3-MANIFOLDS MICHAELMAS 2013

EXAMPLE SHEET 2

1. Let K_1 and K_2 be the knots shown in Figure 1. Show that the result of -1 surgery on K_1 is the same as the result of $+1$ surgery on K_2. (Hint: consider the link at the bottom of the figure.)

2. (The intersection form) Suppose M is an oriented 4-manifold, and let $[M]$ be the generator of $H^4(M, \partial M)$ determined by the orientation. Define a map $q : H^2(M, \partial M) \times H^2(M, \partial M) \to \mathbb{Z}$ by $q(\alpha, \beta) = \langle \alpha \cup \beta, [M] \rangle$. Show that q is a symmetric bilinear form on $H^2(M, \partial M)$. It is called the intersection form. Use the intersection form to show that $\mathbb{CP}^2$ does not have an orientation reversing homeomorphism, and that $S^2 \times S^2$ is not homeomorphic to $\mathbb{CP}^2 \# (-\mathbb{CP}^2)$.

3. Let L be the link in $Y = S^1 \times D^2$ shown in Figure 2. Let Z be the manifold obtained by doing p surgery on L_1 and r surgery on L_2, where $p \in \mathbb{Z}$. Show that $Z \simeq S^1 \times D^2$. What is the class in $H_1(\partial Y)$ that bounds in Z? Deduce that if K is a knot in a 3-manifold and m is a meridian of K, then p surgery on K followed by r surgery on m is homeomorphic to $p - \frac{1}{r}$ surgery on K.

4. Let Y be obtained by surgery on a chain of unknots in S^3, as shown in Figure 3. Use the previous problem to show that Y is a lens space. Use this to give another proof of the fact that $L(p, q) \simeq L(p, q')$, where $qq' \equiv 1$ (p). Show also that any lens space bounds a four-manifold on which the intersection form is negative definite.

5. Suppose Y is a closed orientable 3-manifold with $H_1(Y) = \mathbb{Z}$. Show that there is a 4-manifold M with $\partial M = Y$ and so that the induced map $H_1(Y) \to H_1(M)$ is an isomorphism. If M is a 4-manifold with $\partial M = T^3$, use the cup product to show that the induced map $H_1(T^3) \to H_1(M)$ cannot be injective.

6. (The linking form)

   (a) Let $i^* : H^*(M, \partial M) \to H^*(M)$ be the natural map. If $i^*(\alpha_1) = i^*(\alpha_2)$, show that $q(\alpha_1, \beta) = q(\alpha_2, \beta)$ for all $\beta \in H^2(M, \partial M)$.

   (b) Now suppose Y is a closed connected oriented three-manifold with $|H_1(Y)| = n < \infty$, and choose a simply connected M with $Y = \partial M$ (as oriented manifolds). Show that the induced map $j^* : H^2(M) \to H^2(Y)$ is surjective.

   (c) Given $a \in H^2(Y)$, choose $a_1 \in H^2(M)$ with $j^*(a_1) = a$, and $a_2 \in H^2(M, \partial M)$ with $i^*(a_2) = na_1$. Define the *linking form* $L : H^2(Y) \times H^2(Y) \to \mathbb{R}/\mathbb{Z}$ by letting $L(a, b)$ be image of $q(a_2, b_2)/n^2$ in $\mathbb{R}/\mathbb{Z}$. By part a), L does not depend on the choice of a_2 and b_2. Show that L does not depend on the choice of M. (Hint: if M_1 and M_2 are two such manifolds, consider the intersection form on $M_1 \cup_Y (-M_2)$.)

7. Use the linking form to show that if K is a knot in S^3 and $p > 2$ is an integer, then K_p does not have an orientation reversing homeomorphism.

8. Suppose C is a chain complex defined over $\mathbb{Z}$, and that $C \otimes \mathbb{R}$ is acyclic. Show that the inclusion $C \to C \otimes \mathbb{R}$ defines a canonical choice of generator for $\Lambda^{top}(C \otimes \mathbb{R})$, and that the torsion with respect to this generator is $|H_{odd}(C)|/|H_{even}(C)|$.

9. Compute the multivariable Alexander polynomials of $S^1 \times \Sigma_g$ and of $S^3 - \nu(B)$, where $B \subset S^3$ is the Borromean rings.

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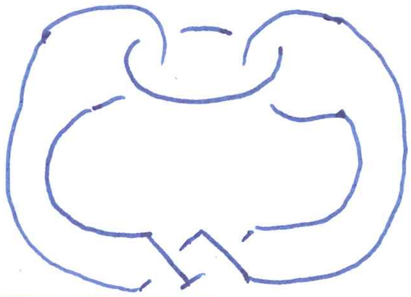
Figure 1



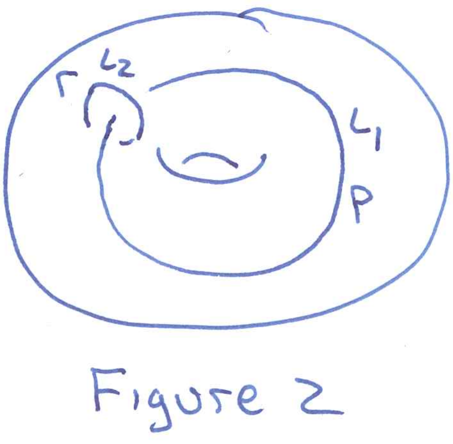


Figure 2

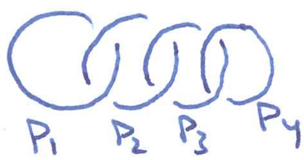


Figure 3