ALGEBRAIC TOPOLOGY (PART III) **MICHAELMAS 2017**

EXAMPLE SHEET 3

1. Suppose $f : X \to Y$. If $\alpha \in H^k(Y)$ and $x \in H_k(X)$, show that $\langle f^*(\alpha), x\rangle = \langle \alpha, f_*(x)\rangle$. Deduce that if $H_*(X)$ and $H_*(Y)$ are free over $\mathbb{Z}$, then $f_* : H_*(X) \to H_*(Y)$ and $f^* : H^*(X) \to H^*(Y)$ are dual maps.

2. Suppose $A \subset X$, and let $\partial : H_n(X,A) \to H_{n-1}(A)$ and $\partial^* : H^{n-1}(A) \to H^n(X,A)$ be the boundary maps in the long exact sequence of a pair. If $\alpha \in H^{n-1}(A), a \in H_n(X,A)$, show that $\langle \partial^*\alpha, a\rangle = \langle \alpha, \partial a\rangle$.

3. Compute $H_*(L^3(12,1) \times \mathbb{RP}^3)$ with coefficients in $\mathbb{Z}$, $\mathbb{Z}/2$, and $\mathbb{Z}/4$.

4. If X is a space, let $\Delta_X : X \to X \times X$ be the *diagonal map* given by $\Delta(x) = (x,x)$. Compute $\Delta_{S^2 *}([S^2]) \in H_2(S^2 \times S^2)$ and $\Delta_{T^2 *}([T^2]) \in H_2(T^2 \times T^2)$.

5. * Let G be a topological group (*i.e.* G is a group and a space, and the actions of multiplication and taking inverse are continuous.) Show there is a map $\Delta : H^*(G) \to H^*(G) \otimes H^*(G)$ satisfying the following properties:

   (a) $\Delta(a \cup b) = \Delta(a) \cup \Delta(b)$, where $(a_1 \otimes a_2) \cup (b_1 \otimes b_2) = (-1)^{|a_2||b_1|}(a_1 \cup b_1) \otimes (a_2 \cup b_2)$.

   (b) $\Delta(a) = a \otimes 1 + 1 \otimes a + \sum_i a_i' \otimes a_i''$, where $|a_i| < |a|$ for all i.

   Deduce that $S^1 \times S^{2n}$ cannot be given the structure of a topological group.

6. Let $U, V \subset X$ be open sets. If $x \in H^*(X,U)$ and $y \in H^*(X,V)$, show that $x \cup y \in H^*(X, U \cup V)$. Using this, show that if X has a covering by n contractible open subsets, then $a_1 \cup a_2 \cup \ldots \cup a_n = 0$ whenever $a_1, \ldots a_n \in H^*(X)$ have grading > 0.

7. Let Σ_g be the surface of genus g. Show that if $g < h$, any map $\Sigma_g \to \Sigma_h$ has degree 0.

8. Let $X_1 = \mathbb{CP}^2 \# \mathbb{CP}^2$, $X_2 = \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$, and $X_3 = S^2 \times S^2$. Show that $H_*(X_1) \simeq H_*(X_2) \simeq H_*(X_3)$, but that no two of the X_i are homotopy equivalent.

9. Given that $f : S^2 \times S^2 \times S^2 \to \mathbb{CP}^3$, what are the possible values of the degree of f?

10. Let M be the Mobius bundle over S^1. Show that $M \oplus M$ is the trivial bundle.

11. Let $E = TS^2$ be the tangent bundle of S^2. Show that the unit sphere bundle $S(E)$ is homeomorphic to $SO(3)$, which is also homeomorphic to $\mathbb{RP}^3$.

12. Identify $S^3 - 0$ with $\mathbb{R}^3$ by stereographic projection. Describe what the fibres of the Hopf fibration look like under this identification. Sketch three distinct fibres.

13. Let $E \to B$ be a real vector bundle equipped with a Riemannian metric, and let $F \subset E$ be a subbundle. Show that $F^\perp$ is a vector bundle, and that $F \oplus F^\perp \cong E$.

14. Let M be a smooth manifold, and let $\Delta \subset M \times M$ be the image of the diagonal embedding $M \to M \times M$ which sends x to (x, x). Show that $\nu_\Delta \simeq TM$.

15. Let M be a triangulated n-manifold. By considering $H_*(M, M-p)$ show that every cell of M has dimension $\leq n$, and that every $n-1$ dimensional face of M is in the boundary of precisely two n dimensional faces.

16. Give a definition of a triangulated n-manifold with boundary. If M is a connected triangulated n-manifold with boundary, show that $H_n(M, \partial M; R)$ is either 0 or R. In the latter case we say that M is R-orientable. If M is R-orientable, show that ∂M is R-orientable. An R-orientation on M is a class $[M, \partial M] \in H_n(M, \partial M)$ whose image in $H_n(M, M-p)$ is a generator of $H_n(M, M-p)$ for all $p \in M$. If $[M, \partial M]$ is an R-orientation on M, show that its image under ∂ is an R-orientation on $[\partial M]$, where ∂ is the boundary map in the long exact sequence of the pair $(M, \partial M)$.

J.Rasmussen@dpmms.cam.ac.uk