π and π' are both unramified, the Hecke eigenvalues of T_l on π, π' (or their images under ι in $\overline{\mathbf{Z}}_p$) have the same image in $\overline{\mathbf{F}}_p$;
- Or, the same requirement but only for all but finitely many primes $l \neq p$.

The existence of a congruence between two (regular algebraic, cuspidal) automorphic representations π, π' places constraints on their local behaviour. For example, suppose that l is a prime such that π_l is unramified and π'_l is an unramified twist of the Steinberg (or special) representation of $\mathrm{GL}_2(\mathbf{Q}_l)$. (In classical terms, this means that if π' corresponds to a newform of level $\Gamma_1(N)$, then $l||N$ and the Nebentypus of π' has conductor prime to l. In terms of the local Langlands correspondence, it means that $\mathrm{rec}_{\mathbf{Q}_l}(\pi'_l)$ is an unramified character twist of the representation given by (2.1).) By hypothesis, there is an isomorphism

$$\overline{r}_\iota(\pi) \cong \overline{r}_\iota(\pi').$$

We see that $r_\iota(\pi')|_{G_{\mathbf{Q}_l}}$ is an infinitely ramified representation whose reduction modulo p is unramified. The characteristic polynomial of $\overline{r}_\iota(\pi')(\mathrm{Frob}_l)$ coincides with the reduction modulo p of the characteristic polynomial of $r_\iota(\pi')(\phi_l)$, where $\phi_l \in G_{\mathbf{Q}_l}$ is any lift of $\mathrm{Frob}_l \in G_{\mathbf{F}_l}$. The eigenvalues of this characteristic polynomial have ratio $\alpha, \alpha l$ for some $\alpha \in \overline{\mathbf{Q}}_l^\times$. We conclude that a necessary condition on π for the existence of π' with the given behaviour at l is that the eigenvalues of $\overline{r}_\iota(\pi)$ in $\overline{\mathbf{F}}_l^\times$ have ratio $l^{\pm 1}$.

It is a fact that, under mild hypotheses, this necessary condition for the existence of a congruence is also sufficient. This was first established by Ribet [Rib84]. His methods can be used to prove:

Theorem 2.5. *Let π be a regular algebraic, cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$ such that π_∞ is of 'weight 2'.*[4] *Suppose given distinct primes p, l satisfying the following conditions:*

1. *The representation $\overline{r}_\iota(\pi)$ is irreducible.*
2. *π_l is unramified.*
3. *The eigenvalues $\alpha, \beta \in \overline{\mathbf{F}}_p^\times$ of $\overline{r}_\iota(\pi)(\mathrm{Frob}_l)$ satisfy $\alpha/\beta = l^{\pm 1}$.*

Then there exists another regular algebraic, cuspidal automorphic representation π' of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$, also of weight 2, such that $\overline{r}_\iota(\pi') \cong \overline{r}_\iota(\pi)$ and π'_l is an unramified twist of the Steinberg representation.

We refer to the result of the theorem as 'level-raising'. If π, π' correspond to classical newforms f, f' of levels $N_f, N_{f'}$, then N_f is prime to l, while $N_{f'}$ is divisible by l. We note that it is easy, by applying the Chebotarev density theorem to the extension of $\mathbf{Q}$ cut out by $\overline{r}_\iota(\pi)$, to find infinitely many primes l satisfying the hypothesis of the theorem. We also note that some global hypothesis (like the irreducibility of $\overline{r}_\iota(\pi)$) is necessary; see again [Rib84] for examples.

Ribet's theorem was proved using automorphic techniques: the theory of Galois representations plays essentially no role. Other authors improved Ribet's results, classifying the possible levels of pairs of congruent newforms and proving level-raising results producing more general ramified local components (see e.g. [DT94]). Mazur and Ribet also proved level-*lowering* results [Rib90] (where one e.g. starts with π' and hopes to produce π), which played an essential role in the first proof of Fermat's Last Theorem [Wil95].

A different and very useful point of view on these problems is given by Galois deformation theory: the necessary local conditions for level-raising at l to occur can be naturally expressed in this framework. Furthermore, Khare–Wintenberger (on their way to the proof of Serre's conjecture, which we will discuss in §4 below) were able to use Galois deformation theory to eventually establish a very strong local-global principle for the existence of congruences. This approach subsumes the results mentioned in the previous paragraph, and also gives an important guide to understanding the much more subtle question of the existence of congruences between modular forms of differing weights.

To describe this, we take up again the notation introduced for Galois deformation theory in the previous section. Let π be a regular algebraic, cuspidal automorphic representation of weight $k \geq 2$. Let p be a prime such that $\overline{r}_\iota(\pi)$ is irreducible, and let S be a finite set of primes containing p and the primes l such that π_l is ramified. At this stage, we suppose for simplicity that π_p is unramified. We introduce a coefficient ring $\mathcal{O} \subset \overline{\mathbf{Q}}_p$ such that $r_\iota(\pi)$ may be defined over $\mathcal{O}$, as well as the following data:

1. The residual representation $\overline{r}_\iota(\pi) : G_{\mathbf{Q},S} \to \mathrm{GL}_2(k)$ and its deformation problem $\mathcal{D} = (S, \det r_\iota(\pi), \{\mathcal{D}_l\}_{l \in S})$, where $\mathcal{D}_p = \mathcal{D}^{cr,(0,k-1)}$ and if $l \in S - \{p\}$ then $\mathcal{D}_l = \mathcal{D}_l^\square$.
2. The Hecke algebra $\mathbf{T}_\pi$: it is a finite flat $\mathcal{O}$-algebra which classifies cuspidal automorphic representations π' with the following properties:
   (a) There is a congruence $\pi \equiv \pi' \bmod p$, i.e. an isomorphism $\overline{r}_\iota(\pi) \cong \overline{r}_\iota(\pi')$.
   (b) π' has weight k and is unramified away from $S - \{p\}$.

[4] That is, of the form associated to holomorphic newforms of weight 2.

(3) A surjective $\mathcal{O}$-algebra homomorphism $R_{\mathcal{D}} \to \mathbf{T}_\pi$.

One conjectures (and can often prove) that the map $R_{\mathcal{D}} \to \mathbf{T}_\pi$ is an isomorphism; in particular, that $R_{\mathcal{D}}$ is a finite flat $\mathcal{O}$-algebra. Even without knowing this, we can associate a discrete invariant to any homomorphism $R_{\mathcal{D}} \to \overline{\mathbf{Q}}_p$, namely the inertial equivalence class of the associated Weil–Deligne representations at primes $l \in S$. We give the relevant definitions.

Definition 2.6. (1) *Let (r, N), (r', N') be Weil–Deligne representations of $W_{\mathbf{Q}_l}$ of rank n over $\mathbf{C}$. We say that they are inertially equivalent if the restrictions $(r|_{I_{\mathbf{Q}_l}}, N)$, $r'|_{I_{\mathbf{Q}_l}}, N')$ are conjugate.*

(2) *Let $[(r, N)]$ be a conjugacy class of pairs consisting of a continuous representation $r : I_{\mathbf{Q}_l} \to \mathrm{GL}_n(\mathbf{C})$ and a nilpotent $n \times n$ matrix $N \in M_n(\mathbf{C})$. We call $[(r, N)]$ an inertial type if (r, N) extends to a Weil–Deligne representation of $W_{\mathbf{Q}_l}$.*

Thus two Weil–Deligne representations of $W_{\mathbf{Q}_l}$ are inertially equivalent if they have the same inertial type. Given the global representation $\overline{r}_\iota(\pi) : G_{\mathbf{Q}} \to \mathrm{GL}_2(k)$, and a prime $l \neq p$, we define $\mathcal{I}_l(\overline{r}_\iota(\pi))$ to be the set of inertial types $[(r, N)]$ over $\overline{\mathbf{Q}}_p$ with the following property: there exists a continuous lift $\rho_l : G_{\mathbf{Q}_l} \to \mathrm{GL}_2(\overline{\mathbf{Z}}_p)$ of $\overline{\rho}|_{G_{\mathbf{Q}_l}}$ such that $\mathrm{WD}(\rho_l)|_{I_{\mathbf{Q}_l}} \cong (r, N)$. This is a finite set. We can now state a version of Khare–Winterberger's local-global principle [KW09a]:

Theorem 2.7. *Let $p \geq 7$ be prime, and suppose that π has weight 2, that π_p is unramified, and that $\overline{r}_\iota(\pi)$ is irreducible. Let S be a finite set of primes, including p and the primes at which π is ramified, and choose for each $l \in S - \{p\}$ an inertial type $[(r_l, N_l)] \in \mathcal{I}_l(\overline{r}_\iota(\pi))$. Then there exists another regular algebraic cuspidal automorphic representation π' of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$ with the following properties:*

(1) *π'_∞ has weight 2 and π'_p is unramified.*
(2) *For each $l \in S - \{p\}$, the inertial type of $\mathrm{WD}(r_\iota(\pi))$ is $[(r_l, N_l)]$.*

Thus $r_\iota(\pi')$ is an automorphic lift (i.e. a lift of the form $r_\iota(\pi')$) of $\overline{r}_\iota(\pi)$ which can have any allowed ramification at each prime $l \neq p$. We leave it to the reader to explain how to deduce Theorem 2.5 from Theorem 2.7 by making an appropriate choice of inertial types. It is important to note that Khare–Wintenberger also prove a version of this theorem which does not have automorphic representations in the statement: instead, one starts with a residual representation $\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ and produces a lift $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{Z}}_p)$ with prescribed local behaviour, which (like an automorphic lift) sits in a compatible system. However, the proof still uses automorphic forms!

Let us sketch the proof of Theorem 2.7. Let $l \in S - \{p\}$ and let $\tau_l = [(r_l, N_l)] \in \mathcal{I}_l(\overline{r}_\iota(\pi))$. The first ingredient is the definition of a suitable deformation problem $\mathcal{D}_l(\tau_l)$. It turns out that a naive definition has good properties. If $x : R_l^\square \to \overline{\mathbf{Q}}_p$ is an $\mathcal{O}$-algebra homomorphism, then we get (by pushforward of the universal lifting) a representation $\rho_x : G_{\mathbf{Q}_l} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$. Let $R_l(\tau_l)$ denote the smallest reduced quotient of $R_l^\square$ such that for any x such that ρ_x has inertial type τ_l, x factors through $R_l(\tau_l)$. We then have the following proposition (see [Sho18, §3.3]):

Proposition 2.8. *With notation as in the previous paragraph:*

(1) $\mathrm{Spec}\, R_l(\tau_l)$ *is a union of irreducible components of* $\mathrm{Spec}\, R_l^\square$.
(2) *For any homomorphism $x : R_l(\tau_l) \to \overline{\mathbf{Q}}_p$ such that $\mathrm{WD}(\rho_x)$ is non-degenerate, the inertial type of $\mathrm{WD}(\rho_x)$ is τ_l.*

We do not define the 'non-degenerate' condition here, but note that it is satisfied by any x such that ρ_x is isomorphic to $r_\iota(\pi')|_{G_{\mathbf{Q}_l}}$, for a regular algebraic cuspidal automorphic representation π' of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$.

To define a global deformation problem, we also need to specify a local deformation problem at the prime p. In the situation of Theorem 2.7, we can simply take $\mathcal{D}_p = \mathcal{D}_p^{cr,(0,1)}$. We then have a global deformation problem $\mathcal{D} = (S, \det r_\iota(\pi), \{\mathcal{D}_l\}_{l \in S})$. Our task is to find a homomorphism $R_\mathcal{D} \to \overline{\mathbf{Q}}_p$ corresponding to an automorphic lift $\rho : G_{\mathbf{Q},S} \to \mathrm{GL}_2(\overline{\mathbf{Z}}_p)$.

Mazur explained how to give a lower bound for the Krull dimension of the unrestricted deformation ring $R_{\overline{\rho},S}$ using obstruction theory and Tate's Euler characteristic formula. The same idea can be adapted to give a lower bound for the Krull dimension of $R_\mathcal{D}$, which turns out in our situation to give the estimate $\dim R_\mathcal{D} \geq 1$. The observation of [KW09a, §3] is that if we can show as well that $R_\mathcal{D}$ is a finite $\mathcal{O}$-algebra, then we will necessarily have $R_\mathcal{D}[1/p] \neq 0$, implying the existence of a homomorphism $R_\mathcal{D} \to \overline{\mathbf{Q}}_p$. One way to establish this finiteness is to show that there is an isomorphism $R_\mathcal{D} \to \mathbf{T}_\pi$ with a suitable Hecke algebra. If we are in the more general situation of a residual representation which is not known to be automorphic, then this step is replaced by an appeal to Taylor's technique of potential automorphy [Tay02], together with an '$R = \mathbf{T}$' theorem over a suitable base extension. We will discuss representations (Galois and automorphic) over more general bases than $\mathbf{Q}$ in §3 below.

In the above discussion we have adopted simplifying hypotheses (in particular, low weight and no ramification) at the prime p. Following Kisin [Kis08], it is possible to define a local deformation problem $\mathcal{D}_p^{pst,(0,k-1),\tau_p}$, corresponding to 'de Rham liftings of Hodge–Tate weights $(0, k-1)$ of inertial type τ_p'. It is then natural to guess that if the associated lifting ring is non-zero then one can again find an automorphic lift of the given residual representation with prescribed inertial type at places $l \in S - \{p\}$ and also prescribed inertial type at p and infinity type π_∞ of weight k. This question is subtle, leading to problems such as 'the weight part of Serre's conjecture' (see e.g. [BDJ10]) and the Breuil–Mézard conjecture [BM02]. Although they have largely been resolved for $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$ (see e.g. [Kis09a]), analogous questions for more general groups are very much open (see [GHS18]).

3. Generalisation to other base fields and higher rank

We now broaden the class of Galois and automorphic representations under consideration. We do this both in order to be able to consider more general questions, but also because the technique of base change plays an essential role even in proofs of theorems (such as Theorem 2.7) in the case of GL_2 over $\mathbf{Q}$.

Let F be a number field, and let $n \geq 1$. A cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$ is an irreducible admissible $\mathrm{GL}_n(\mathbf{A}_F^\infty) \times (\mathfrak{gl}_{F,\mathbf{C}}, K_\infty)$-module which appears in the space of cusp forms on $\mathrm{GL}_n(\mathbf{A}_F)$ (the definition generalising the case of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$, see e.g. [BJ79]). Such a representation π admits a restricted direct product factorisation $\pi = \otimes'_v \pi_v$ running over the set of places v of F; moreover, for each place v we have the local Langlands correspondence rec_{F_v} for the group $\mathrm{GL}_n(F_v)$, the Weil groups W_{F_v} being defined in an analogous way (see [Tat79] for more details). We can give the analogue of Definition 2.1:

Definition 3.1. *A cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$ is said to be algebraic if for each place $v|\infty$ of F, $\mathrm{rec}_{F_v}(\pi_\infty \otimes |\det|^{(1-n)/2})|_{\mathbf{C}^\times}$ is conjugate to a representation of the form $z \mapsto \mathrm{diag}(z^{a_{v,1}}\overline{z}^{b_{v,1}}, \dots, z^{a_{v,n}}\overline{z}^{b_{v,n}})$, where $a_{v,i}, b_{v,i}$ are integers. It is further said to be regular algebraic if for each such v, the numbers $a_{v,1}, \dots, a_{v,n}$ are distinct (which implies that the numbers $b_{v,1}, \dots, b_{v,n}$ are also distinct)*[5].

We make the following conjecture concerning the existence of Galois representations associated to a regular algebraic cuspidal automorphic representation π. (We stick to the *regular* algebraic case as this is the context in which most unconditional results have been proved.)

Conjecture 3.2. *Let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$. Let $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ be an isomorphism. Then there exists a continuous semisimple representation $r_\iota(\pi) : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ such that if $v|p$ then $r_\iota(\pi)|_{G_{F_v}}$ is potentially semistable, in the sense of p-adic Hodge theory, and for any finite place v of F, there is an isomorphism*

$$\mathrm{WD}(r_\iota(\pi)|_{G_{F_v}})^{F-ss} \cong \iota^{-1}\mathrm{rec}_{F_v}(\pi_v|\cdot|^{(1-n)/2}). \tag{3.1}$$

(If $v|p$, we are using the recipe for the Weil–Deligne representation of a potentially semistable representation given by Fontaine [Fon94]. It is further expected that $r_\iota(\pi)$ is always irreducible, although this is not known in general, even in many of those cases that $r_\iota(\pi)$ has been proved to exist.) We now discuss what is known regarding this conjecture, sticking to the case where π is regular algebraic. If $n = 1$, it follows from class field theory. If $n = 2$ and $F = \mathbf{Q}$, then the conjecture is known in its entirety. The same is true if $n = 2$ and F is a totally real number field; in this case most of the associated Galois representations $r_\iota(\pi)$ be may be constructed inside the étale cohomology of Shimura curves defined over the field F [Car86], with the general case completed either using congruences [Tay89] or more complicated Shimura varieties [BR89]. The proof of relation (3.1) in this case, including at the places $v|p$, was completed in [Ski09], using earlier work of Saito and Kisin.

The next case to consider is when $n > 2$ and π is polarizable, in the sense of [BLGGT14]:

Definition 3.3. *Let π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$. We say that π is polarizable if either:*

1. *F is totally real and there is a Hecke character $\chi : F^\times \backslash \mathbf{A}_F^\times \to \mathbf{C}^\times$, with $\chi_v(-1)$ independent of $v|\infty$, such that $\pi \cong \pi^\vee \otimes \chi$.*
2. *F is a CM field (i.e. a totally imaginary quadratic extension of a totally real number field F^+) and there is a Hecke character $\chi : (F^+)^\times \backslash \mathbf{A}_{F^+}^\times \to \mathbf{C}^\times$, with $\chi_v(-1)$ independent of $v|\infty$, such that $\pi^c \cong \pi^\vee \otimes (\chi \circ \mathbf{N}_{F/F^+})$. (Here $c \in \mathrm{Gal}(F/F^+)$ is the non-trivial element.)*

The existence of Galois representations associated to regular algebraic cuspidal automorphic representations which are polarizable can be reduced, by base change and twisting by characters, to the case where F is CM and π is conjugate self-dual (i.e. $\pi^c \cong \pi^\vee$). In this case π often descends (in the sense of Langlands functoriality) to an automorphic representation of a unitary group over the maximal

[5] This equivalence uses the fact that π is cuspidal, in order to able to invoke the 'purity lemma' [Clo90, Lemme 4.9].

totally real subfield $F^+ \subset F$ and the corresponding Galois representations may be constructed inside the étale cohomology of unitary Shimura varieties. For a guide to the (extensive) literature in this case, we refer to [Shi20].

If π is a conjugate self-dual, regular algebraic, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$, then its associated Galois representations $r_\iota(\pi)$ satisfy $r_\iota(\pi)^c \cong r_\iota(\pi)^\vee \otimes \epsilon_p^{1-n}$. It is therefore natural to the consider the deformation theory of n-dimensional representations of G_F satisfying this conjugate-self duality condition. This turns out to be a powerful and flexible context that has many of the good features of the situation for GL_2 over $\mathbf{Q}$. In particular, one can prove analogues of the local-global principle of Khare–Wintenberger (see e.g. [BLGGT14, Theorem 4.3.1]). The theorems we discuss in §4 below are proved in this context.

When F is totally real or CM, but π is not polarizable, the compatible system of Galois representations $r_\iota(\pi)$ associated to π has been constructed [HLTT16, Sch15], but the representations $r_\iota(\pi)$ are not known to be de Rham in general and the local-global compatibility relation (3.1) is known to hold only at all but finitely places v of F (although progress is rapidly being made – see [ACC$^+$23] for some positive results in this direction). This is a reflection of the fact that the Galois representations are not known to have a geometric realization (i.e. a realization inside the étale cohomology of an algebraic variety), but instead may only be constructed by a p-adic limiting process.

It is important to note that the local-global principle (i.e. the analogue of Theorem 2.7) fails in this context! On the Galois side, this may be seen as a reflection of the fact that the 'expected dimension' of the deformation ring $R_{\mathcal{D}}$ that one writes down (which comes from obstruction theory and the Euler characteristic formula) is negative (see e.g. [CG18]). For some local conditions, one may expect that the deformation ring $R_{\mathcal{D}}$ does not have any characteristic 0 points. On the automorphic side, it may be seen as a reflection of the fact that regular algebraic automorphic representations are relatively sparse (see e.g. [Mar12]). Here is an interesting open question in this direction:

Question 3.4. *Let F be an imaginary quadratic field, and let $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ be an irreducible representation. Does there always exist a lift $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbf{Z}}_p)$, unramified almost everywhere and de Rham at the p-adic places of F?*

Finally, we must mention the case when $n \geq 2$ and F is not totally real or CM (for example, when F is a mixed-signature cubic field), when we can say almost nothing. Progress in this direction appears as difficult, at present, as understanding algebraic Maass forms (e.g. proving the analogue of Conjecture 3.2 in the case algebraic cuspidal automorphic representations of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$ not arising from holomorphic modular forms).

4. Applications of congruences to modularity

In this section we will discuss three significant applications of the construction of congruences between modular forms (and automorphic representations) to the modularity (or automorphy) of Galois representations. We first discuss what is expected to hold, restricting to the regular algebraic context introduced in §3. The following conjecture encapsulates conjectures of Langlands, Clozel, and Fontaine–Mazur [FM95]:

Conjecture 4.1. *Let F be a number field, let $n \geq 1$, let p be a prime number, and let $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ be an isomorphism. Then the map $\pi \mapsto r_\iota(\pi)$ of Conjecture 3.2 exists, and defines a bijection between the following two sets:*

(1) *The set of regular algebraic, cuspidal automorphic representations of $\mathrm{GL}_n(\mathbf{A}_F)$.*
(2) *The set of isomorphism classes of irreducible representations $\rho : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ which are geometric, in the sense that they are unramified at all but finitely many places of F and such that for each place $v|p$ of F, $\rho|_{G_{F_v}}$ is de Rham, and which are regular, in the sense that for each embedding $\tau : F \to \overline{\mathbf{Q}}_p$, the set $\mathrm{HT}_\tau(\rho)$ of τ-Hodge–Tate weights has n distinct elements.*

(We use the definition of τ-Hodge–Tate weights given e.g. in [BLGGT14].) A Galois representation ρ which is of the form $r_\iota(\pi)$ is said to be automorphic. The applications we discuss will include some special cases of this theorem. They will all combine the existence of suitable congruences with *automorphy lifting theorems*. The first automorphy lifting theorems were proved in [Wil95, TW95]. Many such theorems now exist in the literature, generally of the following form:

Theorem Schema 4.2. *Let $\rho : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ be a representation satisfying the following conditions:*

(1) *(Residual automorphy) There exists a regular algebraic, cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$ and an isomorphism $\overline{\rho} \cong \overline{r}_\iota(\pi)$ of residual representations $G_F \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$.*
(2) *(Necessary conditions) e.g. ρ is geometric, ρ is irreducible, etc.*
(3) *(Technical conditions) e.g. $\overline{\rho}$ is irreducible, $r_\iota(\pi)$ and ρ have the same Hodge–Tate weights, etc.*

Then ρ is automorphic: there exists a regular algebraic, cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbf{A}_F)$ and an isomorphism $\rho \cong r_\iota(\Pi)$.

Proving an automorphy lifting theorem usually requires controlling certain Selmer groups (e.g. the subgroup of $H^1(F, \mathrm{ad}\,\overline{\rho})$ by suitable local conditions) and understanding the geometry of the local deformation rings associated to ρ (especially at the p-adic places), and these requirements are often the source of the technical conditions. Conversely, automorphy lifting theorems with fewer technical conditions can often have striking applications.

4.3. **First application: modularity and FLT.** Let E be an elliptic curve over a number field F such that $\mathrm{End}_F(E) = \mathbf{Z}$. For any prime p we have a 2-dimensional representation $\rho_{E,p} : G_F \to \mathrm{GL}_2(\mathbf{Q}_p)$, afforded by the 2-dimensional $\mathbf{Q}_p$-vector space $H^1_{\text{ét}}(E_{\overline{F}}, \mathbf{Q}_p)$. This representation is absolutely irreducible and Hodge–Tate regular. Under Conjecture 4.1, there should exist a regular algebraic, cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbf{A}_F)$ such that $\rho_{E,p} \cong r_\iota(\pi)$. When $F = \mathbf{Q}$, this is the conjecture variously known as the Taniyama–Shimura–Weil conjecture or the modularity conjecture for elliptic curves over $\mathbf{Q}$, proved for semistable curves in [Wil95] and in general in [BCDT01]. This application indicates the usefulness of stronger automorphy lifting theorems: the main innovation in [BCDT01] is to analyse the structure of certain potentially crystalline deformation rings, beyond the crystalline or semistable cases used in [Wil95].

Famously, the modularity conjecture implies Fermat's Last Theorem, i.e. the non-existence of a solution to the equation $a^n + b^n = c^n$ in non-zero integers a, b and c and exponent $n \geq 3$. We explain how this follows from the modularity conjecture and the local-global principle Theorem 2.7. We may assume $p \geq 5$. Following the recipe of Frey [Fre87], we write down the elliptic curve

$$E : y^2 = x(x - a^p)(x + b^p).$$

Provided we first permute a, b and c (assumed coprime) so that $a \equiv 3 \bmod 4$ and b is even, a computation (see e.g. [Ser87, §4]) shows that the elliptic curve E is semistable, with minimal discriminant $2^{-8}(abc)^{2p}$. Mazur's classification of torsion subgroups [Maz77] implies that $\overline{\rho}_{E,p}$ is irreducible. We see (using the Tate uniformization introduced above and the fact that the discriminant is a local p^{th} power) that for each prime $l|abc$, $l \nmid 2p$, that $\overline{\rho}_{E,p}|_{G_{\mathbf{Q}_l}}$ is unramified and so $\mathcal{I}_l(\overline{\rho}_{E,p})$ contains the unramified inertial type. Similarly one sees that $\overline{\rho}_{E,p}|_{G_{\mathbf{Q}_p}}$ admits a crystalline weight 2 lift (which we can take to be $\rho_{E,p}|_{G_{\mathbf{Q}_p}}$ if $p \nmid abc$, and construct by hand otherwise). Therefore Theorem 2.7 (with a slight extension if $p|abc$) implies the existence of an automorphic lift $r_\iota(\pi')$ of $\overline{\rho}_{E,p}$, with π' unramified away from 2 and with π'_2 an unramified twist of the Steinberg representation. Such a π' is associated (in classical terms) to a newform of weight 2 and level $\Gamma_0(2)$. However, it is easy to show that no such newform exists, leading to the contradiction that proves the theorem.

4.4. **Second application: Serre's conjecture.** Serre's conjecture, published in 1987 [Ser87], is an antecedent of the Fontaine–Mazur conjecture that concerns representations over $\overline{\mathbf{F}}_p$ (and can therefore be formulated without using p-adic Hodge theory). Here is the statement:

Conjecture 4.5. *Let p be a prime, and let $\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ be a continuous irreducible representation which is odd, in the sense that $\det \overline{\rho}(c) = -1$. (Such a representation is said to be of S-type.) Then:*

(1) *There exists a holomorphic newform f and an isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ such that $\overline{\rho} \cong \overline{\rho}_{f,\iota}$;*
(2) *and moreover, f can be chosen to be of level $N = N(\overline{\rho})$, the prime-to-p Artin conductor of $\overline{\rho}$, and of weight $k = k(\overline{\rho})$, where $k(\overline{\rho})$ depends only on $\overline{\rho}|_{I_{\mathbf{Q}_p}}$ and is given by the recipe in* [Ser87, §2].

It is hard to overstate the importance of this conjecture in the development of the themes discussed in this paper. It has many significant arithmetic consequences, of which we mention a few, referring to [Ser87, Kha10] for further discussion:

- It implies Fermat's Last Theorem, essentially by the route discussed in §4.3 above.
- It implies that for a given prime number p and bound $X > 0$, the set of conjugacy classes of S-type representations $\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ of prime-to-p conductor $N(\overline{\rho}) < X$ is finite.
- It implies the modularity conjecture for elliptic curves, and more generally for abelian varieties of GL_2-type over $\mathbf{Q}$.

Moreover, Serre's conjecture represents an enlarged point of view on the Langlands programme: rather than expecting only a bijection between objects in characteristic 0, it is expected under this conjecture that we should have a similar correspondence in positive characteristic. This in turn makes it reasonable to ask for a correspondence over general bases (such as finite rings), motivating eventually the existence of an isomorphism $R \cong \mathbf{T}$.

Serre's conjecture was proved by Khare–Wintenberger [Kha06, KW09b, KW09c]. The proofs makes essential use of automorphy lifting theorems and the local-global principle Theorem 2.7, which was in fact proved with the application of Serre's conjecture in mind [KW09a].

Let us sketch some ideas of the proof in [KW09b]. First, we note that the the second part of Conjecture 4.5 follows directly from an appropriate version of Theorem 2.7, so we need only focus on the first part, namely the residual modularity of S-type representations. We first treat the level 1 case (i.e. when $N(\overline{\rho}) = 1$), which is proved by induction on the prime p. If $\overline{\rho}$ is an S-type representation, then a suitably souped-up purely Galois-theoretic version of Theorem 2.7 implies the existence of a compatible system $(\rho_\jmath)_\jmath$ of representations containing a p-adic representation that lifts $\overline{\rho}$, each $\rho_\jmath$ being crystalline and unramified away from the residue characteristic. Choosing a prime $l < p$ and an isomorphism $\jmath : \overline{\mathbf{Q}}_l \to \mathbf{C}$, and reducing modulo l, we find that the residual representation $\overline{\rho}_\jmath : G_{\mathbf{Q},\{l\}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_l)$ is modular (by induction). A powerful enough automorphy lifting theorem would imply that $\rho_\jmath$ (and therefore any other member of the compatible system containing it) is automorphic, yielding the modularity of $\overline{\rho}$ by reduction modulo p.

The base case of the induction is the case $p = 2$. Tate [CS15, T.2.7.73] showed that in this case there is no S-type representation of prime-to-p conductor 1, in response to Serre's first steps towards the general conjecture formulated in [Ser87]!

The case of $N > 1$ is treated by induction on N. Given an S-type representation of conductor $N(\overline{\rho})$, we first apply (a souped-up purely Galois-theoretic version of) Theorem 2.7 to lift $\overline{\rho}$ to a compatible system $(\rho_\iota)_\iota$ of conductor $N(\overline{\rho})$. We then choose a prime $l | N(\overline{\rho})$, fix an isomorphism $\jmath : \overline{\mathbf{Q}}_l \to \mathbf{C}$, and consider the reduced representation $\overline{\rho}_\jmath$. This now has conductor $N(\overline{\rho}_\jmath) < N(\overline{\rho})$ (indeed, the conductor is at most the prime-to-l part of $N(\overline{\rho})$), so the modularity of $\overline{\rho}_\jmath$ follows by induction, and the automorphy of the compatible system $(\rho_\iota)_\iota$ should follow on application of a suitable automorphy lifting theorem.

A significant difficulty in the proof is presented by the need to arrange the lifts so that the reduced representations satisfy the technical conditions of available automorphy lifting theorems. Since the publication of [KW09b, KW09c] more general automorphy lifting theorems have been proved, notably in the residually reducible case (see [Pan22]), that make it possible to give a shorter proof (see e.g. [DP22]).

4.6. **Third application: Symmetric power functoriality.** Symmetric power functoriality is a special case of the Langlands functoriality conjectures, which were outlined for the first time in [Lan70], for automorphic representations of a general reductive group. Here we recall the statement of these conjectures for general linear groups. (The general statement requires the notion of L-group and L-homomorphism. These play a fundamental role in the theory of automorphic forms but we do not have space to introduce them here.)

Conjecture 4.7. *Let F be a number field, and let π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$. Let $R : \mathrm{GL}_n \to \mathrm{GL}_m$ be an algebraic representation (i.e. a morphism of linear algebraic groups). Then there exists an automorphic representation $R_*(\pi)$ of $\mathrm{GL}_m(\mathbf{A}_F)$, the 'functorial lift of π along R', such that for every place v of R, there is an isomorphism*

$$\mathrm{rec}_{F_v}(R_*(\pi)_v) \cong R \circ \mathrm{rec}_{F_v}(\pi_v) \tag{4.1}$$

of (Weil–Deligne, if $v \nmid \infty$) representations of W_{F_v}.

Let us discuss some special cases of this conjecture. First, we can restrict, by the theory of Eisenstein series [BJ79, Supplement], to the case that R is irreducible (in other words, is a highest weight representation of the reductive group GL_n over $\mathbf{Q}$).

If $n = 1$, then the irreducible representations of $\mathrm{GL}_1 = \mathbf{G}_m$ are exactly the characters $x \mapsto x^N$, for $N \in \mathbf{Z}$ – while the automorphic representations of $\mathrm{GL}_1(\mathbf{A}_F)$ are simply the continuous characters $\chi : F^\times \backslash \mathbf{A}_F^\times \to \mathbf{C}^\times$. In this case, the functorial lift $R_*(\chi)$ is the power χ^N. The defining relation (4.1) can be checked using the interpretation of rec_{F_v} for $n = 1$ in terms of local class field theory.

If $n = 2$ (which is as far as we will go here), the situation is much harder. Using the $n = 1$ case, we are free to twist R by a power of the determinant. The irreducible algebraic representations of GL_2 are, up to character twist, precisely the symmetric powers $\mathrm{Sym}^m : \mathrm{GL}_2 \to \mathrm{GL}_{m+1}$ ($m \geq 0$) of the standard 2-dimensional representation of GL_2. The truth of the conjecture in this case would have many applications, including most famously the Ramanujan conjecture, which states that if π is a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ with unitary central character, then the parameters $\mathrm{rec}_{F_v}(\pi_v)$ have relatively compact image (suitably interpreted at finite places v with $N_v \neq 0$). It also implies that the family of symmetric power L-functions $L(\pi, R, s)$ associated to π has good properties, including meromorphic continuation to the whole complex plane, an observation which is at the basis of the proof of the Sato–Tate conjecture for elliptic curves over $\mathbf{Q}$ [BLGHT11].

Conjecture 4.7 has been proved for Sym^m and $m = 2$, 3 and 4 by Gelbart–Jacquet [GJ78], Kim–Shahidi [KS02], and Kim [Kim03], respectively. By restricting the class of automorphic representations under consideration, one can go considerably further. Our final goal in this article is to discuss the application of congruences between modular forms to the proof of the following theorem:

Theorem 4.8 (Newton-T., 2022)**.** *Let F be a totally real number field, and let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ which is regular algebraic. Then for each $m \geq 2$, $\mathrm{Sym}^m_*(\pi)$ exists, as an automorphic representation of $\mathrm{GL}_{m+1}(\mathbf{A}_F)$, and is cuspidal if and only if π is not automorphically induced.*

We first proved this theorem in the case $F = \mathbf{Q}$ in 2020 [NT21a, NT21b] using overconvergent modular forms. This proof is surveyed in [New22] and [Tho22b]. Our proof of this theorem in the case of a general totally real field does not make use of overconvergent modular forms or eigenvarieties and is new even in the case $F = \mathbf{Q}$.

The reason we are able to prove anything in this case is because of the connection with Galois representations. Let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ with is not automorphically induced (in the automorphically induced case, the symmetric power liftings can be written down easily by hand.) If $\mathrm{Sym}^m_*(\pi)$ exists, then it is regular algebraic, and for any prime number p and isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$, there is an isomorphism

$$r_\iota(\mathrm{Sym}^m_*(\pi)) \cong \mathrm{Sym}^m r_\iota(\pi)$$

of representations $G_F \to \mathrm{GL}_{m+1}(\overline{\mathbf{Q}}_p)$. Conversely, if the Galois representation $\mathrm{Sym}^m r_\iota(\pi)$ is automorphic, associated to an automorphic representation Π of $\mathrm{GL}_{m+1}(\mathbf{A}_F)$, then Π should be the functorial lift of π (and one would hope to check the key relation (4.1) using local-global compatibility for $r_\iota(\Pi)$). Since automorphy lifting theorems give a way to prove Galois representations are automorphic, this is a new way approach to the problem of functoriality, which will be successful provided we can construct sufficiently many congruences. The main question is then how to force the existence of these congruences.

Our strategy to prove Theorem 4.8 goes back to [CT14], and is by induction on m (taking the cases $m \leq 4$ to be known, as we may). Suppose that $m \geq 5$, and let p be a prime number such that $p < m+1 < 2p$. Such a prime always exists, by Bertrand's postulate.[6] The choice of this prime forces the existence of congruences between symmetric power Galois representations in characteristic p.

Indeed, the representation Sym^m of GL_2, although irreducible in characteristic 0, becomes irreducible in characteristic p when $m+1 > p$. If $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ is an isomorphism and $\overline{r}_\iota(\pi)$ has large enough image[7] (as will generically be the case), then there is an isomorphism

$$(\mathrm{Sym}^m \overline{r}_\iota(\pi))^{ss} \cong (\det \overline{r}_\iota(\pi)^r \otimes \mathrm{Sym}^{p-r-1} \overline{r}_\iota(\pi)) \oplus ({}^{\varphi_p}\overline{r}_\iota(\pi) \otimes \mathrm{Sym}^{r-1} \overline{r}_\iota(\pi)), \tag{4.2}$$

where the summands on the right-hand side are irreducible, and we write $m+1 = p+r$ for some $0 < r < p$, and $\varphi_p \in G_{\mathbf{F}_p}$ for the arithmetic Frobenius, which is acting here on the coefficients of $\overline{r}_\iota(\pi)$. In [Tho15, ANT20] an automorphy lifting theorem that applies to residually reducible Galois representations is proved that could potentially be used to deduce the automorphy of $\mathrm{Sym}^m \overline{r}_\iota(\pi)$, provided one can overcome the following hurdles:

- First, verify the residual automorphy of the two summands on the right-hand side. We have $p-r-1 < m$, so the factor $(\det \overline{r}_\iota(\pi)^r \otimes \mathrm{Sym}^{p-r-1} \overline{r}_\iota(\pi))$ is, by induction, the residual representation of a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_{p-r}(\mathbf{A}_F)$. The same logic applies to $\mathrm{Sym}^{r-1} \overline{r}_\iota(\pi)$. The hard part therefore is to establish the residual automorphy of the tensor product representation.
- Second, bridge the gap to the hypotheses of the main automorphy lifting theorem of [ANT20]. This theorem requires in particular that we verify the residual automorphy of $\mathrm{Sym}^m \overline{r}_\iota(\pi)$ by producing a regular algebraic automorphic representation Π of $\mathrm{GL}_{m+1}(\mathbf{A}_F)$ which is cuspidal (and which even has a local component which is square-integrable).

In [CT14] we were able to deal with these points only under two big assumptions, namely the existence of tensor product Langlands functoriality (for the group $\mathrm{GL}_2 \times \mathrm{GL}_r$, taking care of the first point) and the existence of suitable level-raising congruences (allowing us to construct a *cuspidal* Π, taking care of the second point).

In [NT22], we prove unconditional results by proving enough in the direction of these assumptions. Let us take the second point first: the required level-raising congruences are constructed in [Tho22a] using a study of the geometry of Shimura varieties, a possibility that, at some level, goes back to the work of Ribet [Rib84, Rib90]. We have to take this route, instead of using a result like the powerful Theorem 2.7 (generalised e.g. in [BLGGT14]), because those results require us to be in a situation where we can apply automorphy lifting theorems – which we are not yet in a position to assume.

[6]Curiously, Bertrand's postulate also features in Khare–Wintenberger's proof of Serre's conjecture!

[7]Here 'large' can be taken to mean e.g. 'contains a conjugate of $\mathrm{SL}_2(\mathbf{F}_{p^2})$'.

The key to tackling the first point is ultimately again the construction of sufficiently many congruences. We first choose a congruence $\pi \equiv \pi' \bmod p$ so that there is an isomorphism

$$^{\varphi_p}\overline{r}_\iota(\pi) \otimes \operatorname{Sym}^{r-1} \overline{r}_\iota(\pi) \cong {}^{\varphi_p}\overline{r}_\iota(\pi) \otimes \operatorname{Sym}^{r-1} \overline{r}_\iota(\pi')$$

and such that the p-adic representation $^{\varphi_p} r_\iota(\pi) \otimes \operatorname{Sym}^{r-1} r_\iota(\pi')$ is Hodge–Tate regular (for any choice of lift of φ_p to an element of $G_{\mathbf{Q}_p}$). We then show that this p-adic tensor product representation is automorphic.

We in fact show something stronger. To formulate the stronger statement, we first recall that the group $\operatorname{Aut}(\mathbf{C})$ acts on the set of regular algebraic cuspidal automorphic representations of $\mathrm{GL}_n(\mathbf{A}_F)$. Indeed, given such an automorphic representation Π, and an element $\sigma \in \operatorname{Aut}(\mathbf{C})$, there is a unique representation $^\sigma\Pi$ with the property that $(^\sigma\Pi)^\infty \cong \Pi^\infty \otimes_{\mathbf{C},\sigma^{-1}} \mathbf{C}$ as $\mathbf{C}[\mathrm{GL}_n(\mathbf{A}_F^\infty)]$-modules. This statement is proved in [Clo90] as a consequence of the existence of the rational structure on Π^∞ afforded by the singular cohomology of the locally symmetric spaces associated to $\mathrm{GL}_n(F)$. The proof shows in particular that the representation Π^∞ may be defined over its field of definition K_Π, and thus that $^\sigma\Pi$ only depends on the restriction of σ to K_Π.

It follows from the definitions that there is an isomorphism $^{\varphi_p} r_\iota(\pi) \cong r_\iota(^{\iota\varphi_p\iota^{-1}}\pi)$. The stronger statement that we will prove is that for any $\sigma \in \operatorname{Aut}(\mathbf{C})$, the tensor product

$$r_\iota(^\sigma\pi) \otimes \operatorname{Sym}^{r-1} r_\iota(\pi')$$

is automorphic, or equivalently, that every member of the compatible system

$$(r_\jmath(^\sigma\pi) \otimes \operatorname{Sym}^{r-1} r_\jmath(\pi'))_\jmath \tag{4.3}$$

(where $\jmath$ runs over isomorphisms $\jmath : \overline{\mathbf{Q}}_l \to \mathbf{C}$ for varying primes l) is automorphic. The proof is a kind of induction argument. Let $\widetilde{K}_\pi$ denote the Galois closure of the number field $K_\pi \subset \mathbf{C}$. For any prime number l and place $v|l$ of $\widetilde{K}_\pi$, there is the associated (finite) inertia group $I_{v/l} \subset \operatorname{Gal}(\widetilde{K}_\pi/\mathbf{Q})$. These groups are non-trivial only for the finitely many primes v which are ramified over $\mathbf{Q}$. However, they generate $\operatorname{Gal}(\widetilde{K}_\pi/\mathbf{Q})$: their fixed field would be an everywhere unramified extension of $\mathbf{Q}$, and Minkowski's theorem implies that there is no such extension except $\mathbf{Q}$ itself. In particular, any $\sigma \in \operatorname{Gal}(\widetilde{K}_\pi/\mathbf{Q})$ admits an expression as $\sigma = \delta_1 \dots \delta_s$ for some primes $l_1, \dots, l_s$, places $v_i | l_i$ of $\widetilde{K}_\pi$, and elements $\delta_i \in I_{v_i/l_i}$. We prove the automorphy of the compatible system (4.3) by induction on s.

The base case $s = 0$ occurs when σ is the identity. We prove the automorphy of $r_\iota(\pi) \otimes \operatorname{Sym}^{r-1} r_\iota(\pi')$ by observing that there is an isomorphism of residual representations

$$\begin{aligned}\left(\overline{r}_\iota(\pi) \otimes \operatorname{Sym}^{r-1} \overline{r}_\iota(\pi')\right)^{ss} &\cong \left(\overline{r}_\iota(\pi) \otimes \operatorname{Sym}^{r-1} \overline{r}_\iota(\pi)\right)^{ss} \\ &\cong (\det \overline{r}_\iota(\pi) \otimes \operatorname{Sym}^{r-2} \overline{r}_\iota(\pi)) \oplus \operatorname{Sym}^r \overline{r}_\iota(\pi).\end{aligned} \tag{4.4}$$

The origin of this isomorphism is similar to (4.2), in that both come from reducibilities of representations of GL_2 as an algebraic group. They differ in that (4.2) arises from a reducibility that occurs only in characteristic p, while (4.4) arises from a reducibility in characteristic 0. One might worry that the p-adic representation $r_\iota(\pi) \otimes \operatorname{Sym}^{r-1} r_\iota(\pi')$ is reducible, but happily this tensor product is forced to be irreducible provided the Hodge–Tate weights of π, π' are chosen correctly (see [NT22, Lemma 3.2]). Taking this all in hand, we verify the residual automorphy of the summands of (4.4) by induction (as $r < m$) and get the automorphy of the p-adic representation using the automorphy lifting theorem proved in [ANT20].

Before describing the induction step, let us consider the case $s = 1$, where $\sigma \in I_{v/l}$ for some l-adic place v of $\widetilde{K}_\pi$. In this case, we choose $\jmath : \overline{\mathbf{Q}}_l \to \mathbf{C}$ to be an isomorphism such that $\jmath^{-1}|_{\widetilde{K}_\pi}$ induces the place v. Then there is an isomorphism of residual representations

$$\overline{r}_\jmath({}^\sigma\pi) \cong \overline{r}_\jmath(\pi)$$

(because σ, being an element of the inertia group, acts trivially on the residue field $k(v)$), hence

$$\overline{r}_\jmath({}^\sigma\pi) \otimes \mathrm{Sym}^{r-1}\, \overline{r}_\jmath(\pi') \cong \overline{r}_\jmath(\pi) \otimes \mathrm{Sym}^{r-1}\, \overline{r}_\jmath(\pi').$$

At this point we know (by the case $s = 0$) that the right-hand side is residually automorphic, and we would like to use an automorphy lifting theorem to deduce that $r_\jmath({}^\sigma\pi) \otimes \mathrm{Sym}^{r-1}\, r_\jmath(\pi')$ is automorphic. This looks difficult: it seems impossible to control the set of primes l that ramify in K_π, and we may well have to deal with $l = 2$ (a case which is often particularly troublesome when trying to prove automorphy lifting theorems). We are able to circumvent this difficult by proving what we call a 'functoriality lifting theorem', that is specially adapted to this question. We first proved such a theorem for symmetric powers of 2-dimensional representations in [NT21b]; here we prove a functoriality lifting theorem for tensor products with 2-dimensional representations. The key simplification is that we need control the Galois deformation theory only of the 2-dimensional factor, and this is now reasonably well-understood (even in the case $l = 2$), courtesy of the works of Khare–Wintenberger [KW09c] and Kisin [Kis09b] needed to complete the proof of Serre's conjecture.

It remains to describe the general case. We repeat the argument of the case $s = 1$, linking the compatible systems

$$(r_\jmath(\pi) \otimes \mathrm{Sym}^{r-1}\, r_\jmath(\pi'))_\jmath \text{ and } (r_\jmath({}^\sigma\pi) \otimes \mathrm{Sym}^{r-1}\, r_\jmath(\pi'))_\jmath$$

by a chain of s congruences, the i^{th} congruence taking place in characteristic l_i. Finally, taking $\sigma = \iota\varphi_p\iota^{-1}$ completes the proof.

4.9. **Outlook.** As these examples show, the construction of congruences between modular forms (and, dually, the linking of compatible systems by isomorphisms of residual representations) is an effective tool to establish the modularity of Galois representations. New automorphy lifting theorems make it possible to exploit 'degenerate' congruences, such as those forced to exist by the reducibility of the symmetric powers of the standard representation of GL_2 in positive characteristic. This strategy is particularly effective when combined with mathematical induction.

However, there are important open questions which are not obviously within the scope of these techniques. We end this article by mentioning the analogue of Serre's Conjecture 4.5 over a totally real field F, which might be phrased as follows:

Conjecture 4.10. *Let F be a totally real field, let p be a prime, and let $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ be a continuous, irreducible representation which is totally odd, in the sense that for each place $v|\infty$ of F with associated complex conjugation $c_v \in G_F$, we have $\det \overline{\rho}(c_v) = -1$ (again, we say that such a representation is of S-type). Then $\overline{\rho}$ is modular: there exists a regular algebraic, cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbf{A}_F)$ and an isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ such that $\overline{\rho} \cong \overline{r}_\iota(\pi)$.*

More precisely, this generalises the first part of Conjecture 4.5 – the analogue of the second part is already relatively well understood. The difficulty in establishing this conjecture by induction is that we no longer know how to deal with the base case! Indeed, when $F = \mathbf{Q}$ the base case was given by the non-existence of S-type representations of tame level 1 when $p = 2$. However, over a general base field F, S-type representations of tame level 1 in characteristic 2 certainly can exist (see [Dem09] for an example). Further progress will require new ideas!

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