Proof of Gauss' Theorema Egregium

Let $\sigma : U \to \mathbb{R}^3$ define a parametrized surface S. If $\mathbf{p} \in U$, we write $P = \sigma(\mathbf{p})$ for its image in S. The vectors $\sigma_x = d\sigma_{\mathbf{p}}(1,0)$ and $\sigma_y = d\sigma_{\mathbf{p}}(0,1)$ span the tangent space to $T_P S$, and the map $\mathbf{n} : U \to \mathbb{R}^3$ defined by

$$\mathbf{n} = \frac{\sigma_x \times \sigma_y}{|\sigma_x \times \sigma_y|}$$

is the unit normal to S at P.

Differentiating the relations $\mathbf{n} \cdot \mathbf{n} = 0$,we see that

$$2\mathbf{n}_x \cdot \mathbf{n} = 0 \qquad 2\mathbf{n}_y \cdot \mathbf{n} = 0.$$

The image of $d\mathbf{n}_p$ is spanned by $\mathbf{n}_x$ and $\mathbf{n}_y$. The equations above show that it is contained in $\mathbf{n}(\mathbf{p})^\perp = T_P S$, so we have maps

$$d\sigma_{\mathbf{p}} : \mathbb{R}^2 \to T_P S$$
$$d\mathbf{n}_{\mathbf{p}} : \mathbb{R}^2 \to T_P S$$

We consider the linear map $\Phi_P := d\mathbf{n}_{\mathbf{p}} \circ (d\sigma_{\mathbf{p}})^{-1} : T_P S \to T_P S$. Φ_P has the important property that it is *independent* of the parametrization of S. More precisely, if $\widetilde{\sigma} : U' \to S$ is another parametrization of S, then we can write $\widetilde{\sigma} = \sigma \circ \phi$, where $\phi : U' \to U$ is a diffeomorphism. We compute

$$d\widetilde{\mathbf{n}} \circ (d\widetilde{\sigma})^{-1} = (d\mathbf{n} \circ d\phi) \circ (d\sigma \circ d\phi)^{-1} = d\mathbf{n} \circ d\phi \circ d\phi^{-1} \circ d\sigma^{-1} = d\mathbf{n} \circ d\sigma^{-1}$$

and see that we get the same map whether we use σ or $\widetilde{\sigma}$.

Recall that the first and second fundamental forms are given by

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \sigma_x \cdot \sigma_x & \sigma_x \cdot \sigma_y \\ \sigma_y \cdot \sigma_x & \sigma_y \cdot \sigma_y \end{pmatrix} \qquad \begin{pmatrix} L & M \\ M & N \end{pmatrix} = -\begin{pmatrix} \sigma_x \cdot \mathbf{n}_x & \sigma_x \cdot \mathbf{n}_y \\ \sigma_y \cdot \mathbf{n}_x & \sigma_y \cdot \mathbf{n}_y \end{pmatrix}.$$

Lemma. *With respect to the basis $\langle \sigma_x, \sigma_y \rangle$ of $T_P S$, Φ_P is given by the matrix*

$$-\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

Proof. $\Phi_P(\sigma_x) = d\mathbf{n}_{\mathbf{p}}(1,0) = \mathbf{n}_x(\mathbf{p})$. Similarly, $\Phi_P(\sigma_y) = \mathbf{n}_y(\mathbf{p})$. Thus we must express $\mathbf{n}_x$ and $\mathbf{n}_y$ in terms of σ_x and σ_y. If

$$\mathbf{n}_x = a\sigma_x + b\sigma_y \qquad \mathbf{n}_y = c\sigma_x + d\sigma_y$$

we see that

$$-L = \sigma_x \cdot \mathbf{n}_x = a\sigma_x \cdot \sigma_x + b\sigma_x \cdot \sigma_y = aE + bF$$

etc. so that

$$-\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

□

Thus the *Theorema Egregium* takes the form

Theorem. *If g is the metric induced on U by σ, the Gauss curvature of g is given by*

$$K_{\mathbf{p}}(g) = \det \Phi_P = \frac{LN - M^2}{EG - F^2}.$$

Choose a map $\phi : U' \to U$ which gives geodesic polar coordinates for g near $\mathbf{p}$, so that the induced metric is $\overline{g} = dx^2 + \overline{G}dy^2$. ϕ is an isometry between $\overline{g}$ and g, so $K_{\mathbf{q}}(\overline{g}) = K_{\phi(\mathbf{q})}(g)$. On the other hand we saw above that $\det \Phi$ is unchanged if we replace σ by $\sigma \circ \phi$. Thus to prove the theorem, it suffices to show the relation $K_{\mathbf{p}}(g) = \det \Phi(g)$ in the special case when $g = dx^2 + Gdy^2$. From now on, we assume this is the case. For notational simplicity, we write $A = \sqrt{G}$.

For each $\mathbf{p} \in U$, let

$$\mathbf{e}_1(\mathbf{p}) = \sigma_x(\mathbf{p}) \quad \mathbf{e_2}(\mathbf{p}) = \sigma_y(\mathbf{p})/A \quad \mathbf{e_3}(\mathbf{p}) = \mathbf{n}(\mathbf{p}).$$

Then $\mathbf{e}_1, \mathbf{e_2}, \mathbf{e_3}$ form an orthonormal basis of $\mathbb{R}^3$. We express the partial derivatives $\mathbf{e}_{ix}$ and $\mathbf{e}_{iy}$ in terms of this basis. (This technique is known as the *method of moving frames.*)

Lemma.

$$\begin{pmatrix} \mathbf{e}_{1x} \\ \mathbf{e}_{2x} \\ \mathbf{e}_{3x} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -a \\ 0 & 0 & -bA \\ a & bA & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e_2} \\ \mathbf{e_3} \end{pmatrix} \qquad \begin{pmatrix} \mathbf{e}_{1y} \\ \mathbf{e}_{2y} \\ \mathbf{e}_{3y} \end{pmatrix} = \begin{pmatrix} 0 & A_x & -c \\ -A_x & 0 & -dA \\ c & dA & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e_2} \\ \mathbf{e_3} \end{pmatrix}$$

Proof. Since the $\mathbf{e}_i$ are orthonormal, any vector $\mathbf{v}$ can be written as $\mathbf{v} = \sum(\mathbf{v} \cdot \mathbf{e}_i)\mathbf{e}_i$. Differentiating the relation $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$, we see that

$$\mathbf{e}_{ix} \cdot \mathbf{e}_j + \mathbf{e}_{jx} \cdot \mathbf{e}_i = 0.$$

Moreover

$$\mathbf{e}_{3x} = \mathbf{n}_x = a\sigma_x + b\sigma_y = a\mathbf{e}_1 + bA\mathbf{e}_2.$$

Similarly $\mathbf{e}_{3y} = c\mathbf{e}_1 + dA\mathbf{e}_2$, so we can write

$$\begin{pmatrix} \mathbf{e}_{1x} \\ \mathbf{e}_{2x} \\ \mathbf{e}_{3x} \end{pmatrix} = \begin{pmatrix} 0 & \alpha & -a \\ -\alpha & 0 & -bA \\ a & bA & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e_2} \\ \mathbf{e_3} \end{pmatrix} \qquad \begin{pmatrix} \mathbf{e}_{1y} \\ \mathbf{e}_{2y} \\ \mathbf{e}_{3y} \end{pmatrix} = \begin{pmatrix} 0 & \beta & -c \\ -\beta & 0 & -dA \\ c & dA & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e_2} \\ \mathbf{e_3} \end{pmatrix}$$

To evaluate α, we use the equality of mixed partials. Specifically, $\alpha = \mathbf{e}_{1x} \cdot \mathbf{e}_2 = (\sigma_{xx} \cdot \sigma_y)/A$, and

$$\begin{aligned} \sigma_{xx} \cdot \sigma_y &= (\sigma_x \cdot \sigma_y)_x - \frac{1}{2}(\sigma_x \cdot \sigma_x)_y \\ &= E_x - F_x/2 = 0. \end{aligned}$$

Thus $\alpha = 0$. Similarly $\beta = (\sigma_{xy} \cdot \sigma_y)/A$, and

$$\sigma_{xy} \cdot \sigma_y = \frac{1}{2}(\sigma_y \cdot \sigma_y)_x = G_x/2.$$

Thus $\beta = G_x/(2A) = G_x/(2\sqrt{G}) = (\sqrt{G})_x$. □

We can now compute

$$\begin{aligned} \mathbf{e}_{1x} \cdot \mathbf{e}_{2y} &= (-a\mathbf{e}_3) \cdot (-A_x\mathbf{e}_1 - dA\mathbf{e}_3) = adA \\ \mathbf{e}_{2x} \cdot \mathbf{e}_{1y} &= (-bA\mathbf{e}_2) \cdot (A_x\mathbf{e}_1 - c\mathbf{e}_2) = bcA \end{aligned}$$

so $\mathbf{e}_{1x} \cdot \mathbf{e}_{2y} - \mathbf{e}_{2x} \cdot \mathbf{e}_{1y} = A(ad - bc) = A \det \Phi$.

On the other hand,

$$\begin{aligned} \mathbf{e}_{1x} \cdot \mathbf{e}_{2y} - \mathbf{e}_{2x} \cdot \mathbf{e}_{1y} &= (\mathbf{e}_1 \cdot \mathbf{e}_{2y})_x - (\mathbf{e}_1 \cdot \mathbf{e}_{2x})_y \\ &= (-A_x)_x - (0)_y \\ &= -A_{xx} \end{aligned}$$

Thus $\det \Phi = -A_{xx}/A = -(\sqrt{G})_{xx}/\sqrt{G}$, which is the Gauss curvature of the metric g. This completes the proof of the theorem. □