# EXAMPLE SHEET 4

1. Find the dual Thurston polytopes of the complements of the links shown in Figure 1. Let $Y = S_i - \nu(L)$, and choose $S_i \in H_2(Y, \partial Y)$ with $S_i \cdot m_j = \delta_{ij}$. Sketch norm-minimizing surfaces representing $S_1, S_1 + S_2$,and $S_1 - S_2$ (for a) and $S_1, S_1 + S_2$, and $S_1 + S_2 - S_3$ (for b).

2. Let $L$ be a 2-component link in $S^3$ with linking number $n$, and let $Y = S^3 - \nu(L)$. Define $\phi : H_1(Y) \to (\mathbb{Z}/2)^2$ to be the homomorphism that sends the meridians of $L$ to $(1, 0)$ and $(0, 1) \in (\mathbb{Z}/2)^2$. Show if $v$ is a vertex of the dual Thurston ball $B_T(Y)$, then $\phi(v) = (1, 1)$ if $n$ is odd, and $\phi(v) = (0, 0)$ if $v$ is even. (Hint: If $S$ is a surface, the parity of $\chi(S)$ is determined by the number of boundary components.) Can you generalize to links with more than 2 components?

3. Let $L$ and $Y$ be as above, and assume $n \neq 0$. If $Y'$ is the result of $p/q$ Dehn surgery on one of the components of $L$, show that there is an injection $i : H_2(Y', \partial Y') \to H_2(Y, \partial Y)$. Give a lower bound for the Thurston norm of a class $x \in H_2(Y', \partial Y')$ in terms of the Thurston norm of $i(x)$.

4. Let $Y$ and $Y'$ be as in the previous problem. How is the torsion $T(Y')$ related to $T(Y)$? How does the bound on the Thurston norm of $x \in H_2(Y', \partial Y')$ coming from $T(Y')$ compare with the bound in the previous problem? Show that if twice the Newton polygon of $\Delta(L)$ is equal to $B_T(Y)$ (as opposed to being a proper subset of it), then the bound in problem 2 is sharp for all but finitely many values of the filling slope $p/q$.

5. Suppose $Y$ is a Seifert-fibred 3-manifold with boundary. Show that $\partial Y$ is a union of tori. Let $f \in H_1(T^2)$ be the class of a fiber in a boundary component of $Y$, and suppose we Dehn fill this boundary component by attaching $S^1 \times D^2$ to create a new manifold $Y'$. Show that the Seifert fibration on $Y$ extends to a Seifert fibration on $Y'$ is unless $f \cdot [\partial D^2] = 0$. What is the multiplicity of the new singular fibre?

6. Let $T(p, q) \subset S^3$ be the $(p, q)$ torus knot. Show that $S^3 - T(p, q)$ is Seifert fibred. Deduce that there are infinitely many Dehn surgeries on $T(p, q)$ which give lens spaces. What are their filling slopes? (In contrast, if $K \subset S^3$ is a hyperbolic knot, $K$ has at most two nontrivial lens space surgeries.)

7. Suppose $p, q$ and $r$ are positive integers. Show that
$$\Sigma(p, q, r) = \{(x, y, z) \in \mathbb{C}^3 \mid |x|^2 + |y|^2 + |z|^2 = 1, x^p + y^q + z^r = 0\}$$
is Seifert fibred. What are the multiplicities of the singular fibres? Show that if $p, q$ and $r$ are pairwise relatively prime, then $H_*(\Sigma(p, q, r)) \simeq H_*(S^3)$.

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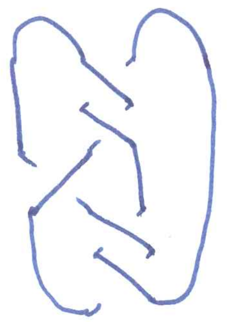
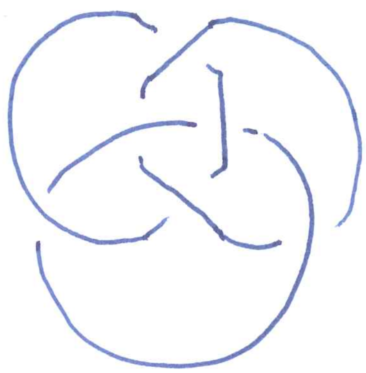
Figure 1

a)



b)



c)