Example Sheet 1

1. Compute (by hand) the Jones polynomial of a) $T(2,n)$, b) the figure 8 knot, c) $T(3,4)$. Deduce that $T(3,4)$ is not alternating.

2. For any $1+1$ dimensional TQFT $\mathcal{A}$ over $\mathbb{Z}$ we can build a chain complex by applying $\mathcal{A}$ to the cube of resolutions of D. If the resulting homology is a knot invariant, show that $\mathcal{A}(S^1)$ must have rank 2.

3. If s is a segment of D, show that $CKh(D)/CKh^r(D,s) \simeq CKh^r(D,s)$.

4. Show that we can assign signs to the edges of the cube $[0,1]^n$ so that every two dimensional face has an odd number of $-$ signs. Show that any two such assigments give to isomorphic chain complexes. (Hint: Consider $C^*_{cell}([0,1]^n;\mathbb{Z}/2)$.)

5. If L is a link in $\mathbb{R}^3$, the *mirror* of L (written $\overline{L}$) is the image of L under an orientation reversing homeomorphism $\mathbb{R}^3 \to \mathbb{R}^3$. If D is a diagram for L, the diagram $\overline{D}$ obtained by changing all overcrossings to undercrossings is a diagram for $\overline{L}$. If L_1 and L_2 are links, their *disjoint union* $L_1 \coprod L_2$ is obtained by embedding them in disjoint balls in $\mathbb{R}^3$. Prove the following relations:

   (a) $V(\overline{L})(q) = V(L)(q^{-1})$.
   (b) $\widetilde{V}(L_1 \coprod L_2) = \widetilde{V}(L_1)\widetilde{V}(L_2)$.
   (c) $V(K_1 \# K_2) = V(K_1)V(K_2)$.

   Categorify each statement.

6. Show that the Jones polynomial satisfies an oriented skein relation:
$$q^2V(D_-) - q^{-2}V(D_+) = (q - q^{-1})V(D_o)$$
   where D_+, D_- and D_o are any three diagrams which are the same away from a single crossing, where they have a positive crossing, and negative crossing, and the oriented resolution, respectively. Deduce that $\widetilde{V}(L)|_{q=1} = 2^{|L|}$, where $|L|$ is the number of components of L. Can you categorify the skein relation?

7. A knot K is *amphichiral* if $K = \overline{K}$. Show that the figure-8 knot is amphichiral, but the trefoil is not. Prove that the crossing number of an alternating amphichiral knot is even. Using a table of Jones polynomials (*e.g.* KnotInfo) find all prime amphichiral knots with at most 8 crossings.

8. If D is a reduced alternating diagram, show that $M(D) + m(D)$ is determined by the writhe $w(D)$. Deduce that any two reduced alternating diagrams of the same knot have the same writhe. This is the second *Tait Conjecture*. (For much of the 20th century, people thought this statement should hold for any two minimal crossing diagrams of the same knot. In the 70's a lawyer by the name of Ken Perko showed that two 10-crossing knots in the knot tables of the day, which had been assumed to be different because they had different writhes, were actually isotopic.)

9. If D is an alternating diagram, show that $|V(D)|_{q=-1}|$ is equal to the number of maximal trees in $B(D)$, which is equal to the number of maximal trees in $W(D)$. If D is any diagram of L, show that the rank of $Kh^r(L, L_i)$ is bounded above by the number of maximal trees in $B(D)$.

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