The following questions are intended as sample exam questions for the online open book exam.

1. (a) Let A be a complete DVR, and let $\pi \in A$ be a uniformizer. Let $X \subset A$ be a set of representatives for the residue field $k = A/(\pi)$ with $0 \in X$. Show that each element $x \in A$ admits a unique *π-adic expansion* $x = \sum_{i=0}^{\infty} a_i \pi^i$ with coefficients $a_i \in X$.

   (b) Now let $A = \mathbb{Z}_p$, $\pi = p$, and $X = \{0, 1, \ldots, p-1\}$, where p is a prime number. Show that if the p-adic expansion of an element $x \in \mathbb{Z}_p$ is eventually periodic, then x is rational.

   (c) Now let $A = \mathbb{Z}_p[\sqrt{p}]$, $\pi = \sqrt{p}$, $X = \{0, 1, \ldots, p-1\}$. Show that if the π-adic expansion of an element $x \in A$ is eventually periodic, then $x \in \mathbb{Q}(\sqrt{p})$.

2. (a) Let $K/\mathbb{Q}_p$ be a finite extension, and let $f(X) \in K[X]$ be a monic polynomial. Define the *Newyon polygon* $N(f)$. State a theorem relating the slopes of the Newton polygon to the factorisation of $f(X)$ in $K[X]$.

   (b) Now let $f(X) \in \mathbb{Q}[X]$ be a monic irreducible polynomial, let $K = \mathbb{Q}(\alpha)$, where α is a root of $f(X)$, and let p be a prime number. State a theorem relating the factorisation of $f(X)$ in $\mathbb{Q}_p[X]$ to the factorisation of the ideal $p\mathcal{O}_K$ as a product of prime ideals of $\mathcal{O}_K$.

   (c) Now let $f(X) = X^4 + 6X^2 - 48 \in \mathbb{Q}[X]$, and let $L/\mathbb{Q}$ denote the splitting field of $f(X)$. Show that $f(X)$ is irreducible and describe $\mathrm{Gal}(L/\mathbb{Q})$ as a subgroup of S_4.

3. (a) Define what it means for a positive definite binary quadratic form to be *reduced*. Prove that if K is an imaginary quadratic field, then there is a bijection between the ideal class group of K and the set of reduced binary quadratic forms of discriminant disc $\mathcal{O}_K$.

   (b) Let $K = \mathbb{Q}(\sqrt{-6})$. Prove that the Hilbert class field of K is $K(\sqrt{2})$.

The following questions may be harder.

4. Use Hensel's lemma to calculate the number of roots of each polynomial in the corresponding field : X^3+1 in $\mathbb{Q}_7$, X^2+2X+4 in $\mathbb{Q}_2$, $3X^3+X+3$ in $\mathbb{Q}_3$.

5. Let $K/\mathbb{Q}_p$ be a finite extension, and let L/K be a Galois totally ramified extension. Let $G = \mathrm{Gal}(L/K)$, and let π_L be a uniformizer.

(a) Recall that we have defined injective maps $G_0/G_1 \to k_L^\times$ and for each $i \geq 1$, $G_i/G_{i+1} \to k_L$. Show that the homomorphism $G_0/G_1 \to k_L^\times$ may be given by the formula

$$\theta_0(s) = s(\pi_L)/\pi_L \bmod (\pi_L),$$

and is independent of the choice of π_L.

(b) Show that the homomorphism $G_i/G_{i+1} \to k_L$ may be given by the formula $s \mapsto (s(\pi_L)/\pi_L - 1)/\pi_L^i$, and does depend on the choice of uniformizer π_L. Show however that the homomorphism $\theta_i : G_i/G_{i+1} \to (\pi_L^i)/(\pi_L^{i+1})$,

$$\theta_i(s) = s(\pi_L)/\pi_L - 1 \bmod (\pi_L^{i+1}),$$

is independent of the choice of π_L.

(c) Show that if $s \in G_0$ and $t \in G_i$, then $\theta_i(sts^{-1}) = \theta_0(s)^i \theta_i(t)$.

6. Let $n \geq 1$ be an integer, $L = \mathbb{Q}(e^{2\pi i/n})$, $K = L \cap \mathbb{R}$. Let $h_L = \#H_L$, $h_K = \#H_K$. Show that $h_K | h_L$. *[Hint: Use class field theory.]*

7. Let $K = \mathbb{Q}(i)$, $L = K(\sqrt[4]{2})$. Then L/K is an abelian extension of degree 4. Calculate the conductor $\mathfrak{m}_{L/K}$, and describe explicitly the kernel of the map $H(\mathfrak{m}_{L/K}) \to \mathrm{Gal}(L/K)$.