ALGEBRAIC TOPOLOGY (PART III) **MICHAELMAS 2017**

EXAMPLE SHEET 2

Questions to be Marked: 4,6,9

1. If $f_0, f_1 : S^{n-1} \to X$ and $f_0 \sim f_1$, show that $X \cup_{f_0} D^n \sim X \cup_{f_1} D^n$.
2. Consider the cell structure on S^n which has two cells of dimension k for each $0 \leq k \leq n$, corresponding to the northern and southern hemispheres of $S^k \subset S^n$. Write out its cellular chain complex and verify that it has the correct homology.
3. Let $X = S^1 \times D^2 / S^1 \times S^1$. Find a cell decomposition of X and compute its homology using the cellular chain complex.
4. Write $S^{2k-1} = \{(z_1, \ldots, z_k) \in \mathbb{C}^k \mid \sum |z_i|^2 = 1\}$ The group $\mathbb{Z}/p$ acts on S^{2k-1} by $a \cdot \mathbf{z} = \lambda^a \mathbf{z}$, where $\lambda = e^{2\pi i/p}$. The *lens space* $L^{2k-1}(p,1)$ is the quotient $S^{2k-1}/(\mathbb{Z}/p)$. Show that $L^{2k-1}(p,1)$ has a cell decomposition with one cell of each dimension between 0 and k. Find $C^{cell}_*(L^{2k-1}(p,1))$ and compute its homology.
5. Suppose $x \in H_n(X)$, where X is an arbitrary topological space. Show that there is a finite cell complex A and a map $f : A \to X$ so that $x \in \operatorname{im} f_*$.
6. Let X and Y be finite cell complexes. A map $f : X \to Y$ is cellular if $f(X_{(n)}) \subset Y_{(n)}$ for all n. Show that such an f induces a chain map $f^{cell} : C^{cell}_*(X) \to C^{cell}_*(Y)$, and that the induced map on homology agrees with f_*.
7. If C is a free finitely generated complex over R, where R is either $\mathbb{Z}$ or a field, show that $\chi(C) = \chi(H(C))$. If C is a finitely generated complex over $\mathbb{Z}$, show that $\chi(H(C \otimes \mathbb{Z}/p)) = \chi(H(C \otimes \mathbb{Z}/q))$ for all primes p, q.
8. Let $f : S^n \to S^n$ have degree k, and let $X = S^n \cup_f D^{n+1}$. Let $\pi : X \to X/S^n \simeq S^{n+1}$ be the quotient map. Compute $\pi^* : H^*(S^{n+1}) \to H^*(X)$ and $\pi_* : H_*(X) \to H_*(S^{n+1})$.
9. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian groups. Show there is a long exact sequence

$$\ldots \to H^n(X, A) \to H^n(X, B) \to H^n(X, C) \to H^{n+1}(X, A) \to \ldots$$

   The boundary map in this sequence is known as the Bockstein homomorphism. Compute this boundary map when $X = \mathbb{RP}^3$ and the short exact sequence of groups is $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2$. Do the same for $X = L^3(4,1)$.
10. Let C_* be a finitely generated chain complex defined over $\mathbb{Z}$. Show that $H_*(C) = 0$ if and only if $H_*(C \otimes \mathbb{Z}/p) = 0$ for every prime p. Now suppose $f : C_* \to C'_*$ is a

chain map. Using problem 5 on Example Sheet 1, show that f is a chain homotopy equivalence if and only if the map $f_* : H_*(C \otimes \mathbb{Z}/p) \to H_*(C' \otimes \mathbb{Z}/p)$ is an isomorphism for every prime p.

11. Let $R = \mathbb{C}[x]/(x^3)$, and for $i = 1, 2$, let M_i be the R-module $\mathbb{C}[x]/(x^i)$. Find a free resolution of M_1 and use it to compute $\mathrm{Tor}_*^R(M_1, M_1)$ and $\mathrm{Tor}_*^R(M_1, M_2)$.

12. Let X be a finite cell complex, and let A be a subcomplex of X. Let $Y = X \times \{0\} \cup A \times I \subset X \times I$. Show that any map $Y \to Z$ extends to a map $X \times I \to Z$. (Hint: start with the case $X = D^n$, $A = S^{n-1}$.)

13. * Let $f : X \to Y$ be a map of finite cell complexes, and suppose $\pi_i(Y, *) = 0$ for all $0 < i \leq n$. Using problem 10, show there is a map $\overline{f} : X/X_n \to Y$ so that $\overline{f} \circ \pi \sim f$, where $\pi : X \to X/X_n$ is the quotient map. Deduce the following consequences:

    (a) $H_i(Y) = 0$ for $0 < i \leq n$.

    (b) The Hurewicz homomorphism $\Phi : \pi_{n+1}(Y, *) \to H_{n+1}(Y)$ is surjective.

    (c) If $\pi_i(Y, *) = 0$ for all $i > 0$, then Y is contractible.

14. * This question assumes some knowledge of π_1.

    (a) Construct a connected finite cell complex X with $\widetilde{H}_*(X) = 0$ but which is not contractible.

    (b) Find a map $f : S^1 \vee S^2 \to S^1 \vee S^2$ which induces the identity map on $H_*(S^1 \vee S^2)$, but is not homotopic to the identity.

J.Rasmussen@dpmms.cam.ac.uk