ALGEBRAIC TOPOLOGY (PART III) **MICHAELMAS 2017**

EXAMPLE SHEET 4

1. Suppose $\pi : E \to B$ is a real vector bundle and that $f : B' \to B$. Describe local trivializations and transition functions for the pullback $f^*(E)$ in terms of local trivializations and transition functions for E.

2. Let M be the Mobius bundle over S^1. Show that $M \oplus M$ is the trivial bundle.

3. Let $E \to B$ be a real vector bundle equipped with a Riemannian metric, and let $F \subset E$ be a subbundle. Show that $F^\perp$ is a vector bundle, and that $F \oplus F^\perp \cong E$. Deduce that if E is an orientable bundle which has a nowhere vanishing section, then $e(E) = 0$.

4. Let $E = TS^2$ be the tangent bundle of S^2. Show that the unit sphere bundle $S(E)$ is homeomorphic to $SO(3)$, which is also homeomorphic to $\mathbb{RP}^3$.

5. Let M be a closed orientable 4-manifold, and write $H_i(M) \simeq F_i \oplus T_i$, where F_i is free and T_i is torsion. Find all relations between F_i and F_j, T_i and T_j for differing values of i and j.

6. If M is a manifold with $H_1(M; \mathbb{Z}/2) = 0$, show that M is orientable.

7. If M is an orientable manifold of dimension $4n+2$, show that the dimension of $H_{2n+1}(M; \mathbb{Q})$ is even.

8. a) Show that there is no orientation reversing homeomorphism $f : \mathbb{CP}^2 \to \mathbb{CP}^2$.

   b) Let $\overline{\mathbb{CP}}^2$ denote $\mathbb{CP}^2$ with the opposite orientation, and define $X_1 = \mathbb{CP}^2 \# \mathbb{CP}^2$, $X_2 = \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$, and $X_3 = S^2 \times S^2$. Show that $H_*(X_1) \simeq H_*(X_2) \simeq H_*(X_3)$, but that no two of the X_i are homotopy equivalent.

9. If M is an oriented $4n$ dimensional manifold, the pairing $(a, b) = \langle a \cup b, [M] \rangle$ defines a symmetric bilinear form on $H^{2n}(M)$. If $H^*(M)$ is free over $\mathbb{Z}$, we can choose a basis of $H^{2n}(M)$ and write $(a, b) = a^T A b$ for some symmetric matrix A with entries in $\mathbb{Z}$. Show $\det A = \pm 1$.

10. If $p \in \mathbb{C}[z_0, z_1, z_2]$ is a homogenous polynomial, we define

$$V_p = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \mid p(z_0, z_1, z_2) = 0\}.$$

   If p and q are chosen such that V_p and V_q are embedded submanifolds of $\mathbb{CP}^2$ which intersect transversely, show that V_p and V_q intersect in precisely $(\deg p)(\deg q)$ points.

11. Let E be the tangent bundle of $\mathbb{CP}^n$. Compute $H_*(S(E))$.

12. Suppose that M is a compact odd-dimensional manifold with boundary. Show that $\chi(M) = \frac{1}{2}\chi(\partial M)$. Conclude that $\mathbb{RP}^2$ does not bound a compact 3-manifold. Does $\mathbb{RP}^3$ bound a compact 4-manifold?

13. Given $\gamma : S^1 \to SO(2)$, let $E_\gamma = D^2_a \times \mathbb{R}^2 \coprod D^2_b \times \mathbb{R}^2 / \sim$, where $\sim$ identifies $(x, v) \in S^1_a \times \mathbb{R}^2$ with $(x, \gamma(x) \cdot v)$. Show that E_γ is an oriented vector bundle over S^2 and compute its Euler class in terms of γ.

14. Construct a 3 dimensional real vector bundle over S^4 which has no nonvanishing section.

15. Suppose B is compact, and that E is a vector bundle over $B \times I$. Show that $E|_{B\times 0} \simeq E|_{B\times 1}$ as follows.

    (a) For each $b \in B$ show there is an open neighborhood V_b of b such that $E|_{V_b\times I}$ is trivial.

    (b) Choose a finite cover $\mathcal{V} = V_1, \ldots, V_m$ of B such that $E|_{V_b\times I}$ is trivial. If $\{\phi_i\}$ is a partition of unity subordinate to $\mathcal{V}$, let $\psi_j = \sum_{i=1}^{j} \phi_i$. Consider the map $f_i : B \to B \times I$ given by $f_i(b) = (b, \psi_i(b))$. Show that $f_i^*(E) \simeq f_{i+1}^*(E)$.

    Deduce that if E' is a vector bundle over B' and $f, g : B \to B'$ are homotopic, then $f^*(E') \simeq g^*(E')$.

16. Let M be an oriented manifold with the property that $H_*(M)$ is generated by embedded submanifolds. Given $f : M \to M$, let $\Lambda = \{(p, f(p)) \in M \times M\}$ be the graph of f. Λ is an embedded submanifold of $M \times M$.

    (a) If $\langle a_i \rangle$ is a basis of $H_i(M; k)$ and $\langle b_i \rangle$ is the dual basis under the cup product pairing, show that $PD(\Lambda) = \sum f^*(a_i) \times b_i$.

    (b) Show that $L(f) := \Lambda \cdot \Delta = \sum_{j=0}^{n} (-1)^j \operatorname{Tr} f_j^*$, where $f_j^* : H^j(M; k) \to H^j(M; k)$.

    (c) Deduce that if $L(f) \neq 0$, f must have a fixed point. (This is the *Lefshetz fixed point theorem.*)

    (d) Show that any map $f : \mathbb{CP}^2 \to \mathbb{CP}^2$ has a fixed point.

17. Given $\phi : S^{2n-1} \to S^n$, let $X_\phi = S^n \cup_\phi D^{2n}$. $H^*(X_\phi) = \langle 1, x_n, x_{2n} \rangle$, where $x_i \in H^i(X_\phi)$. Thus $x_n^2 = H(\phi) x_{2n}$ for some $H(\phi) \in \mathbb{Z}$.

    (a) Show that the map $[\phi] \to H(\phi)$ defines a homomorphism $H : \pi_{2n-1}(S^n) \to \mathbb{Z}$. H is known as the *Hopf invariant.*

    (b) Show that H is surjective for $n = 2, 4$.

    (c) By considering a cell decomposition of $S^n \times S^n$, show that H is nontrivial for every even n. Deduce that $\pi_{4m-1}(S^{2m})$ is infinite for all $m > 0$.

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