CHARACTER SHEAVES ON SYMMETRIC SPACES

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Introduction

This thesis is concerned with the study of a class of K-equivariant perverse sheaves on the symmetric space $X = G/H$. In order to motivate their study, we briefly highlight some properties of Lusztig's theory of character sheaves on G.

Let G be a connected reductive group defined over a finite field F_q, $G(F_q)$ the finite group of its rational points. One can then ask, what is the character table of $G(F_q)$.

This problem is now almost completely solved (there only remains some ambiguity of multiplication by small roots of unity in some characters) as a result of work of Lusztig and Deligne-Lusztig.

One of the key ingredients of this is the theory of character sheaves. These are certain perverse sheaves on G, equivariant with respect to conjugation. Some subset of the character sheaves will be defined over F_q, and for these we can take trace of the Frobenius map F at the points of G^F to give class functions on G^F (a priori, these are only well defined up to homothety, but they can be normalised almost canonically). These *characteristic functions* of the character sheaves give a basis of the class functions on G^F, closely related to the basis of characters of G^F. The change of basis matrix is "almost" diagonal—there are very few off diagonal entries, and these are known explicitly.

These characteristic functions can be explicitly calculated at all points of G^F. This is an involved process, but the fact that it is possible is a consequence of certain geometric properties of the character sheaves—they are the intersection cohomology extension of local systems which come from finite coverings, and these extensions can be calculated in terms of the homology of varieties paved by affine vector spaces.

If $\theta\colon G \to G$ is an involutory automorphism, $K = G^\theta$, we can ask what are the characters of the algebra of double cosets $K^F\backslash G^F/K^F$. This includes the problem of finding the characters of the finite groups G^F as the special case $(G \times G, \theta)$, where $\theta(x, y) = (y, x)$.

In my thesis I study a class of K-equivariant perverse sheaves on the symmetric space $X = G/K$. These sheaves, also called character sheaves, have characteristic functions very closely related to the characters of the algebra $K^F\backslash G^F/K^F$ and so we obtain results about the character table of this algebra.

In the special case $(G \times G, \theta)$, these character sheaves are the same as those defined by Lusztig, but for a general symmetric space the geometric properties of these character sheaves are very different from those on a group. The least group-like symmetric spaces are the "split" ones. For these spaces, the character sheaves involve cohomology along a family of beautiful non-rational varieties—in sharp contrast to the group case, which essentially reduces to that of a point. For example, for SL_3/SO_3, a family of elliptic curves enters the picture.

As a consequence, there are no "elementary" formulas for the values of the characters of this algebra (in contrast with the group case, where we get sums of polynomials in q times roots of unity).

Despite this interesting new feature, many other things remain the same. For example, though the generic character sheaves do not occur as the intersection cohomology extension of a local system obtained from a finite covering, these local systems do have the same rank (if G/K is split), namely the order of the Weyl group. However the monodromy of such local systems is large, and gives representations of the affine Hecke algebra at $q = -1$.

Also these character sheaves are (conjecturally) classified in an analogous way to the group case. I have some results in this direction. In particular, I have a short proof of Lusztig's partition of character sheaves into cells, which avoids the case by case consideration of [L1].

We briefly describe the contents of this thesis, section by section.

Let θ, $\sigma : G \to G$ be two commuting involutions, $K = G^{\theta}$, $H = G^{\sigma}$. This data defines a real symmetric space: G is the complexification of a real group, θ is the Cartan involution, and σ the involution defining the symmetric space structure. We begin the study of certain $K \times H$-equivariant perverse sheaves on G, the character sheaves. If $K = H$, these were defined by Ginsburg [G], generalising the definition of [L1]. Our definition is the same as in [G], we merely observe certain diagrams are equivariant in this more general setup.

Section 1 is concerned with the definition of these character sheaves. In section 2 we introduce induction functors.

In section 3 we describe the partition of character sheaves by two sided cells in $K\backslash\mathcal{B}$, and conjecture a parametrisation of the character sheaves in a cell, along the lines of [L7]. It is perhaps surprising that the partition of character sheaves by two sided cells should be so easy to define.

We then begin a detailed study of generic character sheaves in section 4; this is the heart of this thesis. It is possible to give Lie algebra versions of all these results—one considers the class of perverse sheaves on the Lie algebra obtained from the equivariant perverse sheaves supported on a single orbit closure [L6]. The results are analogous to those of section 4, and the proofs easier.

We make some connections between character sheaves on $(G \times G, G)$ and those on (G, K) in section 5; this gives some insight into the algebra $K^F\backslash G^F/K^F$.

Finally, some small examples are calculated in section 7.

If the base field is $\mathbf{C}$, we have the possibility of using microlocal techniques. As [G], [MV] show, this can simplify proofs significantly. Throughout this thesis, we mention simplifications over $\mathbf{C}$ where appropriate. However, this is in the nature of a sideline. It would be an interesting problem to study the characteristic varieties of character sheaves, but I have not yet attempted this.

Notation

0.1 Let k be an algebraically closed field; all algebraic varieties will be over k. We shall fix a prime number l invertible in k. Let X be an algebraic variety. We denote by DX the bounded derived category of constructible $\bar{\mathcal{Q}}_l$-sheaves on X, a triangulated category with t-structure. We denote by $\mathcal{M}X$ the heart of DX, the abelian category of l-adic perverse sheaves on X [BBD].

If $A \in DX$, we denote by $\mathcal{H}^iA$ its i'th cohomology sheaf, and by ${}^pH^iA$ its i'th perverse cohomology, a complex in $\mathcal{M}X$. If $f : X \to Y$ is a morphism, we have the usual functors $f_*, f_! : DX \to DY$, $f^* : DY \to DX$. We define $f^0 = f^*[\dim X - \dim Y]$; so if f is smooth and all the fibres of f have the same dimension, f^0 restricts to give $f^0 : \mathcal{M}Y \to \mathcal{M}X$.

If we have a stratification of X into finitely many locally closed smooth subvarieties, $X = \coprod X_\alpha$, and $A \in DX$ has the property that $\mathcal{H}^iA|X_\alpha$ is a local system or zero for all i, α, we say A is *constructible with respect to the stratification* $\coprod X_\alpha$.

Given $A \in DX$, by the *constituents* or *perverse constituents* of A, we mean the set of isomorphism classes of simple perverse sheaves which appear as subquotients of $\oplus_i {}^pH^iA$. This is a finite set.

0.2 Now suppose K is a linear algebraic group defined over k, acting on the variety X. We do not suppose K is connected or reductive. Then denote by $D_K(X)$ the equivariant derived category of X, a triangulated category with t-structure, and by $\mathcal{M}_K(X)$ its heart [MV,Appendix]. If L is a subgroup of K, $D_K(X)$ comes equipped with a canonical forgetful functor $D_K(X) \to D_L(X)$; if $L = 1$, there is a canonical equivalence of categories $D_L(X) \to DX$. If K is connected, then $\mathcal{M}_K(X)$ identifies under the forgetful functor with the full subcategory of $\mathcal{M}X$ defined in [L1,1.9].

If L is a subgroup of K, and L acts on the variety X, we may form the induced space $Y = K \times_L X$. Let $i : X \to Y$ be the embedding $x \mapsto$ image of $(1, x)$ in Y. Then there is a functor Γ ("induction") $\Gamma : D_L(X) \to D_K(Y)$ such that Γ, i^0 are inverse equivalences of categories [MV,1.4].

If $f : X \to Y$ is a proper map of K-spaces, and $A \in D_K(X)$ is a split semisimple complex of geometric origin, then $f_! A$ is also. This follows from the decomposition theorem [BBD,6.2.5].

0.3 If k is the algebraic closure of a finite field F_q with q elements, and K and X are defined over F_q, one can define the category $D^m_K(X)$ of mixed complexes in $D_K(X)$ as in [BBD]. If $A \in D^m_K(X)$, ${}^pH^iA$ carries a weight filtration. Denote the j'th piece of the associated graded complex ${}^pH^i_jA$; this is pure of weight j.

Let $\mathcal{K}X$ denote the Grothendieck group of the abelian category $\mathcal{M}_K(X)$. By the *mixed Grothendieck group* of X we mean $\mathcal{K}X \otimes_{\mathbf{Z}} \mathbf{Z}[q^{1/2}, q^{-1/2}]$, where $q^{1/2}$ is a formal symbol. If $A \in D^m_K(X)$, put $[A] = \sum_{i,j} (-1)^i \{{}^pH^i_jA\} \otimes q^{j/2}$, where $\{A'\}$ denotes the image of the pure perverse sheaf A' in $\mathcal{K}X$.

Write $\mathcal{K}'X$ for the subring of $\mathcal{K}X$ spanned by the simple perverse sheaves in $D^m_K(X)$. If $f : X \to Y$ is a morphism defined over F_q, we get maps between $\mathcal{K}'X \otimes_{\mathbf{Z}} \mathbf{Z}[q^{1/2}, q^{-1/2}]$ and $\mathcal{K}'Y \otimes_{\mathbf{Z}} \mathbf{Z}[q^{1/2}, q^{-1/2}]$, by defining $f^*[A] = [f^*A]$, $f_![A] = [f_!A]$, ...

If $F^*A \simeq A$, where F is the Frobenius morphism defining the F_q-rational structure, and if $\phi : F^*A \tilde{\to} A$ is some given isomorphism, we define $\chi_{A,\phi} : X(F_q) \to \bar{\mathcal{Q}}_l$ by $\chi_{A,\phi}(x) = \sum (-1)^i tr(\phi, \mathcal{H}^i_x A)$, the *characteristic function* of A.

0.4 Suppose a torus T acts freely on X. Let $\mathcal{L}$ be a rank one local system on T, and $D^{\mathcal{L}}(X)$ the full subcategory of DX with objects those complexes $A \in DX$ such that there exists some isomorphism $a^*A \simeq \mathcal{L} \boxtimes A$. Then it is easy to check that if $A \in D^{\mathcal{L}}(X)$, ${}^pH^iA \in D^{\mathcal{L}}(X)$; and that if $A \in D^{\mathcal{L}}(X)$, $A' \in D^{\mathcal{L}'}(X)$ with $\mathcal{L}$, $\mathcal{L}'$ distinct local systems on T, that $\mathrm{Ext}_{DX}(A, A') = 0$.

Also, if T acts freely on the variety Y, $f : X \to Y$ is a T-morphism, the usual functors $f^!, f^*, f_!, f_*$ preserve these subcategories.

We call objects of $D^{\mathcal{L}}(X)$ *T-monodromic sheaves with weight* $\mathcal{L}$. If now $K \times T$ acts on X, T acting freely, define $D^{\mathcal{L}}_K(X)$ as the full subcategory of $D_K(X)$ with objects those $A \in D_K(X)$ such that the image of A in DX is in $D^{\mathcal{L}}(X)$.

0.5 Let X' be a closed subset of X, $i : X' \hookrightarrow X$, $j : X \setminus X' \hookrightarrow X$ the inclusions. For each $A \in DX$, we get a distinguished triangle $(j_!j^*A, A, i_!i^*A)$ in DX, and hence a long exact sequence in perverse cohomology. If now $X = X_1 \coprod \cdots \coprod X_N$ is a partition of X into finitely many locally closed pieces such that the inclusion $X_i \hookrightarrow \cup_{j \leq i} X_j$ is the the inclusion of a closed set, then by iterating this construction we get a sequence of long exact sequences which allow us to compute the constituents of $A \in DX$ in terms of the constituents of $A|X_i$ for each i. This is the technique of [L1,2–3]; and we will just refer to it as the *long exact sequence of a partition*. We likewise use the technique of [L1, 12.6–12.7] without comment.

0.6 Let $\{A_\alpha\}$ be a set of simple perverse sheaves on Y. By the *subcategory they generate* we mean the full subcategory $\mathcal{C}$ of $D_K(Y)$ whose objects are those complexes with perverse constituents contained in the set $\{A_\alpha\}$. If $F : D_K(X) \to D_K(Y)$ is some functor with $FA \in \mathcal{C}$ for all simple perverse sheaves A, then $FA \in \mathcal{C}$ for all complexes $A \in D_K(X)$. This follows from repeated use of the distinguished triangle $({}^pH^a[-a], {}^p\tau^{\geq a}, {}^p\tau^{>a})$, where ${}^p\tau^{\geq a}$ denotes the truncation functor with respect to the t-structure.

0.7 Throughout this thesis, G will denote a connnected reductive group defined over k. If L is a subgroup of G, we denote by L^0 the identity component of L, by Z_L the center of L, write U_{L^0} for its unipotent radical, $Z_G(L) = \{g \in G \mid gl = lg, \text{ for all } l \in L\}$, $N_G(L) = \{g \in G \mid gLg^{-1} = L\}$. Write ${}^gx$ for gxg^{-1}.

If S is a torus of L, write $W(S, L) = N_L(S)/Z_L(S)$. In particular, if T is a maximal torus of G, $W(T, G)$ is isomorphic to the Weyl group W of G. Write $\mathcal{B} = \mathcal{B}_G$ for the flag variety of G. If X is a subvariety of G, write X_{uni} for the variety of unipotent elements of X.

If $\theta : G \to G$ is an automorphism of G, write $G^\theta = \{g \in G \mid \theta g = g\}$. If θ is an involutory automorphism, $\theta^2 = 1$, we say (G, θ) is *split* if there exists a maximal torus T of G with $\theta t = t^{-1}$ for all $t \in T$. We say (G, θ) is *essentially split* if $(G/Z_G^0, \theta)$ is split. When we consider an involutory automorphism, we will always assume it is semisimple, i.e. that char $k \neq 2$.

If $\mathcal{L}$ is a rank one local system on T, write $W'_{\mathcal{L}} = \{w \in W(T, G) \mid w^*\mathcal{L} \simeq \mathcal{L}\}$.

0.8 Write $\mathfrak{g}$ for the Lie algebra of G, $\mathfrak{g}^\theta$ (resp. $\mathfrak{g}^{-\theta}$) for the $+1$ (resp. -1) eigenspace of $d\theta$ on $\mathfrak{g}$. If $\mathfrak{h}$ is a vector subspace of $\mathfrak{g}$, write $N(\mathfrak{h})$ for the cone of nilpotent elements in $\mathfrak{h}$.

Suppose $k = \mathbf{C}$. If M is a $\mathfrak{g}$-module, write $\operatorname{Ass}(M)$ for its associated variety. If $A \in DX$ is a complex on a variety X defined over $\mathbf{C}$, write $SS(A) \subseteq T^*X$ for its characteristic variety.

0.9 Suppose $p(x)$ is a polynomial. Then we write $p(q)|_{q\mapsto\lambda}$ for $p(\lambda)$ $(q, \lambda \in \mathbf{Z})$.

1. Definitions

In this section, we define character sheaves and state some of their basic properties. With the exception of (1.6), all the basic notions and results are that of Lusztig [L1], who considered character sheaves on a group. In [G], Ginsburg generalises this to character sheaves on a "complex" symmetric space G/K; we take this slightly further by considering character sheaves in $D_K(G/H)$, i.e. on a "real" symmetric space. The results of this section are thus slight generalisations of well known results.

We recover Lusztig's character sheaves as the diagonal case $(G \times G, G_\Delta, G_\Delta)$, where $G_\Delta \hookrightarrow G \times G$, $g \mapsto (g, g)$.

1.1 For the remainder of this section, we fix a maximal torus T, and a Borel subgroup B containing T with unipotent radical U. Let T act on $G/U \times G/U$ by $(xU, yU).t = (xtU, ytU)$; denote the quotient by this (free) action as $(G/U \times G/U)/T$. It is possible to give a definition of $(G/U \times G/U)/T$ which does not involve a choice of U, T; we do not need this.

We suppose given two groups K, H, and homomorphisms $K \to G$, $H \to G$. Usually K, H will be subgroups of G, but it is sometimes notationally convenient to consider this more general situation. Define functors $ch_G : D_{K\times H}((G/U \times G/U)/T) \to D_{K\times H}(G)$, $hc_G : D_{K\times H}(G) \to D_{K\times H}((G/U \times G/U)/T)$ by $ch_G = p_! q^0$, $hc_G = q_! p^0$, where $p : G \times \mathcal{B} \to G$, $(g, hB) \mapsto g$, and $q : G \times \mathcal{B} \to (G/U \times G/U)/T$, $(g, hB) \mapsto (ghU, hU)T$. (The notation is Ginsburg's).

Here $K \times H$ acts on $G \times \mathcal{B}$ by $(k, h).(g, xB) = (kgh^{-1}, hxB)$, on G by $(k, h).g = kgh^{-1}$, and on $(G/U \times G/U)/T$ by $(k, h).(xU, yU)T = (kxU, hyU)T$; so p, q are $K \times H$-equivariant maps. Also p, q are smooth, and p is proper.

When there is no possibility for confusion we write hc, ch for hc_G, ch_G.

Now, T acts freely on $(G/U \times G/U)/T$ by $t.(xU, yU)T = (xtU, yU)T$; so we can apply the discussion of (0.4) to define T-monodromic sheaves with weight $\mathcal{L}$, $\mathcal{L}$ a (rank one) local system on T. Throughout this thesis, we assume $\mathcal{L}^{\otimes n} \simeq \bar{\mathcal{Q}}_l$ for some integer n invertible in the ground field k; when we refer to a local system on T we always mean such a "tame" local system.

In this way we get the category $D^{\mathcal{L}}_{K\times H}((G/U \times G/U)/T) \simeq D^{\mathcal{L}}_{K\times H\times T}(G/U \times G/U)$.

1.2 Proposition. *$hc \circ ch : D^{\mathcal{L}}_{K\times H}((G/U \times G/U)/T) \to \oplus_{w\in W/W'_{\mathcal{L}}} D^{w^*\mathcal{L}}_{K\times H}((G/U \times G/U)/T)$*

Proof.Let $Z = G \times \mathcal{B} \times \mathcal{B}$, $q_i : Z \to (G/U \times G/U)/T$, $(g, h_1B, h_2B) \mapsto (gh_iU, h_iU)T$, for $i = 1, 2$. Then $K \times H$ acts on Z by $(k, h).(g, h_1B, h_2B) = (kgh^{-1}, hh_1B, hh_2B)$ so q_1, q_2 are equivariant, and $hc \circ ch = (q_1)_! {q_2}^0 : D_{K\times H}((G/U \times G/U)/T) \to D_{K\times H}((G/U \times G/U)/T)$ (base change).

Weights of the torus action are defined in the (non-equivariant) derived category, and the image of $(q_1)_! {q_2}^0$ there is the usual $(q_1)_! {q_2}^0$, so we need only show $(q_1)_! {q_2}^0 : D^{\mathcal{L}}(G/U \times G/U)/T) \to \oplus_{w\in W/W'_{\mathcal{L}}} D^{w^*\mathcal{L}}(G/U \times G/U)/T)$.

Partition Z by the G-orbits on $\mathcal{B} \times \mathcal{B}$; i.e. put $Z_w = \{(g, h_1B, h_2B) \mid h_1^{-1}h_2 \in BwB\}$. Then applying the long exact sequence in perverse cohomology to this partition, we see it is enough to show $(q_{1w})_! q^0_{2w} : D^{\mathcal{L}}(G/U \times G/U)/T \to D^{w^*\mathcal{L}}(G/U \times G/U)/T)$, where q_{iw} denotes the restriction of q_i to Z_w.

Let $Z'_w = \{((xU, yU)T, (aU, bU)T) \in (G/U \times G/U)/T \times (G/U \times G/U)/T \mid Ux^{-1}aU = Uy^{-1}bU \subseteq BwB\}$. Then under the map $Z_w \to Z'_w$, $(g, h_1B, h_2B) \mapsto ((gh_1U, h_1U)T, (gh_2U, h_2U)T)$, Z_w becomes an affine space bundle over Z'_w with fibres isomorphic to $U \cap {}^{w^{-1}}U$; and if $q'_{iw} : Z'_w \to (G/U \times G/U)/T$ denotes the i'th projection $(i = 1, 2)$, we have $(q_{1w})_! q^*_{2w} = (q'_{1w})_!(q'_{2w})^*[-2d](-d)$, where $d = \dim(U \cap {}^{w^{-1}}U)$.

Finally, define a T-action on Z'_w by $t.((xU, yU)T, (aU, bU)T) = ((xw^{-1}twU, yU)T, (atU, bU)T)$. Then q_2 is T-equivariant, and q_1 is also, with respect to the "twisted by w" T-action: $t.(xU, yU)T = (xw^{-1}twU, yU)T$. The proposition is immediate from the remarks following the definition of T-monodromic.

The next proposition is precisely [G,8.5.1], [MV,3.6] in our $K \times H$-equivariant setting. The proof is the same (one must only check the diagrams used can be made $K \times H$-equivariant).

Proposition. *The identity functor is a direct summand of $ch \circ hc : D_{K\times H}(G) \to D_{K\times H}(G)$.*

Indeed, for $\mathcal{A} \in D_{K\times H}(G)$, $ch \circ hc(\mathcal{A}) \simeq \mathcal{A} * \mathrm{Spr}$, where Spr is the complex in $D_G(G)$ given by $\pi_! \bar{\mathcal{Q}}_l$, for $\pi : \{(g, B') \in G^{uni} \times \mathcal{B} \mid g \in B'\} \to G^{uni}$, $(g, B') \mapsto g$ the Springer resolution of the unipotent variety, G acts on G by conjugation, and $*$ is the convolution defined below.

1.3 Define $(G, K, H)^\wedge_\mathcal{L}$ to be the full subcategory of $D_{K\times H}(G)$ consisting of those complexes whose perverse constituents are the constituents of the complexes $\{ch(\mathcal{A}) \mid \mathcal{A} \in D^{w^*\mathcal{L}}_{K\times H}(G/U \times G/U)/T), w \in W\}$. It is clear that if (A, B, C) is a distinguished triangle in $D_{K\times H}(G)$ with $A, C \in (G, K, H)^\wedge_\mathcal{L}$ then $B \in (G, K, H)^\wedge_\mathcal{L}$.

Write $\mathcal{M}(G, K, H)^\wedge_\mathcal{L}$ for the abelian subcategory of equivariant perverse sheaves in $(G, K, H)^\wedge_\mathcal{L}$; we call the simple sheaves the *character sheaves* of (G, K, H). For a fixed $\mathcal{L}$ there are only finitely many (isomorphism classes of) character sheaves, as there are only finitely many isomorphism classes of simple perverse sheaves in $D^\mathcal{L}_{K\times H}((G/U \times G/U)/T)$. Also, every simple character sheaf arises as a constituent of $ch(\mathcal{A})$ for some *simple* perverse sheaf $\mathcal{A}$, and hence as a summand of $ch(\mathcal{A})$ (decomposition theorem).

The following proposition follows from the definition and the above two propositions.

Proposition. *i) $(G, K, H)^\wedge_\mathcal{L}$ depends only on the Weyl group orbit of $\mathcal{L}$. ii) The functors ch, hc restrict to give functors between $(G, K, H)^\wedge_\mathcal{L}$ and $\oplus_{w\in W/W'_\mathcal{L}} D^{w^*\mathcal{L}}_{K\times H}((G/U \times G/U)/T)$. iii) If $W\mathcal{L} \neq W\mathcal{L}'$, $Ext_\mathcal{C}((G, K, H)^\wedge_\mathcal{L}, (G, K, H)^\wedge_{\mathcal{L}'}) = 0$, where $\mathcal{C}$ is the category DG or $D_{K\times H}(G)$. iv) The Verdier dual D takes $(G, K, H)^\wedge_\mathcal{L}$ to $(G, K, H)^\wedge_{\mathcal{L}^\vee}$, where $\mathcal{L}^\vee$ is the dual local system to $\mathcal{L}$.*

For notational simplicity we write $(G, K)^\wedge_\mathcal{L}$, $\mathcal{M}(G, K)^\wedge_\mathcal{L}$ for $(G, K, K)^\wedge_\mathcal{L}$, $\mathcal{M}(G, K, K)^\wedge_\mathcal{L}$; and $G^\wedge_\mathcal{L}$, $\mathcal{M}G^\wedge_\mathcal{L}$ for $(G \times G, G_\Delta)^\wedge_\mathcal{L}$, $\mathcal{M}(G \times G, G_\Delta)^\wedge_\mathcal{L}$ (here $G_\Delta \hookrightarrow G \times G$ is the diagonal embedding). This notation is in slight contradiction with the notation of [L1], where $G^\wedge_\mathcal{L}$ denotes the simple objects of what we write $\mathcal{M}G^\wedge_\mathcal{L}$.

We call the complexes $ch(\mathcal{A})$, for $\mathcal{A} \in D^\mathcal{L}_{K\times H}((G/U \times G/U)/T)$, *standard sheaves.*

We say a character sheaf in $(G, K, H)^\wedge_\mathcal{L}$ has *central character* $\mathcal{L}$. If the base field is $\mathbf{C}$, this can be defined in terms of the action of $Z(\mathfrak{g})$ on global sections of the $\mathcal{D}$-module corresponding to the character sheaf [G], hence the name— $Z(\mathfrak{g})$ will have weights in an affine Weyl group orbit on $\mathfrak{t}^*$ corresponding to $\mathcal{L}$.

1.4 Define a functor, "convolution", $* : D(G/U \times G/U) \times D(G/U \times G/U) \to D(G/U \times G/U)$, by $A_1 * A_2 = q_!(p^0_1 A_1 \otimes p^0_2 A_2)$, where $p_{ij} : G/U \times G/U \times G/U \to G/U \times G/U$, $p_{ij}(h_1U, h_2U, h_3U) = (h_iU, h_jU)$, and $p_1 = p_{12}$, $p_2 = p_{23}$, $q = p_{13}$. This is an associative operation.

For the remainder of this paragraph, we will write D_X for $D_X(G/U \times G/U)$, where X is a group acting on $G/U \times G/U$. Then the multiplication $*$ gives multiplication functors $D_{G_\Delta} \times D_{K\times H} \to D_{K\times H}$, $D_{K\times H} \times D_{G_\Delta} \to D_{K\times H}$ by composing the forgetful functor $D_{G_\Delta} \to D_{K_\Delta}$ with $*$ (and making $K \times H$ act on $G/U \times G/U \times G/U$ so as to make p_1, p_2, q $K \times H$-equivariant), as well as multiplications $D_{G_\Delta} \times D_{G_\Delta} \to D_{G_\Delta}$, $D_{K\times K} \times D_{K\times K} \to D_{K\times K}$. Denote by $D^{\mathcal{L},\mathcal{L}'}_{G_\Delta}$ those G_Δ-equivariant complexes which have $\mathcal{L} \boxtimes \mathcal{L}'$ monodromy weight with respect to the $T \times T$-action. It is clear that this category is empty unless $\mathcal{L}$, $\mathcal{L}'$ are in the same W-orbit, and that there are precisely $W'_{\mathcal{L}}$ simple perverse sheaves in this category if $\mathcal{L} \in W\mathcal{L}'$. Then $\oplus_{(\mathcal{L}_1,\mathcal{L}_2)\in W\mathcal{L}\times W\mathcal{L}} D^{\mathcal{L}_1,\mathcal{L}_2}_{G_\Delta}$ forms a tensor category with respect to $*$, with $D^{\mathcal{L},\mathcal{L}}_{G_\Delta}$ a tensor subcategory. The mixed Grothendieck group of this category, and the induced algebra structure, is almost precisely that of the algebra of [MS,3.3]; that of the subcategory is almost that of the Hecke algebra $H'_{\mathcal{L}}$ of [L1,6.1]. Likewise, we can write explicit formulae for the action of these algebras on the mixed Grothendieck group of $D_{K\times H}$ (when K, H are as in (1.5)), generalising [LV,3.5] and [MS,3.3]. We do not need these formulae, except in special cases. Observe $D^{\mathcal{L}_1,\mathcal{L}_2} * D^{\mathcal{L}_3,\mathcal{L}_4} = 0$ unless $\mathcal{L}_2 = \mathcal{L}_3$.

We can also define a multiplication functor $* : DG \times DG \to DG$, by $A_1 * A_2 = m_!(A_1 \boxtimes A_2)$, where $m : G \times G \to G$ $(x,y) \mapsto xy$ is the multiplication map. This also gives multiplications $D_{G_\Delta}(G) \times D_{G_\Delta}(G) \to D_{G_\Delta}(G)$, $D_{G_\Delta}(G) \times D_{K\times H}(G) \to D_{K\times H}(G)$, $D_{K\times H}(G) \times D_{G_\Delta}(G) \to D_{K\times H}(G)$, $D_{K\times K}(G) \times D_{K\times K}(G) \to D_{K\times K}(G)$. These in fact preserve the category of character sheaves, giving functors $G^\wedge_{\mathcal{L}} \times G^\wedge_{\mathcal{L}} \to G^\wedge_{\mathcal{L}}$, $G^\wedge_{\mathcal{L}} \times (G,K,H)^\wedge_{\mathcal{L}} \times G^\wedge_{\mathcal{L}} \to (G,K,H)^\wedge_{\mathcal{L}}$, $(G,K)^\wedge_{\mathcal{L}} \times (G,K)^\wedge_{\mathcal{L}} \to (G,K)^\wedge_{\mathcal{L}}$. (This follows from the last proposition of (1.2) and the lemma below).

As a consequence of [L9] one can compute the multiplication $G^\wedge_{\mathcal{L}} \times G^\wedge_{\mathcal{L}} \to G^\wedge_{\mathcal{L}}$ explicitly on the level of mixed Grothendieck groups. (Characters are mutually orthogonal idempotents for the multiplication, so by using the relation between characters and the characteristic functions of character sheaves [L1,3,4,9], one can describe the multiplication of character sheaves).

The following lemma, and its obvious variants, are easy.

Lemma. *i) If $A_1 \in G^\wedge_{\mathcal{L}}$, $A_2 \in (G,K,H)^\wedge_{\mathcal{L}}$, then $A_1 * A_2 \simeq A_2 * A_1$. ii) If $A_1 \in G^\wedge_{\mathcal{L}}$, $A_2 \in (G,K,H)^\wedge_{\mathcal{L}}$, then $hc(A_1 * A_2) \simeq hc(A_1) * hc(A_2)$. iii) If $C_1 \in D^{\mathcal{L},\mathcal{L}}_{G_\Delta}$, $C_2 \in D^{\mathcal{L},\mathcal{L}}_{K\times H}$, then $ch(C_1 * C_2) \simeq ch(C_2 * C_1)$. iv) If $C \in D^{\mathcal{L},\mathcal{L}}_{G_\Delta}$, $A \in (G,K,H)^\wedge_{\mathcal{L}}$, then $C * hc(A) \simeq hc(A) * C$. (Regarding $D((G/U \times G/U)/T) \simeq D_T(G/U \times G/U)$ for the purposes of multiplication).*

Unfortunately we are unable to exploit this structure on the character sheaves in any interesting way.

1.5 We suppose given two commuting involutions $\theta, \sigma : G \to G$, such that K (resp. H) is a subgroup of finite index in the fixpoints of θ (resp. σ), i.e. $(G^\theta)^0 \le K \le G^\theta$, $(G^\sigma)^0 \le H \le G^\sigma$. Let $X = \{x \in G \mid \theta x = x^{-1}\}^0 \simeq G^\theta\backslash G$, $X' = \{x \in G \mid \sigma x = x^{-1}\}^0 \simeq G/G^\sigma$.

Consider the disconnected group C_2G obtained as the semidirect product of G with the cyclic group of order 2 with generator $\theta\sigma$ (with G normal, and $\theta\sigma.g.(\theta\sigma)^{-1} = \theta\sigma(g)$). Let G^1 be the non-identity component of C_2G. Then in [L2,2] there is defined a partition of G^1 into finitely many constructible sets, each piece of which is smooth and G-invariant for conjugation. The intersection of $\theta\sigma.X$ with the pieces of the partition of G^1 give a partition of X. Taking the inverse image of the pieces of this partition under the map $G \to \theta\sigma.X$, $g \mapsto \theta\sigma.g^{-1}\theta g$, we get a partition of G into finitely many constructible sets, each of which is smooth and $K \times H$-invariant. Taking the connnected components of this partition, we get a stratification of G, the *Lusztig stratification* of (G,K,H).

We could also define a $K \times H$-invariant stratification using X' rather than X; these two stratifications coincide, as $g\sigma g^{-1}.\sigma\theta$ is conjugate to $(\theta\sigma.g^{-1}\theta g)^{-1}$, and the map $g \mapsto g^{-1}$ permutes the pieces of the partition of G^1.

Proposition. *Character sheaves are constructible with respect to the Lusztig stratification.*

Proof.If $k = \mathbf{C}$, this follows immediately from (1.7) below. In general, the proof is exactly the same, but one uses the vanishing cycles functor ϕ_f rather than the characteristic variety. For lack of a reference, we sketch the details; I'm grateful to M. Finkelberg for significant help with them.

Let $\mathcal{A}$ be a character sheaf, Y a piece of the Lusztig stratification, $g \in Y$. If f is a function defined on a Zariski neighborhood of g, such that $f(g) = 0$ and f is smooth at g, then we want to show that $(\phi_f\mathcal{A})_g \ne 0$ implies $df(g) \in T^*_Y X$. If this is so for all such g, Y, f, it follows that $\mathcal{A}$ is constructible with respect to the Lusztig stratification.

As every character sheaf is a summand of a standard sheaf $ch(\mathcal{A}')$, $\mathcal{A}' \in D^{\mathcal{L}}_{K\times H}((G/U \times G/U)/T)$, it is enough to show this for $ch(\mathcal{A}') = p_!q^*\mathcal{A}'$. In what follows, to simplify notation we assume the central character $\mathcal{L}$ is trivial; this is no restriction—what matters is that $\mathcal{A}'$ is locally constant along the T-orbits.

Now, p is proper, so $\phi_f(p_! q^* \mathcal{A}')_g = (p_! \phi_{fp}(q^*\mathcal{A}'))_g$. We assume $\phi_f(ch(\mathcal{A}'))_g \neq 0$. Then there is a $B' \in \mathcal{B}$ with $\phi_{fp}(q^*\mathcal{A}')_{(g,B')} \neq 0$, and as p is smooth, fp is smooth at (g, B').

Also, q is smooth and $\phi_{fp}(q^*\mathcal{A}')_{(g,B')} \neq 0$. It follows there is some function f' on a neighborhood of $({}^gB', B')$ in $\mathcal{B} \times \mathcal{B}$ such that $f'({}^gB', B') = 0$, f' is smooth at $({}^gB', B')$, $d(f'q)(g, B') = d(fp)(g, B)$, and $\phi_{f'q}(q^*\mathcal{A}')_{(g,B')} = \phi_{f'}(\mathcal{A}')_{({}^gB',B')} \neq 0$.

Finally, $\mathcal{A}'$ is $K \times H$-equivariant on $\mathcal{B} \times \mathcal{B}$, so locally constant along the $K \times H$-orbits, and hence $(\phi_{f'}\mathcal{A}')_{(B'',B')} \neq 0$ implies $df(B'', B')$ is in the conormal bundle to the orbit of (B'', B'). To finish the proof one merely calculates these conormal bundles, and checks that this simple estimate implies $df(x) \in T^*_Y X$. If the characteristic of k is good for G, this is precisely as in the complex case (see (1.6)). If the characteristic of k is bad, one must be slightly careful; I've yet to do this computation.

1.6 Suppose the base field k is $\mathbf{C}$. We recall (and slightly generalise) the results of [G], [MV].

If $f : Y \to X$ is a map, we denote as in [KS,4.3] by ${}^tf'$, f_π the induced maps

$$T^*Y \xleftarrow{{}^tf'} Y \times_X T^*X \xrightarrow{f_\pi} T^*X. \tag{1.6.1}$$

Identify $\mathfrak{g}$ with the left invariant vector fields on G, so $TG \simeq G \times \mathfrak{g}$, and identify $\mathfrak{g} \simeq \mathfrak{g}^*$ via a non-degenerate G-invariant symmetric bilinear form, so $T^*G \simeq G \times \mathfrak{g}$. Write $N(\mathfrak{g})$ for the nilpotents of G, and $\mu : G \times \mathfrak{g} \to \mathfrak{g}$ for the second projection—the *moment map.* Then $T^*\mathcal{B} \simeq \{(\lambda, B) \in N(\mathfrak{g}) \times \mathcal{B} \mid \lambda \in \mathrm{Lie}(B)\} = \dot{\mathfrak{g}}_N$. Also, if $X = \{x \in G \mid \theta x = x^{-1}\}^0 \simeq G^\theta \backslash G$, $X' = \{x \in G \mid \sigma x = x^{-1}\}^0 \simeq G/G^\sigma$, then we have $T^*X \simeq \{(x,\xi) \in X \times \mathfrak{g} \mid \theta\xi = -{}^x\xi\}$, $T^*X' \simeq \{(x,\xi) \in X' \times \mathfrak{g} \mid \sigma\xi = -{}^x\xi\}$ as subbundles of T^*G.

Let $p : G \times \mathcal{B} \to G$, $(g, B) \mapsto g$, $q : G \times \mathcal{B} \to \mathcal{B} \times \mathcal{B}$, $(g, B) \mapsto ({}^gB, B)$, $\pi : G \to X$, $g \mapsto g^{-1}\theta g$, $\pi' : G \to X'$, $g \mapsto g\sigma g^{-1}$.

Lemma. *For each of the maps p, q, π, π', the diagram (1.6.1) becomes i)* $G \times \mathfrak{g} \times \dot{\mathfrak{g}}_N \xleftarrow{{}^tp'} G \times \mathfrak{g} \times \mathcal{B} \xrightarrow{p_\pi} G \times \mathfrak{g}$, $(g, \lambda, (0, B)) \leftarrow (g, \lambda, B) \mapsto (g, \lambda)$. *ii)* $G \times \mathfrak{g} \times \dot{\mathfrak{g}}_N \xleftarrow{{}^tq'} \{(g, B, n_1, n_2) \in G \times \mathcal{B} \times N(\mathfrak{g}) \times N(\mathfrak{g}) \mid n_1 \in Lie^gB, n_2 \in LieB\} \xrightarrow{q_\pi} \dot{\mathfrak{g}}_N \times \dot{\mathfrak{g}}_N$, $(g, {}^{g^{-1}}n_1, (n_2, B)) \leftarrow (g, B, n_1, n_2) \mapsto ((n_1, {}^gB), (n_2, B))$. *iii)* $G \times \mathfrak{g} \xleftarrow{{}^t\pi'} \{(g, \lambda) \mid \theta\lambda = -{}^{g^{-1}\theta g}\lambda\} \xrightarrow{\pi_\pi} T^*X$, $(g, 2\theta\lambda) \leftarrow (g, \lambda) \mapsto (g^{-1}\theta g, \lambda)$. *iv)* $G \times \mathfrak{g} \xleftarrow{{}^t(\pi')'} \{(g, \lambda) \mid \theta\lambda = -{}^{g\sigma g^{-1}}\lambda\} \xrightarrow{\pi'_\pi} T^*X'$, $(g, -2\,{}^{g^{-1}}\sigma\lambda) \leftarrow (g, \lambda) \mapsto (g\sigma g^{-1}, \lambda)$.

As a consequence of the lemma (i), (ii), and the estimates for the behaviour of SS under direct image by a proper map and inverse image by a smooth map [KS,5.4.4,5.4.5], we get $SS(ch(\mathcal{A})) \subseteq \{(g, \lambda) \in G \times N(\mathfrak{g}) \mid \theta({}^g\lambda) = -{}^g\lambda\}$. By considering $p_! q_2^* = i^* p_! q^*$, where $i : G \to G$, $g \mapsto g^{-1}$, and $q_2 : G \times \mathcal{B} \to \mathcal{B} \times \mathcal{B}$, $(g, B) \mapsto (B, {}^gB)$ we get analogously $SS(ch(\mathcal{A})) \subseteq G \times N(\mathfrak{g}^{-\sigma})$. So if $\mathcal{A} \in (G, K, H)^\wedge_{\mathcal{L}}$ is a character sheaf, $SS(\mathcal{A}) \subseteq \Lambda^G_{K,H}$, where $\Lambda^G_{K,H} = \{(g, \lambda) \in G \times N(\mathfrak{g}) \mid \theta({}^g\lambda) = -{}^g\lambda, \sigma\lambda = -\lambda\}$.

If $K = G^\theta$, $H = G^\sigma$, and we identify $D_{K \times H}(G)$ with $D_H(X)$ (resp. $D_K(X')$), then the characteristic variety of a character sheaf is contained in $\Lambda^X_H = \{(x, \lambda) \in X \times N(\mathfrak{g}) \mid \theta(\lambda) = -{}^x\lambda, \sigma\lambda = -\lambda\}$ (resp. $\Lambda^{X'}_K = \{(x, \lambda) \in X' \times N(\mathfrak{g}) \mid \theta(\lambda) = -\lambda, \sigma\lambda = -{}^x\lambda\}$). Alternately, we could identify G with the symmetric space $(G \times G, G)$, so T^*G is isomorphic to the subbundle $\{(g^{-1}, g, \xi_1, \xi_2) \in G \times G \times \mathfrak{g} \times \mathfrak{g} \mid {}^g\xi_2 = -\xi_1\}$ of $T^*(G \times G)$. Then the characteristic variety of a character sheaf is contained in $\Lambda^{G \times G}_{K \times H} = G \times G \times N(\mathfrak{g}^{-\theta}) \times N(\mathfrak{g}^{-\sigma}) \cap T^*G$.

1.7 Write Λ for the union of the conormal bundles to the Lusztig stratification. It is easily shown that $\Lambda^G_{K,H} \subseteq \Lambda$, so [KS,8.4.1] character sheaves are constructible with respect to the Lusztig stratification.

Character sheaves are $K \times H$-equivariant perverse sheaves with characteristic variety contained in $\Lambda^G_{K,H}$. As in [G], this implies the corresponding $\mathcal{D}(K\backslash G)$-module (resp. $\mathcal{D}(G/H)$-module)

M of a character sheaf is "admissible" in the sense of [G], and that if $V \subset \Gamma(K\backslash G, M)$ (resp. $V \subset \Gamma(G/H, M)$) is a finitely generated $\mathcal{U}(\mathfrak{g})$-submodule of $\Gamma(M)$, then it is actually a $(\mathfrak{g}, \mathfrak{h})$-module (resp. a $(\mathfrak{g}, \mathfrak{k})$-module), and $\operatorname{Ass}(V) = \mu(SS(M))$ if V generates M as a $\mathcal{D}$-module, and G is of adjoint type.

Or, identifying G with $G \times G/G$, a $\mathcal{D}(G)$-module M corresponding to a character sheaf is admissible, and if $V \subset \Gamma(G, M)$ is a finitely generated $\mathcal{U}(\mathfrak{g} \times \mathfrak{g})$-submodule of $\Gamma(M)$, then it is actually a $(\mathfrak{g} \times \mathfrak{g}, \mathfrak{k} \times \mathfrak{h})$-module, with $\operatorname{Ass}(V) = \mu(SS(M))$ if V generates M as a $\mathcal{D}$-module, and G is of adjoint type.

Also arguing as in [G], we know the converse of these statements. In particular, a $K \times H$-equivariant perverse sheaf on G with characteristic variety contained in $\Lambda^G_{K,H}$ is necessarily a character sheaf. We can clearly weaken this to: a $K \times H$-equivariant perverse sheaf $\mathcal{A}$ with $SS(\mathcal{A}) \subseteq G \times N(\mathfrak{g})$ is necessarily a character sheaf; or (as in [MV]) to: If K, H are connected and simply connected, and $\mathcal{A}$ is an irreducible perverse sheaf with $SS(\mathcal{A}) \subseteq \Lambda^G_{K,H}$, then $\mathcal{A}$ is a character sheaf.

We will make a more interesting observation about the characteristic variety of a character sheaf in section 3.

2. Induction

In this section we define certain "induction functors" from character sheaves on Levi subgroups L with $\theta L = L$, $\sigma L = L$ to character sheaves on G. In the diagonal case these include the induction functors defined by Lusztig [L1]; they take perverse sheaves to perverse sheaves, and monodromy comes from finite covers. In general, we need the condition $W'_{\mathcal{L}} \subseteq L$ to obtain perverse sheaves; and monodromy, even for induction from a torus, can be very large. Nonetheless, this is enough to give a bijection between $(L, \theta)^\wedge_{\mathcal{L}}$ and $(G, \theta)^\wedge_{\mathcal{L}}$ for such $\mathcal{L}$; and hence to reduce the classification of character sheaves to the classification of those with central character in a finite set.

2.1 Throughout this section, let G be a connected reductive group, $B = TU_B$ a Borel subgroup, T a maximal torus of B, $P \supset B$ a parabolic subgroup with $P = LU_P$ its Levi decomposition, $T \subset L$. Write $W_* \subset W$ for the Weyl groups of L, G respectively, and $pr_L : P \to L$, $lu \mapsto l$ for the projection onto L. We further suppose given two groups K, H and a map $K \times H \to G \times G$ as in (1.1). We then get groups K_P, H_P as the inverse image of $P \times P$ under the map $K \times H \to G \times G$, and hence data (L, K_P, H_P), (P, K_P, H_P).

If $K \cap U_P$, $H \cap U_P$ are connected, and we set $K_L = pr_L(K_P)$, $H_L = pr_L(H_P)$ then $D_{K_P \times H_P}(L) \simeq D_{K_L \times H_L}(L)$, as then K_L, H_L differ from K_P, H_P by a connected unipotent group which acts trivially on L. This is the case when K, H are the fixpoints of involutions.

The following lemma is straightforward.

Lemma. *i) The map $K \times_{K_P} P/U_B \to KP/U_B$ induced by $(k, pU_B) \mapsto kpU_B$ is a $K \times T$-equivariant isomorphism. ii) The map $H \times_{H_P} P/U_B \to HP/U_B$ induced by $(h, pU_B) \mapsto hpU_B$ is a $H \times T$-equivariant isomorphism. iii) The map $(K \times H) \times_{(K_P \times H_P)} L \to (KP/U_P \times HP/U_P)/L$ induced by $(k, h, l) \mapsto (klU_P, hU_P)L$ is a $K \times H$-equivariant isomorphism.*

As a result of (i) and (ii), we get an equivalence of categories $D_{K \times H \times T}(KP/U_B \times HP/U_B) \to D_{K_P \times H_P \times T}(P/U_B \times P/U_B)$, given by restriction (and shift). Write this $A \mapsto \tilde{A}$. As a result of (iii), we get an equivalence of categories $D_{K_P \times H_P}(L) \to D_{K \times H}((KP/U_P \times HP/U_P)/L)$, given by the induction functor (0.2). Write this $A \mapsto \Gamma A$.

2.2 Definition. *Write $Ind^G_{L,P} : D_{K_P \times H_P}(L) \to D_{K \times H}(G)$ for the composition $(pr_1)_! q^0 j_! \Gamma$ where*

$$G \xleftarrow{pr_1} G \times G/P \xrightarrow{q} (G/U_P \times G/U_P)/L \xleftarrow{j} (KP/U_P \times HP/U_P)/L$$

and pr_1 is the first projection, j the inclusion, and $q(g, hP) = (ghU_P, hU_P)L$.

Proposition (Transitivity of Induction). $Ind^G_{L,P} \circ ch_L \circ {}^\sim = ch_G$

Proof.Consider the following commutative cartesian diagram, where the second row is obtained from the first by applying the functor $(K \times H) \times_{(K_P \times H_P)} (\cdot)$.

$$\begin{tikzcd}
L \arrow[d, "i_0"] & P \arrow[l, "pr_L"'] \arrow[d, "i_1"] & P \times P/B \arrow[l, "pr_1"'] \arrow[r, "q'"] \arrow[d, "i_2"] & (P/U_B \times P/U_B)/T \arrow[d, "i_3"] \\
(KP/U_P \times HP/U_P)/L & Z_1 \arrow[l, "\alpha"'] \arrow[d, "pr_1"] & Z_2 \arrow[l, "\pi"'] \arrow[r, "q"] & (KP/U_B \times HP/U_B)/T \\
 & G & &
\end{tikzcd}$$

Here $Z_1 = \{(g, hP) \mid (ghP, hP) \in KP/P \times HP/P\}$, $Z_2 = \{(g, hB) \mid (ghP, hP) \in KP/P \times HP/P\}$, and $q(g, hB) = (ghU_B, hU_B)T$, $q'(g, hB) = (ghU_B, hU_B)T$, $\alpha(g, hP) = (ghU_P, hU_P)L$, $\pi(g, hB) = (g, hP)$, $i_0(l) = (lU_P, U_P)L$, $i_1(p) = (p, P)$, i_2, i_3 are the obvious inclusions, and pr_1 is projection onto the first factor.

Now we wish to compare $\alpha^0 \Gamma ch_L(\tilde{A})$ with $\pi_! q^0 A$, for $A \in D_{K\times H\times T}(KP/U_B \times HP/U_B)$. As $i_1^0 : D_{K\times H}(Z_1) \to D_{K_P \times H_P}(P)$ is an equivalence of categories, it is enough to compare $i_1^0 \alpha^0 \Gamma ch_L(\tilde{A})$ with $i_1^0 \pi_! q^0 A$. But $i_1^0 \alpha^0 \Gamma ch_L(\tilde{A}) = pr_L^0 i_0^0 \Gamma ch_L(\tilde{A}) = pr_L^0 ch_L(\tilde{A})$, as i_0^0, Γ are inverse equivalences of categories. Also $i_1^0 \pi_! q^0 A = (pr_1)_! i_2^0 q^0 A = (pr_1)_! q'^0 i_3^0 A = ch_P(\tilde{A})$, as $\tilde{A} = i_3^0 A$, by definition. Finally, $pr_L^0 ch_L \tilde{A} = ch_P \tilde{A}$.

Corollary. *$Ind^G_{L,P}$ takes character sheaves to complexes whose perverse constituents are character sheaves. More precisely, $Ind^G_{L,P} : (L, K_P, H_P)^\wedge_{\mathcal{L}} \to (G, K, H)^\wedge_{\mathcal{L}}$.*

We also have the following variant of the proposition.

Proposition. *If $Q = MU_Q$ is a parabolic subgroup with $Q \subseteq P$, $M \subseteq L$, then*

$$Ind^G_{L,P} \circ Ind^L_{M,Q\cap L} = Ind^G_{M,Q}$$

Now suppose $A \in D_{K\times H\times T}(KP/U_B \times HP/U_B)$ is such that $j_! A$ is perverse, where j is the inclusion $KP/U_B \times HP/U_B \hookrightarrow G/U_B \times G/U_B$. Then $ch_G(j_! A)$ is semisimple, $ch_L(\tilde{A})$ is semisimple (both by the decomposition theorem), and so if A' is a summand of $ch_L(\tilde{A})$, $Ind^G_{L,P} A'$ is semisimple. If in fact we have $j_! : \mathcal{M}^{\mathcal{L}}_{K\times H\times T}(KP/U_B \times HP/U_B) \to \mathcal{M}^{\mathcal{L}}_{K\times H\times T}(G/U_B \times G/U_B)$ then we say $(P, \mathcal{L})$ is *good induction data*. If this is the case we clearly have

$Ind^G_{L,P} : (L, K_P, H_P)^\wedge_{\mathcal{L}} \to (G, K, H)^\wedge_{\mathcal{L}}$ takes semisimple complexes to semisimple complexes

as then every semisimple complex in $(L, K_P, H_P)^\wedge_{\mathcal{L}}$ is a summand of some $ch_L(\tilde{A})$, with $j_! A$ simple perverse on $G/U_B \times G/U_B$.

2.3 Write $d_P = \dim(U_P) - \dim(K/K_P) - \dim(H/H_P)$. We define an adjoint functor to induction. Define

$$Res^G_{L,P} : D_{K\times H}(G) \to D_{K_P \times H_P}(L)$$

as the composition $i'^* j^* q_! pr_1^* [d_P]$, where

$$G \xleftarrow{pr_1} G \times G/P \xrightarrow{q} (G/U_P \times G/U_P)/L \xleftarrow{j} (KP/U_P \times HP/U_P)/L \xleftarrow{i'} L$$

is as in (2.2), and $i'(l) = (lU_P, U_P)L$. Then it is clear that $Res^G_{L,P} = (pr_L)_! i^* [d_P]$, where $i : P \hookrightarrow G$ is the inclusion, and that if $(P, \mathcal{L})$ is good (so $j_! = j_*$), that $\mathrm{Hom}(Res^G_{L,P} A', A) = \mathrm{Hom}(A', Ind^G_{L,P} A)$ for all $A \in D_{K_P \times H_P}(L)$, $A' \in D_{K\times H}(G)$.

Proposition. $Res^G_{L,P} : (G,K,H)^\wedge_{\mathcal{L}} \to \bigoplus_{w\in W_*\backslash W}(L,K_P,H_P)^\wedge_{w^*\mathcal{L}}$.

PROOF.It is enough to show for all $A \in D^{\mathcal{L}}_{K\times H}((G/U\times G/U)/T)$ that $Res^G_{L,P}ch_G A$ has perverse constituents character sheaves on (L,K_P,H_P) with central characters of the form $w^*\mathcal{L}$. Hence (by base change) we are calculating $p_! q^* A$, where $L \xleftarrow{p} P\times G/B \xrightarrow{q} (G/U_B\times G/U_B)/T$, $pr_L(p) \leftarrow (p,gB) \mapsto (pgU_B, gU_B)T$.

Partition G/B into P-orbits, $P\backslash G/B \simeq W_*\backslash W$ which we identify with the elements of shortest length in each coset of W_*. For each such $w \in W_*\backslash W$, consider the diagram of $K_P\times H_P$ spaces

$$\begin{tikzcd} L & P\times PwB/B \arrow[l, "p"'] \arrow[r, "f"] & Z \arrow[r, "\alpha"] \arrow[d, "\gamma"] & (PwB/U_B\times PwB/U_B)/T \arrow[r, "i_w"] \arrow[d, "\delta_w"] & (G/U_B\times G/U_B)/T \\ & L & L\times L/B_* \arrow[l, "pr_1"'] \arrow[r, "\beta"] & (L/U_{B_*}\times L/U_{B_*})/T & \end{tikzcd}$$

where $B_* = pr_L({}^wB\cap P)$ is a Borel of L, $U_{B_*} = pr_L({}^wU_B\cap P)$ is its unipotent radical, $\beta(l_l, l_2B_*) = (l_1l_2U_{B_*})T$, $\delta_w(p_1wU_B, p_2wU_B) = (pr_L(p_1)U_{B_*}, pr_L(p_2)U_{B_*})$, $(\alpha f)(p_1,p_2wB) = (p_1p_2wU_B, p_2wU_B)T$, $(\gamma f)(p_1,p_2wB) = (pr_L(p_1), pr_L(p_2)U_{B_*}$. Here, Z is the fibre product of β with δ_w, and f is a locally trivial fibration with fibres isomorphic to $U_P\cap {}^wU_B$, an affine space of dimension $d = dim(U_P) - l(w)$.

We wish to understand $\Phi_w A = p_!(\alpha f)^* i^*_w A$. As $p = pr_1\gamma f$, and $f_! f^* A' = A'[-2d](-d)$ for $A' \in D(Z)$ (f is a locally trivial fibration with affine space fibres of dimension d), we have $\Phi_w A = (pr_1)_!\beta^*(\delta_w)_! i^*_w A[-2d](-d) = ch_L(\delta_{w!} i^*_w A)[-2d](-d)$. But it is clear that $\delta_{w!} i^*_w : D^{\mathcal{L}}_{K\times H}((G/U\times G/U)/T) \to D^{w^*\mathcal{L}}_{K_P\times H_P\times T}(L/U_{B_*}\times L/U_{B_*})$. So $\Phi_w A \in (L,K_P,H_P)^\wedge_{w^*\mathcal{L}}$.

The proposition is immediate from the long exact sequence in perverse cohomology.

REMARK. If $W'_{\mathcal{L}} \subset W_*$, then the long exact sequence in perverse cohomology associated to this partition degenerates into a sequence of short exact sequences.

We now consider a similar analysis of $Res^G_{L,P} Ind^G_{L,P}$. This is the functor $p_! q^* j_! \Gamma[d_P]$, where j, Γ are as in (2.1), and p, q are as in $L \xleftarrow{p} P\times G/P \xrightarrow{q} (G/U_P\times G/U_P)/L$. Again, partition G/P by P orbits which we index by the shortest length coset representatives in $W_*\backslash W/W_*$. We can then study the diagram

$$\begin{tikzcd} L & P\times PwP/P \arrow[l, "p"'] \arrow[r, "\alpha"] \arrow[d, "\gamma"] & (PwP/U_P\times PwP/U_P)/({}^{w^{-1}}P\cap L) \arrow[r, "i_w"] \arrow[d, "\delta_w"] & (G/U_P\times G/U_P)/L \\ & L\times L/P_* \arrow[r, "\beta"] & (L/U_{P_*}\times L/U_{P_*})/L_* & \end{tikzcd}$$

where $P_* = pr_L(P\cap {}^wP)$ is a parabolic of L, $P_* = L_*U_{P_*}$ with $L_* = L\cap {}^wL$, $U_{P_*} = L\cap {}^wU_P$. Then this diagram is again "almost" cartesian—$P\times PwP/P$ is a locally trivial fibration over the actual fibre product of β and δ_w, with fibres isomorphic to $U_P\cap {}^wU_P$. Thus $\gamma_!\alpha^* = \beta^*\delta_{w!}[-2d](-d)$, with $d = dim(U_P\cap {}^wU_P)$. So we need to understand $\delta_{w!} i^*_w$. This is rather complicated, and for our purposes the following will be enough.

Proposition. *Put* $\Phi_w = pr_{1!}(i_w\alpha)^* j_!\Gamma[d_P] : D_{K_P\times H_P}(L) \to D_{K_P\times H_P}(L)$. *Then* Φ_w *maps* $(L,K_P,H_P)^\wedge_{\mathcal{L}}$ *to* $\oplus_{v\in W_*wW_*}(L,K_P,H_P)^\wedge_{v^*\mathcal{L}}$. *Further,* Φ_1 *is the identity functor.*

PROOF.The last statement is clear. As to the first, it is enough to show that $\Phi_w\circ ch_L$ lands in the claimed category. This will follow if we show $hc_L\circ\Phi_w\circ ch_L$ maps $D^{\mathcal{L}}_{K_P\times H_P\times T}(L/U_{B_*}\times L/U_{B_*})$ to $\oplus_{v\in W_*wW_*} D^{v^*\mathcal{L}}_{K_P\times H_P\times T}(L/U_{B_*}\times L/U_{B_*})$, as by the definition of character sheaves, ch_L takes this last category to the desired category of character sheaves, and $\Phi_w\circ ch_L$ is a constituent of $ch_L hc_L \Phi_w ch_L$.

We omit the rest of the proof, as it merely consists of a routine sequence of base changes and the long exact sequence of a partition; the key point is the partition $PwP/B = \coprod_{v\in W_*wW_*} BvB/B$. The previous propositions give ample description of this technique.

As a consequence, we get the following (compare [L1,17.12]).

Theorem. *If $W'_{\mathcal{L}} \subset W_*$, and $(P,\mathcal{L})$ is good induction data, then*

$$Ind^G_{L,P} : (L, K_P, H_P)^\wedge_{\mathcal{L}} \to (G,K,H)^\wedge_{\mathcal{L}}$$

is an equivalence of categories with t-structures. In particular, the simple perverse sheaves correspond.

Proof. First, suppose W_*xW_* and W_*yW_* are distinct double cosets. Then $(L, K_P, H_P)^\wedge_{x^*\mathcal{L}}$, $(L, K_P, H_P)^\wedge_{y^*\mathcal{L}}$ are distinct. For, if not, $x^*\mathcal{L}$ and $y^*\mathcal{L}$ are in the same W_* orbit; say $w^*x^*\mathcal{L} = y^*\mathcal{L}$ for $w \in W_*$. Then $xw = vy$ for some $v \in W'_{\mathcal{L}} \subset W_*$, contradicting $W_*xW_* \neq W_*yW_*$. Hence $\mathrm{Ext}(\Phi_x A, \Phi_y A) = 0$, where Φ_w is as in the previous proposition, and the long exact sequence in perverse cohomolgy for ${}^pH^* Res^G_{L,P} Ind^G_{L,P}$ associated to the partition of $P \times G/P$ by P-orbits breaks up into a sequence of short exact sequences. In other words, $Res^G_{L,P} Ind^G_{L,P} A = \oplus_{w \in W_* \backslash W / W_*} \Phi'_w A$ (a direct sum) where Φ'_w is a map between the two categories of character sheaves in the proposition, and ${}^pH^i \Phi'_w A = {}^pH^i \Phi_w A$. However, for A split, we also have $\Phi'_1 A = \Phi_1 A$, as a complex A' with ${}^pH^i A' = 0$ for all $i \neq 0$ has $A' = {}^pH^0 A'$. We can now conclude that for A split $\mathrm{Hom}(Ind^G_{L,P} A', Ind^G_{L,P} A) = \mathrm{Hom}(A', Res^G_{L,P} Ind^G_{L,P} A) = \mathrm{Hom}(A', \oplus \Phi'_w A) = \mathrm{Hom}(A', A)$.

So $Ind^G_{L,P}$ is fully faithful on the heart $\mathcal{M}(L, K_P, H_P)^\wedge_{\mathcal{L}}$. But further, as we assume $(P,\mathcal{L})$ is good, if A is simple perverse, $Ind^G_{L,P} A$ is semisimple, and this calculation shows $Ind^G_{L,P} A$ is simple perverse, up to shift. But ${}^pH^j Ind^G_{L,P} A = {}^pH^{-j} Ind^G_{L,P} A$ (relative hard Lefschetz theorem); so in fact there must be no shift. From this we can conclude $Ind^G_{L,P}$ sends arbitrary perverse sheaves to perverse sheaves. Finally, if A' is a simple in $(G,K,H)^\wedge_{\mathcal{L}}$, then transitivity of induction shows it is a constituent of $Ind^G_{L,P} A$ for some simple A in $(L, K_P, H_P)^\wedge_{\mathcal{L}}$ (as $(P,\mathcal{L})$ is good). But then $Ind^G_{L,P} A$ is simple, so $Ind^G_{L,P} A = A'$.

This shows $Ind^G_{L,P}$ is an equivalence of categories between $\mathcal{M}(L, K_P, H_P)^\wedge_{\mathcal{L}}$ and $\mathcal{M}(G,K,H)^\wedge_{\mathcal{L}}$. We deduce the theorem from this and [B,1.4].

This result, together with the following results, reduces the classification of character sheaves with arbitrary central character to the classification of character sheaves with central character in a small finite set. This is unsatisfactory—the results of [L1] show that one should be able to reduce to the case $\mathcal{L} = \bar{\mathcal{Q}}_l$.

Remark. If $W'_{\mathcal{L}} \not\subseteq W_*$ we usually have $Ind^G_{L,P} A \neq Ind^G_{L,P'} A$, if P, P' are distinct parabolic subgroups with Levi subgroup L, and $A \in (G,K,H)^\wedge_{\mathcal{L}}$.

2.4 Let G^{der} be the derived group of G, $\mathcal{E}_0$ a rank one local system in $D_{K \times H}(G/G^{der})$. Then if $\mathcal{E}$ is the pullback of $\mathcal{E}_0$ under $G \to G/G^{der}$, and $\tilde{\mathcal{E}}$ the pullback of $\mathcal{E}_0$ under the map $(G/U_B \times G/U_B)/T \to G/G^{der}$, $(gU_B, hU_B)T \mapsto gh^{-1}G^{der}$ we have

Lemma [L1, 17.9]. *i) $A \mapsto A \otimes \mathcal{E}$ defines an equivalence of categories $(G,K,H)_{\mathcal{L}} \to (G,K,H)_{\mathcal{L}\otimes\mathcal{E}}$. ii) $A' \mapsto A' \otimes \tilde{\mathcal{E}}$ defines an equivalence of categories $D^{\mathcal{L}}_{K\times H}((G/U \times G/U)/T) \to D^{\mathcal{L}\otimes\mathcal{E}}_{K\times H}((G/U \times G/U)/T)$. iii) $ch_G(A') \otimes \mathcal{E} = ch_G(A' \otimes \tilde{\mathcal{E}})$, $hc_G(A) \otimes \tilde{\mathcal{E}} = hc_G(A \otimes \mathcal{E})$.*

Here, we've written $\mathcal{L} \otimes \mathcal{E}$ for $\mathcal{L} \otimes (\mathcal{E}|T)$.

2.5 Now suppose we are given two commuting involutions $\theta, \sigma : G \to G$, such that K (resp. H) is a subgroup of finite index in the fixpoints of θ (resp. σ), i.e. $(G^\theta)^0 \leq K \leq G^\theta$, $(G^\sigma)^0 \leq H \leq G^\sigma$. The following result allows us to apply the theorem.

Proposition. *Let $P = LU_P$ be any parabolic subgroup with Levi subgroup L, where $\theta L = L$, $\sigma L = L$. Let $\mathcal{L}$ be a local system with $W'_{\mathcal{L}} \subseteq W_*$. Then $(P,\mathcal{L})$ is good.*

Proof. This follows from the calculation of the perverse extensions of local systems in $D^{\mathcal{L}}_K(G/U)$. The case of $(G \times G, G)$ is due to Lusztig, in [L4]; the general case is due to Vogan.

3. Unipotent Character Sheaves

We would like to understand $(G,K,H)^\wedge_{\mathcal{L}}$; in particular to parameterise the simple objects. In this section we partition the character sheaves according to two sided cells in $K\backslash\mathcal{B}$. When the symmetric space is $(G\times G,G)$, we recover the partition of [L1]; our proof is simpler. For simplicity we state everything for $\mathcal{L}=\bar{\mathcal{Q}}_l$, i.e. for "unipotent character sheaves". The results however have (obvious) generalisations true at any central character.

We also state a conjecture about the parametrisation of character sheaves, after [L7].

We suppose given two commuting involutions $\theta,\sigma:G\to G$, and subgroups K, H such that K (resp. H) is a subgroup of finite index in the fixpoint set of θ (resp. σ), i.e. $(G^\theta)^0\le K\le G^\theta$, $(G^\sigma)^0\le H\le G^\sigma$. Also fix $\mathcal{L}=\bar{\mathcal{Q}}_l$ throughout this section.

3.1 We first recall the structure of $D_K^{\mathcal{L}}(G/U)\simeq D_K(\mathcal{B})$. The results are due to [LV], [V1] when the base field is $\mathbf{C}$, and are (almost certainly) true for arbitrary algebraically closed fields of characteristic different from 2, though no published proofs exist.

We define several orders on the simple perverse sheaves in $D_K^{\mathcal{L}}(G/U)$. Let $\mathcal{E}_1$, $\mathcal{E}_2$ be two irreducible local systems in $D_K^{\mathcal{L}}(G/U)$, supported on the orbits $\mathcal{O}_1$, $\mathcal{O}_2$ respectively, and write $\mathcal{E}_1^\sharp$, $\mathcal{E}_2^\sharp$ for their perverse extensions. Define $\mathcal{E}_1^\sharp\le\mathcal{E}_2^\sharp$ if their exists an i such that $\mathcal{H}^i\mathcal{E}_2^\sharp|\mathcal{O}_1$ contains $\mathcal{E}_1$ as a constituent. The partial order this induces (it must be made transitive in general) is called *$\mathcal{G}$-Bruhat order.*

Define $\mathcal{E}_1^\sharp\le_{LR}\mathcal{E}_2^\sharp$ if there exists a perverse sheaf $\mathcal{A}$ in $D_G(\mathcal{B}\times\mathcal{B})$ such that $\mathcal{E}_1^\sharp$ is a perverse constituent of $\mathcal{A}*\mathcal{E}_2^\sharp$. (In general, when $\mathcal{L}\ne\bar{\mathcal{Q}}_l$, replace $D_G(\mathcal{B}\times\mathcal{B})$ with the tensor category of (1.4)). The equivalence classes of simple perverse sheaves defined by the preorder $\le_{LR}$ are called (two-sided) cells. For a two sided cell $\mathbf{c}$, write $D^{\mathcal{L}}(\le\mathbf{c})$ for the subcategory generated by the perverse sheaves $\mathcal{E}_1^\sharp$ with $\mathcal{E}_1^\sharp\le_{LR}\mathbf{c}$. This category is stable under the Hecke algebra action $*$. (Recall that given a collection of simple perverse sheaves in $D_K(G/U)$, by the "subcategory they generate" we mean the full subcategory of $D_K(G/U)$ whose objects are those complexes with perverse constituents among the given collection.)

If $\mathcal{A}\in D^{\mathcal{L}}(\le\mathbf{c})$, but $\mathcal{A}\notin D^{\mathcal{L}}(\le\mathbf{c}')$, for all $\mathbf{c}'\le_{LR}\mathbf{c}$, $\mathbf{c}'\ne\mathbf{c}$, we say $\mathcal{A}$ is a complex *in the cell* $\mathbf{c}$.

These two orders $\le$, $\le_{LR}$ define two graphs on the simple objects in $D_K^{\mathcal{L}}(G/U)$, which may have many connected components. However,

Proposition [LV,V1]**.** *The connected components of the $\le_{LR}$ and of the $\le$ graph agree. Given a component, call the subcategory it generates a block. Then if B, B' are two distinct blocks, we have $Ext_{D_K(G/U)}(B,B')=0$. Conversely, a block is indecomposable—the graph on the simple objects of B given by joining $\mathcal{A}$ to $\mathcal{A}'$ if $Ext_{D_K(G/U)}(\mathcal{A},\mathcal{A}')\ne0$ is connected.*

In the diagonal case $(G\times G,G)$ there is only one block. Combinatorial descriptions of $\le$, two sided cells and the structure of blocks can be found in the works of Vogan. We summarise some of this information. Given a cell $\mathbf{c}$, we define a W-graph associated to $\mathbf{c}$, giving a representation of W with basis corresponding to the simple perverse sheaves in $\mathbf{c}$, as in [LV]. For each cell $\mathbf{c}$, its W-graph contains precisely one special representation of W, conjecturally with multiplicity one. All the representations carried by such a W-graph belong to a single two-sided cell of W. In particular, we have a map from cells $\mathbf{c}$ of $D_K(\mathcal{B})$ to two sided cells of $D_G(\mathcal{B}\times\mathcal{B})$. (Recall that in the case of $(G\times G,G)$, distinct cells carry distinct representations; and the partition of $W^\wedge$ so obtained is called the partition by two-sided cells of W).

3.2 We apply the above discussion to $(G\times G,K\times H)$. The partition of $D^{\mathcal{L}}_{K\times H}((G/U\times G/U)/T)$ into blocks gives a partition of $(G,K,H)^\wedge_{\mathcal{L}}$ into "flag manifold blocks"; this is well defined as $hc\circ ch(B')\subseteq B'$, for B' a block of $D^{\mathcal{L}}_{K\times H}((G/U\times G/U)/T)$. We also have $\mathrm{Ext}_{D_{K\times H}(G)}(\mathcal{A},\mathcal{A}')=0$ if $\mathcal{A}$, $\mathcal{A}'$ are in different flag manifold blocks of $(G,K,H)^\wedge_{\mathcal{L}}$. However, the flag manifold blocks of $(G,K,H)^\wedge_{\mathcal{L}}$ are not indecomposable. For example, the indecomposable summands of $G^\wedge_{\mathcal{L}}$ are parametrised by pairs $(L,\mathcal{E})$ (up to conjugacy), where L is a Levi subgroup of G and $\mathcal{E}$ is a cuspidal local system on L/Z_L^0 with unipotent central character [L1].

3.3 Write M for the mixed Grothendieck group of $D_{K\times H}(\mathcal{B}\times\mathcal{B})$, $\underline{H}$ for that of $D_G(\mathcal{B}\times\mathcal{B})$. $\underline{H}$ is an $\mathbf{Z}[q^{1/2},q^{-1/2}]$-algebra with respect to $*$, the *Iwahori-Hecke algebra*, and M is a $\underline{H}\otimes\underline{H}^{op}$-module. Then we have

$$hc\circ ch(m)=\sum_{w\in W}q^{-l(w)}T_w m T_{w^{-1}} \tag{3.3.1}$$

as a consequence of the proof of proposition (1.2) (the partition of Z in the proposition is a partition into $K\times H$-spaces). If we further identify M with $M_K\otimes M_H$, where M_K (resp. M_H) is the mixed Grothendieck group of $D_K(\mathcal{B})$ (resp. $D_H(\mathcal{B})$), then $h_1(m_1\otimes m_2)h_2^T=h_1m_1\otimes h_2m_2$, where ${}^T:\underline{H}\to\underline{H}^{op}$ is the algebra anti-automorphism $T_w\mapsto T_{w^{-1}}$, and $\underline{H}$ acts on M_K, M_H on the left as in [LV]. Then $hc\circ ch(m_1\otimes m_2)=\Delta(m_1\otimes m_2)$, where $\Delta=\sum_{w\in W}q^{-l(w)}T_w\otimes T_w\in\underline{H}\otimes\underline{H}$. Observe $h_1\otimes h_2\Delta=\Delta h_2\otimes h_1$.

For E an irreducible $\underline{H}$-module, let $r_E=\sum_{w\in W}q^{-l(w)}tr(T_w,E)T_{w^{-1}}\in\underline{H}$ (note $tr(T_w,E)\in\mathbf{Z}[q^{1/2},q^{-1/2}]$). Then r_E is central; if E, E' are distinct irreducible $\underline{H}$-modules $r_Er_{E'}=r_{E'}r_E=0$, and $r_E^2=(\sum q^{l(w)})D_E(q)^{-1}\dim(E)r_E$, where $D_E(q)$ is the formal degree of E. In particular, r_E acts as a non-zero scalar on E, and as 0 on E', for $E'\not\simeq E$; and the elements r_E form a basis for the center of $\underline{H}$. Also, if E occurs in the W-graph of a two sided cell $\mathbf{c}$ of W, then r_E represents a complex in the cell $\mathbf{c}$.

Now, as $ch(hm)=ch(mh)$ for all $m\in M, h\in\underline{H}$ (1.4), it is clear that on the level of mixed Grothendieck groups, ch factors $M\xrightarrow{triv}M^{triv}\hookrightarrow M\xrightarrow{ch}$ (Grothendieck group of character sheaves), where $M\to M^{triv}$, $m\mapsto m^{triv}$ is the projection onto the isotypical component of the trivial (Lie algebra) $\underline{H}$-module, with $\underline{H}$ acting on M by $ad(h)m=hm-mh$.

In particular, if $\mathcal{A},\mathcal{A}'\in D_{K\times H}(\mathcal{B}\times\mathcal{B})$ have $[\mathcal{A}]^{triv}=[\mathcal{A}']^{triv}$, where $[\mathcal{A}]$ denotes the image of $\mathcal{A}$ in the mixed Grothendieck group, then a character sheaf X appears with non-zero coefficient in $ch[\mathcal{A}]$ if and only if it appears with non-zero coefficient in $ch[\mathcal{A}']$.

If $\mathcal{A}$ is a simple perverse sheaf, then X is a constituent of $ch(\mathcal{A})$ if and only if it appears with non-zero coefficent in $[ch(\mathcal{A})]=ch[\mathcal{A}]$ (as by the decomposition theorem, $ch(\mathcal{A})$ is pure split semisimple).

As a consequence, if $m_1,\dots,m_r$ is a basis of M^{triv}, every unipotent character sheaf occurs with non-zero coefficient in $ch(m_i)$ for some i (compare [L1,14.12]), as we can always find perverse sheaves $\mathcal{A}_1,\dots,\mathcal{A}_s$ with $[\mathcal{A}_1]^{triv},\dots,[\mathcal{A}_s]^{triv}$ spanning M^{triv}.

The following special case of this will be used in section 7: If $\mathcal{A}$, $\mathcal{A}'$ are simple perverse sheaves in the cell $\mathbf{c}_1\boxtimes\mathbf{c}_2$, and the W-graph for this cell carries the representation $E\otimes E$, with E an irreducible W-module, then the set of constituents of $ch(\mathcal{A})$ are the same as the set of constituents for $ch(\mathcal{A}')$, modulo character sheaves that arise as constituents of $ch(\mathcal{C})$, for $\mathcal{C}$ in a cell strictly LR-smaller then $\mathbf{c}_1\boxtimes\mathbf{c}_2$. (This is immediate, as $(E\otimes E)^{triv}$ is one dimensional).

We also conclude

Proposition. *Suppose $\mathbf{c}_1\boxtimes\mathbf{c}_2$ is a cell of $D_{K\times H}(\mathcal{B}\times\mathcal{B})$, and $\mathbf{c}_1$, $\mathbf{c}_2$ are associated to distinct two sided cells of W. Then if $\mathcal{A}$ is a constituent of $ch(\mathcal{A}')$, with $\mathcal{A}'\in D^{\mathcal{L}}(\leq\mathbf{c}_1\boxtimes\mathbf{c}_2)$, there are cells $\mathbf{c}_1'$, $\mathbf{c}_2'$ with $\mathbf{c}_1'\boxtimes\mathbf{c}_2'\leq_{LR}\mathbf{c}_1\boxtimes\mathbf{c}_2$, $(\mathbf{c}_1',\mathbf{c}_2')\neq(\mathbf{c}_1,\mathbf{c}_2)$, and a complex $\mathcal{A}''\in D^{\mathcal{L}}(\leq\mathbf{c}_1'\boxtimes\mathbf{c}_2')$ such that $\mathcal{A}$ is a perverse constituent of $ch(\mathcal{A}'')$.*

PROOF.Let $M^{<\mathbf{c}_1\boxtimes\mathbf{c}_2}$ (resp. $M^{\leq\mathbf{c}_1\boxtimes\mathbf{c}_2}$) be the mixed Grothendieck group of the category generated by the perverse sheaves $\mathcal{E}^\sharp$ with $\mathcal{E}^\sharp\leq_{LR}\mathbf{c}_1\boxtimes\mathbf{c}_2$ (resp. $\mathcal{E}^\sharp\leq_{LR}\mathbf{c}_1\boxtimes\mathbf{c}_2$ and $\mathcal{E}^\sharp\not\simeq_{LR}\mathbf{c}_1\boxtimes\mathbf{c}_2$), and write $M^{\mathbf{c}_1\boxtimes\mathbf{c}_2}$ for the $\underline{H}\otimes\underline{H}^{op}$-module $M^{\leq\mathbf{c}_1\boxtimes\mathbf{c}_2}/M^{<\mathbf{c}_1\boxtimes\mathbf{c}_2}$. By assumption, $M^{\mathbf{c}_1\boxtimes\mathbf{c}_2}\simeq M^{\mathbf{c}_1}\otimes M^{\mathbf{c}_2}$ for $\underline{H}$-modules $M^{\mathbf{c}_1}$, $M^{\mathbf{c}_2}$ which carry disjoint representations of H. So $(M^{\mathbf{c}_1\boxtimes\mathbf{c}_2})^{triv}=0$, and $(M^{\leq\mathbf{c}_1\boxtimes\mathbf{c}_2})^{triv}=(M^{<\mathbf{c}_1\boxtimes\mathbf{c}_2})^{triv}$. The proposition follows from the discussion preceding it. (Note that although it is clear that $\Delta(M^{\leq\mathbf{c}_1\boxtimes\mathbf{c}_2})\subseteq M^{<\mathbf{c}_1\boxtimes\mathbf{c}_2}$, we cannot use this to prove the proposition).

3.4 Write $(G,K,H)^\wedge_{\mathcal{L},\leq\mathbf{c}}$ for the full subcategory of $(G,K,H)^\wedge_{\mathcal{L}}$ generated by the simple character sheaves $\mathcal{A}$ such that $hc(\mathcal{A})\in D^{\mathcal{L}}(\leq\mathbf{c})$. If $\mathcal{A}'\in(G,K,H)^\wedge_{\mathcal{L},\leq\mathbf{c}}$, then $hc(\mathcal{A}')\in D^{\mathcal{L}}(\leq\mathbf{c})$ (this follows from the distinguished triangles $({}^pH^a\mathcal{A}'[-a],\ {}^p\tau^{\geq a}\mathcal{A}',\ {}^p\tau^{>a}\mathcal{A}'))$.

We say $\mathcal{X} \in (G,K,H)^{\wedge}_{\mathcal{L},\leq \mathbf{c}}$ is a character sheaf *in the cell* $\mathbf{c}$ if $\mathcal{X} \notin (G,K,H)^{\wedge}_{\mathcal{L},\leq \mathbf{c}'}$ for all cells $\mathbf{c}' \leq_{LR} \mathbf{c}$ with $\mathbf{c}' \neq \mathbf{c}$. If $\mathcal{X} \in (G,K,H)^{\wedge}_{\mathcal{L}}$ is simple, then $\mathcal{X}$ is in the cell $\mathbf{c}$ if and only if $hc(\mathcal{X})$ is in the cell $\mathbf{c}$ of $D^{\mathcal{L}}_{K\times H}((G/U \times G/U)/T)$.

Theorem. *i) Let $\mathcal{X} \in (G,K,H)^{\wedge}_{\mathcal{L}}$ be a simple character sheaf. Then there exists a (unique) cell $\mathbf{c}$ with $hc(\mathcal{X})$ in the cell $\mathbf{c}$. This gives a partition of the simple character sheaves by two sided cells; $\mathcal{X}$ is in the cell $\mathbf{c}$ if and only if a) there exists a simple perverse sheaf $\mathcal{A} \in D^{\mathcal{L}}_{K\times H}((G/U\times G/U)/T)$ in the cell $\mathbf{c}$ with $\mathcal{X}$ a summand of $ch(\mathcal{A})$, and b) if $\mathcal{A}' \in D^{\mathcal{L}}_{K\times H}((G/U\times G/U)/T)$ is a simple perverse sheaf such that $\mathcal{X}$ is a summand of $ch(\mathcal{A}')$, then $\mathcal{A} \leq_{LR} \mathcal{A}'$. ii) There exists a character sheaf in the cell $\mathbf{c}$ if and only if $Hom_W(E_{\mathbf{c}_1}, E_{\mathbf{c}_2}) \neq 0$, where $\mathbf{c} = \mathbf{c}_1 \boxtimes \mathbf{c}_2$, and $E_{\mathbf{c}_i}$ is the representation of W carried by the W-graph of $\mathbf{c}_i$; i.e. if and only if the two sided cell of W associated to $\mathbf{c}_1$ is the same as the two sided cell of W associated to $\mathbf{c}_2$.*

Proof.First observe that $hc \circ ch(D^{\mathcal{L}}(\leq \mathbf{c})) \subseteq D^{\mathcal{L}}(\leq \mathbf{c})$. This follows from the proof of proposition (1.2) and the definition of $D^{\mathcal{L}}(\leq \mathbf{c})$ (see (3.3.1)). Then if $\mathcal{A} \in D^{\mathcal{L}}_{K\times H}((G/U \times G/U)/T)$ is a simple perverse sheaf in the cell $\mathbf{c}$ and $\mathcal{X}$ is a constituent of $ch(\mathcal{A})$, we have $hc(\mathcal{X}) \in D^{\mathcal{L}}(\leq \mathbf{c})$ (as $ch(\mathcal{A})$ is semisimple, so $hc(\mathcal{X})$ is a summand of $hc \circ ch(\mathcal{A})$). Now, if $\mathcal{X}$ is any simple character sheaf, $\mathcal{X}$ is a constituent of $ch(hc(\mathcal{X}))$, so $\mathcal{X}$ is a constituent of $ch(\mathcal{A})$ for some perverse constituent $\mathcal{A}$ of $hc(\mathcal{X})$. Thus $hc(\mathcal{X}) \in D^{\mathcal{L}}(\leq \mathbf{c})$, where $\mathbf{c}$ is the cell containing $\mathcal{A}$; and indeed $hc(\mathcal{X})$ is in the cell $\mathbf{c}$.

This proves (i), and (ii) is immediate from the preceding proposition.

In the case of the symmetric space $(G\times G, G, G)$, this coincides with the partition of [L1]: there $\mathcal{X}$ is in the cell $\mathbf{c}$ if and only if $\mathcal{X}$ occurs as a constituent of $ch(r_E)$ for some representation E in the W-graph of $\mathbf{c}$. As r_E is a complex in the cell $\mathbf{c}$, this coincides with our partition. However, Lusztig's description is more precise than ours: he essentially decomposes M as a $\underline{H} \otimes \underline{H}^{op}$-module in such a way that if M_1, M_2 are distinct summands, then $ch(m_1)$, $ch(m_2)$ have no constituents in common for all $m_1 \in M_1$, $m_2 \in M_2$. Such a decomposition is not possible in general—it already fails for (SL_2, SO_2, SO_2).

3.5 Suppose $k = \mathbf{C}$. We describe a cruder partition of the character sheaves than that into cells. We suppose that G is of adjoint type, and $K = G^\theta$, $H = G^\sigma$.

First recall the situation on the flag variety. If $\mathcal{A} \in D_G(\mathcal{B} \times \mathcal{B})$ is a perverse sheaf in the two sided cell $\mathbf{c}$, then $\mu(SS(\mathcal{A}))$ is the closure of a single nilpotent G-orbit on $\mathfrak{g}$ (embedded in $\mathfrak{g} \times \mathfrak{g}$ by $x \mapsto (x, -x)$). This nilpotent orbit is the special nilpotent associated by the Springer correspondence to the unique special representation in the W-graph of $\mathbf{c}$. In particular, there is a 1-1 correspondence between two sided cells of W and special nilpotents in $\mathfrak{g}$.

If now $\mathcal{A} \in D_K(\mathcal{B})$ is an irreducible perverse sheaf, $\mu(SS(\mathcal{A})) \subseteq N(\mathfrak{g}^{-\theta})$. It is not true that $\mu(SS(\mathcal{A}))$ is irreducible; however if $\mu(SS(\mathcal{A})) = \bar{\mathcal{O}}_1 \cup \cdots \cup \bar{\mathcal{O}}_m$ for some K-orbits $\mathcal{O}_i$ on $N(\mathfrak{g}^{-\theta})$ with m minimal, we have that $\mathcal{O}_1, \ldots, \mathcal{O}_m$ are all contained in a single G-orbit on $N(\mathfrak{g})$, and if the codimension of $\bar{\mathcal{O}}_i \setminus \mathcal{O}_i$ is at least two for some (equivalently any) $\mathcal{O}_i$, then $m = 1$; i.e. $\mu(SS(\mathcal{A}))$ is irreducible. Further, the G-orbits in $N(\mathfrak{g})$ obtained in this way are special nilpotent orbits. More precisely, if $\mathcal{A} \in D_K(\mathcal{B})$ is in the cell $\mathbf{c}$, then the G-orbit obtained from $\mu(SS(\mathcal{A}))$ corresponds to the two sided cell of W which corresponds to $\mathbf{c}$ [V2].

Also recall that if $\mathcal{A}_i \in D_K(\mathcal{B})$ is a perverse sheaf in the cell $\mathbf{c}_i$, $i = 1, 2$, and $\mathbf{c}_1 \leq_{LR} \mathbf{c}_2$, then $\mu(SS(\mathcal{A}_1)) \subseteq \mu(SS(\mathcal{A}_2))$, and hence if $\mathbf{c}_1 = \mathbf{c}_2$ that $\mu(SS(\mathcal{A}_1)) = \mu(SS(\mathcal{A}_2))$. However the maps from cells in $D_K(\mathcal{B})$ to subsets of K-orbits in $N(\mathfrak{g}^{-\theta})$ is not injective in general. (For example, if $(G, K) = (SL_3, SO_3)$ there are two cells which map to the subregular nilpotents; see (7.2)).

We further recall that if V is a $(\mathfrak{g}, \mathfrak{k})$-module, then $\mathrm{Ass}(V) = \mu(SS(\mathcal{V}))$, where $\mathcal{V}$ is the Beilinson-Bernstein localisation of V, a complex in $D_K(G/U)$.

Let us call a collection $\{\mathcal{O}_1, \ldots, \mathcal{O}_m\}$ a *special K-subset of nilpotent orbits* if there is some irreducible $\mathcal{A} \in D_K(\mathcal{B})$ with $\mu(SS(\mathcal{A})) = \bar{\mathcal{O}}_1 \cup \cdots \cup \bar{\mathcal{O}}_m$, m minimal.

Proposition. *Suppose $\mathcal{A}$ is an irreducible unipotent character sheaf, regarded as a sheaf on $G^\theta \backslash G$ (resp. on G/G^σ). Then $\mu(SS(\mathcal{A})) = \bar{\mathcal{O}}_1 \cup \cdots \cup \bar{\mathcal{O}}_m$ (resp. $\mu(SS(\mathcal{A})) = \bar{\mathcal{O}}'_1 \cup \cdots \cup \bar{\mathcal{O}}'_{m'}$) where*

$\{\mathcal{O}_1, \dots, \mathcal{O}_m\}$ (resp. $\{\mathcal{O}'_1, \dots, \mathcal{O}'_{m'}\}$) is a special H-subset of nilpotent orbits (resp. special K-subset). Moreover, $\mathcal{O}_1 \cup \dots \cup \mathcal{O}_m \cup \mathcal{O}'_1 \cup \dots \cup \mathcal{O}'_{m'}$ is contained in a single G-orbit in $N(\mathfrak{g})$.

Proof.Let M be the $\mathcal{D}(K\backslash G)$-module corresponding to the character sheaf $\mathcal{A}$. We can choose an irreducible finitely generated $\mathcal{U}(\mathfrak{g})$-submodule of $\Gamma(K\backslash G, M)$ (choose any f.g. submodule; it must contain an irreducible submodule). Now, as remarked in (1.7), it follows that $\mu(SS(\mathcal{A})) = \mathrm{Ass}(V)$. But V is an irreducible $(\mathfrak{g}, \mathfrak{h})$-module, so $\mathrm{Ass}(V)$ is a special H-subset. We can repeat this argument for G/H or $G \times G/G$. Doing it for $G \times G/G$ we obtain the last statement of the proposition.

Remark. $\mathcal{D}(K\backslash G) \simeq \oplus E_i$, for some finite dimensional $\mathfrak{g}$-modules E_i [G]. Hence we have a surjective homomorphism of $\mathfrak{g}$-modules, $\sum(E_i \otimes V) \to \Gamma(K\backslash G, M)$, where $V \subseteq \Gamma(K\backslash G, M)$ is an irreducible $\mathcal{U}(\mathfrak{g})$-module. Then $\mathrm{Ass}(E_i \otimes V) = \mathrm{Ass}(V)$, so one can deduce $\mathrm{Ass}(V') = \mathrm{Ass}(V)$ if $V' \subseteq \Gamma(K\backslash G, M)$ is another irreducible $\mathcal{U}(\mathfrak{g})$-submodule. Then [G,4.3.3] explains this phenomenon.

We consider the relation between this partition of character sheaves and the one into two sided cells, restricting attention to the case $(G \times G, G, G)$ out of sheer laziness.

Lemma. *Let $\mathcal{A} \in G^\wedge_{\mathcal{L}}$, regarded as a perverse sheaf on $G \times G$, via $g \mapsto (g^{-1}, g)$. Let $\mathcal{A}' \in D_G(\mathcal{B} \times \mathcal{B})$. Having thus identified T^*G as a subbundle of $T^*(G \times G)$ we have i) $\mu(SS(ch(\mathcal{A}'))) \subseteq \mu(SS(\mathcal{A}'))$ ii) $\mu(SS(hc(\mathcal{A}))) \subseteq \mu(SS(\mathcal{A}))$.*

The lemma is immediate from (1.6) and [MV,B2] (recall hc involves direct image for a non-proper map). As a consequence, if $\mathcal{A} \in G^\wedge_{\mathcal{L}}$, as $\mathcal{A}$ is a summand of $ch \circ hc(\mathcal{A})$, we get $\mu(SS(\mathcal{A})) \subseteq \mu(SS(ch \circ hc(\mathcal{A}))) \subseteq \mu(SS(hc(\mathcal{A}))) \subseteq \mu(SS(\mathcal{A}))$. As the map from cells to nilpotent orbits is injective, the partition defined in this way coincides with that defined in (3.4) (and hence with Lusztig's).

3.6 We suppose again that k is arbitrary.

Conjecture. *Given a cell $\mathbf{c} = \mathbf{c}_1 \boxtimes \mathbf{c}_2$ of $D^{\mathcal{L}}_{K \times H}((G/U \times G/U)/T)$ with $(G, K, H)^\wedge_{\mathcal{L}, \mathbf{c}} \neq 0$, let $\mathcal{G}_{\mathbf{c}}$ be the small finite group associated in* [L1] *to the two-sided cell associated to $\mathbf{c}_1$. Then we conjecture there exist subgroups $\mathcal{H}_{\mathbf{c}}$, $\mathcal{K}_{\mathbf{c}}$ of $\mathcal{G}_{\mathbf{c}} \times \mathcal{G}_{\mathbf{c}}$ such that the simple objects in $(G, K, H)^\wedge_{\mathcal{L}, \mathbf{c}}$ are in 1-1 correspondence with the irreducible $\mathcal{K}_{\mathbf{c}} \times \mathcal{H}_{\mathbf{c}}$-equivariant vector bundles on $\mathcal{G}_{\mathbf{c}} \times \mathcal{G}_{\mathbf{c}}$.*

For the symmetric space $(G \times G, G, G)$ this conjecture is precisely theorem [L1,17.8.3] in the beautiful reformulation of [L7].

4. Generic Character Sheaves

In this section K is a connected reductive group, $\theta : G \to G$ an involutory automorphism, and K a group of finite index in the fixpoints of θ; $(G^\theta)^0 \leq K \leq G^\theta$. We also fix a maximal θ-split torus A, and a maximal torus T containing A. Write $L = Z_G(A)$, and W_* for the Weyl group of L. Also write $T_K = K_T = T \cap K$.

A local system $\mathcal{L}$ on T such that $W'_{\mathcal{L}}$ is contained in the Weyl group of L (we write this as $W'_{\mathcal{L}} \subseteq L$) and $D^{\mathcal{L}}_{K_T \times K_T}(T)$ is non-empty is called *generic*. Throughout this section we suppose fixed a generic local system $\mathcal{L}$. A character sheaf in $(G, K)^\wedge_{\mathcal{L}}$ is also called *generic*.

4.1 Let $P = LU_P$ be any parabolic with Levi subgroup L. The results of section 2 imply that, for $\mathcal{L}$ generic,

$$Ind^G_{L,P} : (L, K_L)^\wedge_{\mathcal{L}} \to (G, K)^\wedge_{\mathcal{L}}$$

is an equivalence of categories, sending simple perverse sheaves to simple perverse sheaves.

The following lemma, applied to (L, K_L), gives a complete description of $(L, K_L)^\wedge_{\mathcal{L}}$.

Lemma. *Let (G, K) be such that $G^{der} \subseteq K$, so $D_{K \times K}(G) \simeq D_K(T/T_K)$, where K acts trivially on T/T_K. Then for $\mathcal{A} \in D^{\mathcal{L}}_{K_T \times K_T}(T)$, $ch_G(\mathcal{A}) = Ind^G_{T,B}(\mathcal{A}) = H^*(\mathcal{B}) \otimes \dot{\mathcal{A}}$, where the image of*

$\dot{\mathcal{A}}$ under the forgetful functor $D_K(T/T_K) \to D_{K_T}(T/T_K)$ is $\mathcal{A}$, and $H^*(\mathcal{B})$ is the pure complex of constant sheaves given by $p^*p_!\bar{\mathbb{Q}}_l$, where $p : \mathcal{B} \to pt$ is the projection to a point.

As a consequence, if B is any Borel contained in P, $A \subset B$, then $Ind^G_{T,B} = Ind^G_{L,P} \circ Ind^L_{T,B\cap L} = H^*(\mathcal{B}_L) \otimes Ind^G_{L,P}$, and ${}^pH^0 Ind^G_{T,B} : D^{\mathcal{L}}_{K_T\times K_T}(T) \to (G,K)^\wedge_{\mathcal{L}}$ takes simple perverse sheaves to simple character sheaves bijectively. In particular, there are $|T_K/T^0_K|^2$ simple character sheaves with central character $\mathcal{L}$, for $\mathcal{L}$ generic.

4.2 Theorem. *The restriction of a generic character sheaf $\mathcal{A}$ to the strata of regular semisimple elements of X (i.e. to the $g \in G$ such that $Z^0_G(g^{-1}\theta g)$ is conjugate to L) is a local system of rank $|W(A,K)|$. Moreover, if (G,K) is split, then*

$$\sum_i (-1)^i \dim \mathcal{H}^i_g\mathcal{A} = (-1)^{\dim G} \sum_i (-1)^i \dim H^{2i}_c(\mathcal{B}_{g^{-1}\theta g})$$

We will determine the local system explicitly below. Also, if (G,K) is not split, we will obtain more precise information about the stalks of generic character sheaves by reducing to the split case.

Proof.As we know character sheaves are constructible with respect to the Lusztig stratification, the first statement is equivalent to showing $\sum_i(-1)^{i+\dim G}\dim\mathcal{H}^i_g\mathcal{A} = |W(A,K)|$, for g in the generic strata. As the irreducible character sheaves in $(L,K_L)^\wedge_{\mathcal{L}}$ are rank one local systems with finite monodromy, this alternating sum is (up to sign) the Euler characteristic of ${}^{g^{-1}}KP/P \cap KP/P$, where P is any parabolic subgroup with Levi subgroup L. (By the definition of $Ind^G_{L,P}$, and (4.1)). Choose a parabolic P so $P \cap \theta P = L$. Then a parabolic $P_1 \in G/P$ is in the K-orbit of P if and only if P_1 is opposed to θP_1.

Write $P' = \theta P \in G/P$, and put $Z = \{(x, P_1, P_2) \in G\times G/P\times G/P \mid P_1 \text{ opposed to } P_2, P_2 \text{ opposed to } {}^{x^{-1}}P_1\}$, $\hat{Z} = \{(x, h_1U_P, h_2U_{P'}) \in G \times G/U_P \times G/U_{P'} \mid h_1^{-1}h_2 \in PP', h_2^{-1}xh_1 \in P'P\}$. Define maps $G \xleftarrow{pr_1} Z \xleftarrow{\pi} \hat{Z} \xrightarrow{\alpha} L$, where pr_1 is the first projection, π is the obvious morphism making $\hat{Z} \to Z$ an $L \times L$-bundle, $\alpha(x, h_1U_P, h_2U_{P'}) = \beta_1(h_1^{-1}h_2)\beta_2(h_2^{-1}xh_1)$, where $\beta_1 : PP' \to L$, $ulu' \mapsto l$, and $\beta_2 : P'P \to L$, $u'lu \mapsto l$. Also define involutions $\psi : G \to G$, $g \mapsto \theta g^{-1}$; $\psi : \hat{Z} \to \hat{Z}$, $(x, h_1U_P, h_2U_{P'}) \mapsto (\theta x^{-1}, \theta h_2U_P, \theta h_1U_{P'})$, $\psi : Z \to Z$, $(x, P_1, P_2) \mapsto (\theta x^{-1}, \theta P_2, \theta P_1)$, $\psi : L \to L$, $g \mapsto \theta g^{-1}$. Then ψ intertwines pr_1, π, α; i.e. this is a diagram of $\mathbf{Z}/2$-spaces.

Then $G^\psi = X$, and the fibres of the the map $Z^\psi \to X$ at $x = g^{-1}\theta g \in X$, which we will denote Z^ψ_x, are isomorphic to ${}^{g^{-1}}KP/P \cap KP/P$. Also observe $\alpha(Z^\psi) \subseteq Z^0_L$.

Let $\dot{\mathcal{L}}$ be the local system on Z^0_L obtained by restricting $\mathcal{L}$ from T to Z^0_L; regard $\dot{\mathcal{L}}$ as a complex on L by extension by zero. It is clear that $\psi^*\dot{\mathcal{L}} \simeq \dot{\mathcal{L}}$, so if $\ddot{\mathcal{L}}$ denotes the unique G-equivariant local system on Z such that $\pi^0\ddot{\mathcal{L}} = \alpha^0\dot{\mathcal{L}}$, we have $\sum(-1)^i tr(\psi, \mathcal{H}^i_x((pr_1)_!\ddot{\mathcal{L}})) = \sum(-1)^i tr(1, H^i_c(Z^\psi_x))$ which is the Euler characteristic we are trying to calculate.

Now,

$$(pr_1)_!\ddot{\mathcal{L}} \simeq Ind^G_L\dot{\mathcal{L}}, \tag{4.2.1}$$

where Ind^G_L is the induction functor of conjugation equvariant sheaves on the group G [L1,4.1]. Assuming this for the moment, we proceed as follows. The complex $Ind^G_L\dot{\mathcal{L}}$ is equal to the intersection cohomology extension of $p_!q^*\mathcal{L}$, where $G \xleftarrow{p} \dot{Z} = \{(g, hL) \mid h^{-1}gh \in Z^{0\,reg}_L\} \xrightarrow{q} Z^0_L$, $q(g, hL) = h^{-1}gh$, $p(g, hL) = g$ [L2,2.6], and p is a $N_G(L)/L$ Galois cover, with action $(g, hL).w = (g, hwL)$, $w \in N_G(L)/L$.

Define actions $\psi : \dot{Z} \to \dot{Z}$, $(g, hL) \mapsto (\theta g^{-1}, \theta hL)$, $\psi : Z^0_L \to Z^0_L$, $z \mapsto \theta z^{-1}$. Again ψ intertwines p, q and the $N_G(L)/L$ action on $\dot{Z}$ commutes with the action of ψ.

Now if x is regular semisimple in X, $((pr_1)_!\ddot{\mathcal{L}})_x \simeq (p_!q^*\dot{\mathcal{L}})_x$ as $\langle\psi\rangle$-modules, and so $\sum tr(\psi, \mathcal{H}^*_x(p_!q^*\mathcal{L})) = |(N_G(L)/L)^\theta|$, the fixpoints of θ on $N_G(L)/L$. But $W(A,K) = N_{K^0}(A)/Z_{K^0}(A) = (N_G(L)/L)^\theta$, and so the first assertion of the theorem holds.

For the second assertion, we assume (G,K) is split, i.e. $T = L$, $\dot{\mathcal{L}} = \mathcal{L}$, $W(A,K) = W(T,G)$. As $\mathcal{L}$ is a local system with finite monodromy, $tr(\psi, (Ind_T^G \mathcal{L})_x) = tr(\psi, (Ind_T^G \bar{\mathcal{Q}}_l)_x)$. Choose a rational structure on G so that T becomes an F_q-split torus: $FT = T$ and $Ft = t^q$ for all $t \in T^F$. Then there is a canonical isomorphism $\phi : F^* Ind_T^G \bar{\mathcal{Q}}_l \tilde{\rightarrow} Ind_T^G \bar{\mathcal{Q}}_l$, which comes from the obvious isomorphism $F^* \bar{\mathcal{Q}}_l \tilde{\rightarrow} \bar{\mathcal{Q}}_l$. We thus get isomorphisms $(\psi F^i)^* Ind_T^G \bar{\mathcal{Q}}_l \tilde{\rightarrow} Ind_T^G \bar{\mathcal{Q}}_l$; write $tr(\psi F^i, Ind_T^G \bar{\mathcal{Q}}_l)$ for the characteristic function on $G^{\psi F^i}$ so obtained.

We will calculate $tr(\psi, Ind_T^G \bar{\mathcal{Q}}_l)$ by calculating $tr(\psi F^i, Ind_T^G \bar{\mathcal{Q}}_l)$ for all i; $tr(\psi, Ind_T^G \bar{\mathcal{Q}}_l) = -\lim_{t\to\infty} \sum tr(\psi F^i, Ind_T^G \bar{\mathcal{Q}}_l) t^i$.

Now, ψF is the Frobenius morphism for a rational structure on G with $G^{\psi F}$ not a group—$\psi F = F' \circ i$, where $F'(x) = F\theta(x)$ is a group homomorphism, and $i(x) = x^{-1}$. We can consider $G^{\psi F}$ as the symmetric space $(G \times G, G)$ with the Frobenius $(x,y) \mapsto (F'y, F'x)$. Then the argument sketched here is repeated in (7.4.5–7.4.9) below, perhaps with more detail. Regardless, the results of [L1,8-10,24] on the properties of the characteristic functions of character sheaves require very little modification to this twisted situation—we have for example that

$$\sum_{u \in G^{\psi F}} tr(\psi F, (Ind_{T_1}^G \bar{\mathcal{Q}}_l)_u)^2 = |T_1^{F'}|^{-2} |N_G(T_1)^{F'}| |G^{F'}| \tag{4.2.2}$$

for T_1 any ψF-stable maximal torus.

We also have an analogue of the character formula [L1,8.5]. As a consequence of this, and the actual [L1,8.5] applied to $tr(F, Ind_T^G \bar{\mathcal{Q}}_l)$, we see that if we can prove the second assertion of the theorem for $g^{-1}\theta g = x$ unipotent, it will follow for general g.

To do this, we use the fact that the "orthogonality relations" between the constituents of $Ind_T^G \bar{\mathcal{Q}}_l$ determine their characteristic function [L1,24.4]. Let i, i' denote two ψF-stable irreducible constituents of $Ind_T^G \bar{\mathcal{Q}}_l$, $\omega_{i,i'}^{(\psi F)}$ their inner product with respect to ψF $(= \sum_{u \in G_{\text{uni}}^{\psi F}} X_i(u) \tilde{X}_{i'}(u)$, where X_i is the characteristic function of i, $\tilde{X}_{i'}$ that of $Di')$, and let $\omega_{i,i'}$ denote the inner product of i, i' with respect to the split Frobenius F [L1,24.2–24.3]. It is then clear from (4.2.2) that $\omega_{i,i'}^{(\psi F)} = (-1)^{\dim T} \omega_{i,i'}|_{q \mapsto -q}$.

This means that if $\psi u = u$, $Fu = u$, there exists some u' in the same conjugacy class as u, with $Fu' = u'$, and $tr(\psi F, (Ind_T^G \bar{\mathcal{Q}}_l)_u) = (-1)^{\dim G} tr(F, (Ind_T^G \bar{\mathcal{Q}}_l)_{u'})|_{q \mapsto -q}$ (Ennola duality). The second assertion of the theorem follows.

It remains to prove (4.2.1). In the case (G,K) is split (or even quasi-split), $\dot{\mathcal{L}}$ is a character sheaf on $L = T$, $W'_{\mathcal{L}} = 1$, and (4.2.1) is immediate from [L1,2.12-2.15]. In general, $\dot{\mathcal{L}}$ is not a character sheaf on L (in the language of [L2] it is a "quasi-admissible" complex). Nonetheless, the proof of [L3, 4.6] applies word for word to give (4.2.1), with the following modifications. Instead of [L3,4.7b] one should use

Let Ω be a good $P - P$ double coset such that for some $y \in \Omega$, $P \cap {}^yP$ contains a common Levi subgroup conjugate to L. ($L = Z_G(A)$, P as above). Then if $\dim(P\backslash\Omega) > 0$, there exists a sequence $\Omega^1, \ldots, \Omega^{m-1}$ of elementary $P - P$ double cosets such that $\Omega^1 * (\Omega^2 * (\cdots * \Omega^{m-1})) \cdots)$ is defined and equal to Ω.

This also follows from [H] because of the assumptions on L. The point is that for an elementary good $P - P$ double coset Ω', $\int_{Z_g} \phi^* \dot{\mathcal{L}} = 0$ (notation as in [L3,3.5], applied to the group generated by ${}^yP$, P, $y \in \Omega'$; with $g \in \langle {}^yP, P \rangle$).

4.3 It is clear that the preceding result is significantly simpler when (G,K) is split. We now describe a reduction to the split case.

Choose a Borel B' containing T, so $B' \cap \theta B'$ has maximal possible dimension. Then the dense K-orbit on $\mathcal{B}$ is a vector bundle over the orbit KB'/B', and there is a unique minimal θ-stable parabolic P' containing B'. Then $P' = L'U_{P'}$, $T \subseteq L'$ with $L'/Z_{L'}$ split, $\theta P' = P'$. Note L' is usually different from L above; it is the Levi subgroup with root system the real roots of T, whereas L is the Levi subgroup with roots the compact imaginary roots of T. Also, in general we cannot find a Borel subgroup $B_0 \supset T$ such that L, L' are both standard Levi subgroups with respect to B_0; i.e. we cannot find parabolics $P_0 = LU_{P_0}$, $P'_0 = L'U_{P'_0}$ with $P_0 \cap P'_0$ containing a Borel subgroup of G.

Despite this, we still have

$$ {}^pH^0 Ind^G_{T,B'} : D^{\mathcal{L}}_{K_T \times K_T}(T) \to (G,K)^{\wedge}_{\mathcal{L}} $$

takes simple perverse sheaves to simple character sheaves bijectively.

Indeed, write $Ind^G_{T,B'} = Ind^G_{L',P'} \circ Ind^{L'}_{T,B\cap L'}$, and observe that $Ind^{L'}_{T,B\cap L'} : D^{\mathcal{L}}_{K_T\times K_T}(T) \to (L', K\cap L')^{\wedge}_{\mathcal{L}}$ is an equivalence of categories, as $W'_{\mathcal{L}} \subseteq W_*$, and $W_* \cap L' = 1$. As L' is essentially split, we need to study $Ind^G_{L',P'}$.

As KP'/P' is a closed orbit in G/P', $(P',\mathcal{L})$ is good induction data for any local system $\mathcal{L}$; i.e. $Ind^G_{L',P'}$ takes semisimple complexes to semisimple complexes.

Lemma. *Fix $x \in A$, put $Y_x = \{hP' \in KP'/P' \mid x \in {}^hP'\}$. i) $Y_x = \bigcup_{w\in W(A,K)} Z_{K^0}(x)wP'/P'$ ii) If $x \in A^{reg}$, i.e. if $Z^0_G(x) = L$, then $Z_{K^0}(x)wP'/P' \simeq \mathcal{B}_L$, and so Y_x is the disjoint union of $|W(A,K)/W(A,K)\cap L'|$ copies of $\mathcal{B}_L$. iii) If $g^{-1}\theta g = x \in A$, then $Y_x \simeq \{hP' \in KP'/P' \mid ghP' \in KP'/P'\}$*

Proof.In (i), it is clear that the right hand side is in Y_x. Conversely, suppose $x \in {}^hP'$, $h \in K$. Then $h^{-1}xh$ is contained in some maximal θ-split torus of P'; all such are conjugate to $A \subseteq P'$ by some element $k \in K\cap P'$, so for such a k, $k^{-1}h^{-1}xhk \in A$. Two elements of A which are conjugate are conjugate under some element of $N_K(A)$, so $hk \in Z_K(x)N_K(A)$. Hence $h \in Z_K(x)N_K(A)(K\cap P') = Z_{K^0}(x)N_{K^0}(A)(K\cap P')$, as $K = K^0(K\cap A)$.

For (ii), $L \supseteq Z_{K^0}(x) \supseteq L^{der}$, so $Z_{K^0}(x)wP' = LwP'$. So the result follows if we know $L\cap {}^wP'$ is a Borel of L; which is equivalent to $L\cap P'$ being a Borel of L (as ${}^wL = L$) which is clear ($L\cap P'$ is a parabolic of L with Levi subgroup $L\cap L' = T$).

Finally, in (iii), write $(G/P')^\theta = \mathcal{O}_1 \coprod \mathcal{O}_2$, where $\mathcal{O}_1 = KP'/P'$ and $\mathcal{O}_2$ is a (possibly empty) collection of K-orbits, all of which are closed. Put $Y^i_x = \{hP' \in KP'/P' \mid ghP' \in \mathcal{O}_i\}$, $i = 1,2$. It is clear that $Y^1_x \coprod Y^2_x = Y_x$, as $\theta({}^{gh}P') = {}^{gh}P' \iff x \in {}^hP'$, and that each Y^i_x is a union of connected components of Y_x. But we can write $x = t^{-1}\theta t$, with $t\in A$, $gt^{-1} \in K$, and then $twP = wP \in \mathcal{O}_1$, for $w \in K$. So, by (i), each component of Y_x lies in Y^1_x, so $Y_x = Y^1_x$.

Recall that by the regular semisimple elements of (G,K), denoted X^{rss}, (respectively the regular semisimple elements of $(L', K\cap L')$, denoted $X^{\text{rss}}_{L'}$) we mean those $g\in G$ (resp. $g \in L'$) such that $g^{-1}\theta g$ is semisimple and $Z_G(g^{-1}\theta g)$ has minimal dimension. It follows from the lemma and the definition of $Ind^G_{L',P'}$ that if $\mathcal{A} \in (L', K\cap L')^{\wedge}_{\mathcal{L}}$ is a simple character sheaf, then $({}^pH^0 Ind^G_{L',P'}\mathcal{A})|X^{\text{rss}}$ is obtained as if from an unramified $|W(A,K)/W(A,K)\cap L'|$ cover of X^{rss}, by inducing the local system $\mathcal{A}|X^{\text{rss}}_{L'}$ (which is of rank $|W(A,K)\cap L'|$).

As we know the generic character sheaves are perverse extensions of their restrictions to the regular semisimple elements, and that these restrictions are local systems of rank $|W(A,K)|$, it follows that ${}^pH^0 Ind^G_{L',P'}\mathcal{A}$ is irreducible. Hence ${}^pH^0 Ind^G_{T,B'} : D^{\mathcal{L}}_{K_T\times K_T}(T) \to (G,K)^{\wedge}_{\mathcal{L}}$ takes simple perverse sheaves to simple perverse sheaves bijectively.

Alternately, if we explicitly know the monodromy representation of $\mathcal{A}|X^{\text{rss}}_{L'}$, we know that of ${}^pH^0 Ind^G_{L',P'}\mathcal{A}$, and we can verify that it is irreducible directly. This will in fact follow just from knowing $\mathcal{A}|X^{\text{rss}}_{L'}$ as a representation of $\pi_1(T)$ (see below). This is our claimed reduction to the split case.

4.4 We assume (G,K) is split; T a maximal θ-split torus, $B = TU$ a Borel containing it, $W(A,K) \simeq W$. We will calculate the monodromy representation of the local system obtained by restricting a generic character sheaf to the regular semisimple elements of X.

The following refinement of the results of (2.3) in this case is the key lemma in this calculation; it is a slight variant of the key point of [L8], that the Euler characteristic of $KB/B \cap Uw_0B/B$ is $(-1)^{\dim U}$ [L8,5.4] (where w_0 is the longest element in W).

Identify $D^{\mathcal{L}}_{K_T\times K_T}(T) \simeq D^{\mathcal{L}}_{K\times K}(KB/U \times KB/U)$ via Γ as in (2.1).

Lemma. *If $\dot{\mathcal{L}} \in D^{\mathcal{L}}_{K_T \times K_T}(T)$ is a simple local system, $\mathcal{L}$ generic, we have*

$$hc \circ ch(\dot{\mathcal{L}}) \simeq Res^G_{T,B} Ind^G_{L,P} \dot{\mathcal{L}} \simeq \oplus_{w \in W} w^* \dot{\mathcal{L}}$$

PROOF. We use the notation of proposition 1.2. As $\mathcal{L}$ is generic, $hc \circ ch(\dot{\mathcal{L}}) \simeq \oplus_{w\in W}(q'_{1w})_!(q'_{2w})^* \dot{\mathcal{L}}[2\dim U - 2d_w](-d_w)$, where q'_{iw} is as in (1.2), $d_w = \dim(U \cap {}^{w^{-1}}U)$, and $(q'_{1w})_!(q'_{2w})^*\dot{\mathcal{L}} \in D^{w*\mathcal{L}}_{K\times K}((G/U \times G/U)/T)$. So the lemma follows if we show $(q'_{1w})_!(q'_{2w})^*\dot{\mathcal{L}} = w^*\dot{\mathcal{L}}[2d_w - 2\dim U](d_w - \dim U)$. (Note that it is clear that this complex is supported on $KB/U \times KB/U$).

This follows by induction on $l(w)$; writing Z'_w as a bundle over Z'_{ws} with fibres isomorphic to Z'_s in the usual way, where $l(ws) = l(w) - 1$, $l(s) = 1$. (Roughly, $(q'_{1w})_!(q'_{2w})^*\dot{\mathcal{L}} = T_w \dot{\mathcal{L}} T_{w^{-1}}$). It then suffices to understand $(q'_{1s})_!(q'_{2s})^*\dot{\mathcal{L}}$ for $s \in W$ simple. As KB/B is the open orbit, and (G, K) is split, this reduces to a calculation in (SL_2, SO_2), and ultimately to the computation of $H^*_c(\mathbf{P}^1\backslash 3 \text{ points}, \tilde{\mathcal{L}})$, where $\tilde{\mathcal{L}}$ is a local system on $\mathbf{P}^1\backslash 3$ points with non-trivial monodromy around each of the points. We omit the details.

Corollary. *If $T \subset L_1$ is any Levi subgroup, $B \subset P_1$ a parabolic with Levi subgroup L_1, W_1 the Weyl group of L_1, and $\dot{\mathcal{L}} \in D^{\mathcal{L}}_{K_T\times K_T}(T)$ a simple local system with $\mathcal{L}$ generic; then*

$$Res^G_{L_1,P_1} Ind^G_{T,B} \dot{\mathcal{L}} \simeq \oplus_{w\in W_1\backslash W} Ind^{L_1}_{T,B\cap L_1} w^* \dot{\mathcal{L}}$$

PROOF. If $\mathcal{A} \in (G,K)^\wedge_{\mathcal{L}}$ is a generic character sheaf such that $hc(\mathcal{A}) \simeq \oplus w^*\dot{\mathcal{L}}$, then $\mathcal{A} \simeq ch(\dot{\mathcal{L}})$, by the lemma. The corollary is then immediate from the first proposition of (2.3), transitivity of restriction, and the lemma.

5. Relation to the characters of $K^F\backslash G^F/K^F$

If G^F is a finite group, and K^F is subgroup, one can consider the algebra $K^F\backslash G^F/K^F$ of double cosets, equivalently the algebra of $K^F \times K^F$-equivariant functions from G^F to $\bar{\mathcal{Q}}_l$, with convolution as multiplication. It is well known that the characters of this algebra are obtained by averaging the characters of G^F: if χ is a character of G^F, then $g \mapsto \sum_{k\in K^F} \chi(gk)$ is either 0 or a character of $K^F\backslash G^F/K^F$, and all characters are obtained in this way.

For this reason it is natural to study the behaviour of character sheaves on G when averaged by K.

5.1 Consider the map $\pi : G \to K\backslash G$, $g \mapsto Kg$. This is a K-equivariant map, where K-acts on G by conjugation and on $K\backslash G$ by right multiplication. We thus get a functor $\pi_! : D_G(G) \to D_K(K\backslash G)$ by first mapping $D_G(G) \to D_K(G)$ via the forgetful functor.

Proposition. $\pi^0\pi_! : G^\wedge_{\mathcal{L}} \to (G,K)^\wedge_{\mathcal{L}}$

PROOF. The proposition is immediate over $\mathbf{C}$, as a consequence of (1.7). In general, it is enough to show $\pi^0\pi_! ch'(\mathcal{A}) \in (G,K)^\wedge_{\mathcal{L}}$ for $\mathcal{A} \in D^{\mathcal{L}}_G((G/U \times G/U)/T)$ (where ch' refers to the functor ch for $(G \times G, G)$). By base change, this is equivalent to showing $(p_1)_! q_1^* \mathcal{A} \in (G,K)^\wedge_{\mathcal{L}}$, where $G \xleftarrow{p} K \times G \times \mathcal{B} \xrightarrow{q} (G/U \times G/U)/T$, $p(k,g,hB) = g$, $q(k,g,hB) = (kghU, hU)T$. We "factor" this into the following diagram

$$\begin{tikzcd}
 & K \times G \times \mathcal{B} \arrow[r, "q_2"] \arrow[d, "\pi_1"] & K \times (G/U \times G/U)/T \arrow[r, "\alpha"] \arrow[d, "\beta"] & (G/U \times G/U)/T \\
G & G \times \mathcal{B} \arrow[l, "p"'] \arrow[r, "q"] & (G/U \times G/U)/T &
\end{tikzcd}$$

where p, q are the maps in the definition of ch_G; $\pi_1(k,g,hB) = (g,hB)$, $\beta(k,(g_1U,g_2U)T) = (g_1U, g_2U)T$, $q_2(k,g,hB) = (k,(ghU,hU)T)$, $\alpha(k,(g_1U,g_2U)T) = (kg_1U, g_2U)T$.

Then $p_1 = p\pi_1$, $q_1 = \alpha q_2$, and the diagram is Cartesian, so $(p_1)_! q_1^* \mathcal{A} = ch_G(\beta_!\alpha^*\mathcal{A})$. But it is clear that if $\mathcal{A} \in D^{\mathcal{L}}_G((G/U \times G/U)/T)$, then $\beta_!\alpha^*\mathcal{A} \in D^{\mathcal{L}}_{K\times K}((G/U \times G/U)/T)$. The proposition is immediate.

Let us write $Av = \beta_! \alpha^* : D^{\mathcal{L}}_G((G/U \times G/U)/T) \to D^{\mathcal{L}}_{K\times K}((G/U \times G/U)/T)$. We have shown $\pi^0 \pi_! ch'(\mathcal{A}) = ch_G(Av(\mathcal{A}))$. It is easy to check that Av intertwines the (left and right) Hecke algebra actions of (1.4). As a consequence of this and (3.4) we get

Corollary. *If $\mathcal{X} \in G^\wedge_{\mathcal{L},\le \mathbf{c}}$ is a character sheaf in the two sided cell $\mathbf{c}$ of W, then the perverse constituents of $\pi^0\pi_!\mathcal{X}$ belong to cells in $K\backslash\mathcal{B} \times K\backslash\mathcal{B}$ whose associated two sided cells of W are LR-smaller ($\le_{LR}$) than $\mathbf{c}$.*

If the base field is $\mathbf{C}$, it is also clear that the nilpotents attached to $\pi^0\pi_!\mathcal{A}$ are in the closure of the nilpotent orbits in $N(\mathfrak{g}^{-\theta})$ associated to $\mathbf{c}$.

Also, if we do this for the symmetric space $(G\times G, G)$ then $\pi_!(\mathcal{A}_1 \boxtimes \mathcal{A}_2) = \mathcal{A}_1 * i^*\mathcal{A}_2$, where $\mathcal{A}_1, \mathcal{A}_2 \in G^\wedge_{\mathcal{L}}$, and $i : G \to G$, $g \mapsto g^{-1}$ is the inverse map.

We calculate $\pi^0\pi_! Ind^G_T\mathcal{L}$ more explicitly, where $Ind^G_T\mathcal{L} \in G^\wedge_{\mathcal{L}}$ is the character sheaf on the group G defined in [L1,4.1].

Given a $v \in G$ with $\theta({}^vT) = {}^vT$, consider the complex on $(G/U \times G/U)/T$ obtained as $r_!s^*\mathcal{L}$, where $(G/U \times G/U)/T \xleftarrow{r} \dot{Y}_v \xrightarrow{s} T$, with $\dot{Y}_v = (K/K\cap {}^vU \times K/K\cap {}^vU)/(K\cap {}^vT)\times T$, and $r((k_1vU, k_2vU), t) = (k_1vtU, k_2vU)T$, $s((k_1vU, k_2vU), t) = t$.

If $\mathcal{L}|(T\cap {}^{v^{-1}}K)^0 \not\simeq \bar{\mathcal{Q}}_l$, then $r_!s^*\mathcal{L} = 0$ (It is enough to show that the fibre at every point is 0, and this is clear as it involves the homology of a torus with coefficients in a non-constant local system). If $\mathcal{L}|(T\cap {}^{v^{-1}}K)^0 \simeq \bar{\mathcal{Q}}_l$, then $r_!s^*\mathcal{L} \simeq H^*_c(({}^vT\cap K)^0)\otimes \tilde{\mathcal{L}}_v$, where $\tilde{\mathcal{L}}_v$ is a local system on the orbit $(KvB \times KvB)/T$, of rank $({}^vT\cap K)/({}^vT\cap K)^0$.

Proposition. *In the mixed Grothendieck group of $(G,K)^\wedge_{\mathcal{L}}$, we have*

$$\pi^0\pi_! Ind^G_T\mathcal{L} = \sum_v H^*_c(({}^vB\cap K)^0)\otimes ch(\tilde{\mathcal{L}}_v)[\dim(K\cap {}^vU)]$$

where the sum is over all orbits $v \in K\backslash G/B$ such that $\mathcal{L}|({}^vT\cap K)^0 \simeq \bar{\mathcal{Q}}_l$

PROOF.Regard $\mathcal{L}$ as a sheaf $\mathcal{A}$ on $(G/U\times G/U)/T$ supported on the closed G_Δ-orbit (via $G\times T \to (G/U\times G/U)/T$, $(g,t)\mapsto (gtU, gU)T$). Then $ch'(\mathcal{A}) \simeq Ind^G_T\mathcal{L}$. It is clear that the support of $Av(\mathcal{A})$ is contained in $\coprod \mathcal{O}\times\mathcal{O}$, as $\mathcal{O}$ runs through the $K\times T$-orbits on G/U. Given such an $\mathcal{O}$, pick a $v\in G$ with $\theta({}^vT) = {}^vT$ and $vU \in \mathcal{O}$. It is then easy to see, after identifying $\mathcal{O}$ with $K\times T/K\cap {}^vB$ (where $K\cap {}^vB$ is embedded $k\mapsto (k, pr_T(v^{-1}kv))$) that $Av(\mathcal{A})|\mathcal{O}\times\mathcal{O} \simeq H^*_c(K\cap {}^vU)\otimes r_!s^*\mathcal{L}[\dim(K\cap {}^vU)]$. The proposition is immediate from the identity $\pi^0\pi_!ch'(\mathcal{A}) = ch_G(Av(\mathcal{A}))$.

For example, if (G,K) is split, T a split torus, and we denote the simple character sheaves in $(G,K)^\wedge_{\mathcal{L}}$ as $A^{\mathcal{L}}_{\chi_1\chi_2}$, $\chi_i \in T^\wedge_K$, then $\pi^0\pi_! Ind^G_T\mathcal{L} \simeq \oplus_{\chi\in T^\wedge_K} A^{\mathcal{L}}_{\chi\chi}$ (an isomorphism in $D_K(G)$). It is also easy to see that in this case

$$A^{\mathcal{L}}_{\chi_1\chi_2} * A^{\mathcal{L}}_{\chi_3\chi_4} = \begin{cases} 0 & \text{if } \chi_2 \ne \chi_3 \\ H^*_c(K)\otimes H^*_c(T)\otimes A^{\mathcal{L}}_{\chi_1\chi_4} & \text{if } \chi_2 = \chi_3. \end{cases}$$

where $H^*_c(K)\otimes H^*_c(T)$ is the complex $p^*p_!\bar{\mathcal{Q}}_l$, p the map from the open $K\times T$ orbit to a point. So the generic character sheaves multiply "almost" like matrix units.

Now suppose G,K are defined over a finite field F_q, and let F denote the Frobenius morphism. If $\phi_0 : F^*\mathcal{L} \tilde{\to} \mathcal{L}$ is an isomorphism, we can define $\phi : F^* Ind^G_T\mathcal{L} \tilde{\to} Ind^G_T\mathcal{L}$, and Lusztig proves [L3] that $\chi_{Ind^G_T\mathcal{L},\phi} = R^G_T(\chi_{\mathcal{L},\phi_0})$. If in addition $\mathcal{L}$ is generic, then $R^G_T(\chi_{\mathcal{L},\phi_0})$ is an irreducible character of G^F.

This gives an alternate approach to the results of [L8]. Of more interest, this implies that if (G,K) is split, a generic character of the double coset algebra $K^F\backslash G^F/K^F$, when evaluated at a regular semisimple element, is the sum of $|W||T^F_K|$ complex numbers of absolute value one. These numbers are not roots of unity, though some power of q times them will be algebraic integers, and there are no "elementary" formulae for them. (See section 7 for examples). This is in sharp contrast to the characters of a group.

7. Examples

In this section we explicitly compute a few small examples. Each example arises as the complexification of some real group (K is the complexification of a maximal compact) and will be labelled by this real group.

After this section was written, J. Saxl informed me of the results of R. Lawther on the structure of (E_6, F_4), and (D_n, B_{n-1}), the two symmetric spaces, other than $(G\times G, G)$ and (A_{2n}, C_n), which have one conjugacy class of θ-stable tori. It is then an easy exercise to describe both the character sheaves, and the change of basis matrix to the irreducible characters in these cases. This will be done in an expository paper after this thesis is finished. Honest.

In that paper, we will also prove the following (reassuring) triviality. If (G, K) is a symmetric space with $G = GL_n$, and G, K are defined over F_q and F_q-split, and if $\mathcal{A} \in G^{\wedge}_{\bar{\mathcal{Q}}_l}$ is the unique character sheaf in the cell $\mathbf{c}$, then $\sum_{k\in K^F} \chi_{\mathcal{A}}(k)$ is the multiplicity of the W-module E in the W-graph of the flag manifold $D_K(\mathcal{B})$, where E is the W-graph carried by $\mathbf{c}$.

7.1 $SL_2(\mathbf{R})$

Let $(G, K) = (SL_2, SO_2)$. There are 3 K-orbits on $\mathcal{B} \simeq \mathbf{P}^1$: two closed orbits p_1, p_2 (these are points), and the dense orbit Ω. The dense orbit supports a non-constant local system, denoted $\mathcal{E}$, which is its own perverse extension. It forms a block on its own. There are three cells in the block of the constant sheaf on $\mathcal{B}$.

There are $10 = 1 + 3^2$ unipotent character sheaves: the constant sheaf, the constant sheaf on each component of $X_{\text{uni}} \cup (-1).X_{\text{uni}}$ (these are of the form $ch(p_ip_j)$, $1 \le i, j \le 2$), $ch(\mathcal{E}\mathcal{E})$ which is the perverse extension of a rank 2 local system on X^{rss}, and the four sheaves $ch(\mathcal{E}p_i)$, $ch(p_i\mathcal{E})$ ($i = 1, 2$), which have support all of X, and are rank one local systems with $\mathbf{Z}/2$-monodromy where they are non-zero ($ch(\mathcal{E}p_i)$ is zero on the support of $ch(p_1p_i) \oplus ch(p_2p_i)$, and $ch(p_i\mathcal{E})$ is zero on the support of $ch(p_ip_1) \oplus ch(p_ip_2)$).

If $\mathcal{L}$ is a local system on T with $\mathcal{L}^{\otimes 2} \not\simeq \bar{\mathcal{Q}}_l$, the $\mathcal{L}$ is a generic central character. We get 4 character sheaves for each such $\mathcal{L}$. If $\mathcal{L}$ is a local system with $\mathcal{L} \not\simeq \bar{\mathcal{Q}}_l$, $\mathcal{L}^{\otimes 2} \simeq \bar{\mathcal{Q}}_l$, then we still get 4 character sheaves at this central character.

If we choose a rational structure on G, K so that both G, K are F_q-split, then there are $2q + 6$ K^F-orbits in X^F. The characteristic functions of the character sheaves fixed by Frobenius form a basis (which is *not* orthogonal) of the K^F-invariant functions on X^F.

7.2 $SL_3(\mathbf{R})$

Let $(G, K) = (SL_3, SO_3)$. There are 4 K-orbits on $\mathcal{B}$, and 7 irreducible perverse sheaves in $D_K(\mathcal{B})$. We describe them geometrically. The quadratic form defining SO_3 gives a quadric Q in $\mathbf{P}^2$. Then the orbits are $cl = \{(p \subset l) \mid p \in Q, l \parallel Q\} \simeq \mathbf{P}^1$, $v_1 = \{(p \subset l) \mid p \notin Q, l \parallel Q\}$, $v_2 = \{(p \subset l) \mid p \in Q, l \not\parallel Q\}$, $\Omega = \{(p \subset l) \mid p \notin Q, l \not\parallel Q\}$, where $p \in \mathbf{P}^2$, l is a line in $\mathbf{P}^2$, and we write $l \parallel Q$ to mean l is tangential to Q. This picture of the orbits is due to Lusztig.

The open orbit Ω supports four local systems. These are $\mathcal{E}_1$, obtained from the double cover $\{(p \subset l, l') \mid p \subset l', l' \parallel Q\}$, $\mathcal{E}_2$, obtained from the double cover $\{(p \subset l, p') \mid p' \in l \cap Q\}$, $\mathcal{E}$, obtained from the fibre product of these two covers, and the constant local system. From this description it is clear that if $\mathcal{E}_i^\sharp$ denotes the perverse extension of $\mathcal{E}_i$, we have $\mathcal{E}_i^\sharp = \mathcal{E}_i + v_i$, $\mathcal{E}^\sharp = \mathcal{E}$.

There are 5 cells, $\mathcal{E}^\sharp$, cl, $\{v_1^\sharp, \mathcal{E}_1^\sharp\}$, $\{v_2^\sharp, \mathcal{E}_2^\sharp\}$, $\Omega^\sharp$, whose W-graphs carry (respectively) the trivial, trivial, reflection, reflection and sign representations of W.

There are $9 = 2^2 + 2^2 + 1$ unipotent character sheaves, one for each pair of cells in $K\backslash\mathcal{B}$ which carry the same W-graph. There is $ch(cl \boxtimes cl)$, which is the perverse extension of the constant local system on the strata of regular elements of X which are neither unipotent or semisimple. This sheaf, when restricted to the unipotent variety, is the perverse extension of the constant sheaf on the regular unipotents. The other unipotent character sheaves have support all of X. The sheaf associated to $\{v_i^\sharp, \mathcal{E}_i^\sharp\} \boxtimes \{v_i^\sharp, \mathcal{E}_i^\sharp\}$, $i = 1, 2$, has rank 3 and finite monodromy, though the covering it arises from is not Galois. The sheaf associated to $\{v_i^\sharp, \mathcal{E}_i^\sharp\} \boxtimes \{v_j^\sharp, \mathcal{E}_j^\sharp\}$, $i \neq j$, has rank 2 and large monodromy—the fibre over a point $g \in G$ is $H^1_c({}^g\bar{v}_i \cap \bar{v}_j, \bar{\mathcal{Q}}_l)$ and ${}^g\bar{v}_i \cap \bar{v}_j$ is an elliptic curve for generic g ($\bar{v}_i = v_i \cup cl$). The sheaves $ch(cl \boxtimes \mathcal{E})$, $ch(\mathcal{E} \boxtimes cl)$, $ch(\mathcal{E} \boxtimes \mathcal{E})$, all have rank 6 and large monodromy. Finally, there is the constant sheaf.

7.3 $Sp_4(\mathbf{R})$

Let $(G, K) = (Sp_4, GL_2)$. We describe the unipotent character sheaves in the flag manifold block of the constant sheaf. There are six cells in this block; β_+, β_-, whose W-graph carries the trivial representation, ρ_+, ρ_0, ρ_-, whose W-graph carries the relection representation plus the one dimensional representation ($s_1 \mapsto -1, s_2 \mapsto 1$), where s_1 is the short simple reflection in W, s_2 the long simple reflection, and Ω which carries the sign representation.

Let reg_+, reg_- denote the two K-orbits of regular unipotents in X, sub_+, sub_0, sub_- the three orbits of subregular unipotents, with $\overline{reg_\pm} \supset sub_\pm \cup sub_0$. Write $short_+$ (resp. $short_-$) for the strata of elements of X with Jordan decomposition $x = su$, such that $Z_G(s)$ is K-conjugate to a Levi subgroup with roots the short simple roots, and u is a minimal unipotent in $\overline{reg_+}$ (resp. $\overline{reg_-}$).

Let ϵ denote a semisimple element of X with $Z_G(\epsilon) \simeq SL_2 \times SL_2$; write ϵu for the strata $K.\epsilon u$ (there are 9 such). Write u_{++} (resp. u_{--}) for an element of $Z_G(\epsilon) \cap sub_+$ (resp. $Z_G(\epsilon) \cap sub_-$), and u_{+-}, u_{-+} for some representatives of the $Z_K(\epsilon)$ orbits of $Z_K(\epsilon) \cap sub_0$ (normalised so that if $g^{-1}\theta g = \epsilon u_{++}$, then $g \theta g^{-1} = \epsilon u_{+-}$).

Write $long_0$ for the strata of elements of X with Jordan decomposition $x = su$, such that $Z_G(s)$ is K-conjugate to a Levi subgroup with roots the long simple roots, and $u \in sub_0$.

We list the cells, followed by the character sheaves in that cell. If Y is a subvariety of X, we write Y to mean the perverse extension of the constant sheaf on Y.

$\Omega\Omega : X$, $\beta_+\beta_+ : reg_+$, $\beta_-\beta_+ : -1.reg_+$, $\beta_-\beta_- : reg_-$, $\beta_+\beta_- : -1.reg_-$, $\rho_+\rho_+ : sub_+, short_+$, $\rho_-\rho_+ : -1.sub_+, -1.short_+$, $\rho_-\rho_- : sub_-, short_-$, $\rho_+\rho_- : -1.sub_-, -1.short_-$, $\rho_0\rho_+ : \epsilon u_{++}, \mathcal{A}_1$, $\rho_0\rho_- : \epsilon u_{--}, \mathcal{A}_2$, $\rho_+\rho_0 : \epsilon u_{+-}, \mathcal{A}_3$, $\rho_-\rho_0 : \epsilon u_{-+}, \mathcal{A}_4$, where $\mathcal{A}_1, \dots, \mathcal{A}_4$ are the perverse extensions of rank 1 local systems with $\mathbf{Z}/2$ monodromy on the regular semisimple elements of X; and $\rho_0\rho_0$ which contains two character sheaves. One is the perverse extension of a rank two local system on the regular semisimple elements, with generic stalk given by H_c^1 of an elliptic curve; the other is the perverse extension of a rank one local system on $long_0$.

7.4 $GL_n(\mathbf{H})$.

In the remainder of this section, G is a direct product of general linear groups, $G = \prod GL_{2n_i}$, $K = \prod Sp_{2n_i}$ and $\theta : G \to G$ is an involution with fixpoints K. Put $\tilde{L} = \prod GL_{n_i} \times GL_{n_i}$, a Levi subgroup of G with $\theta(x_i, y_i) = (y_i, x_i)$ for $(x_i, y_i) \in \tilde{L}$, and let $\tilde{P}$ be a θ-stable parabolic of G with Levi subgroup $\tilde{L}$. Then $X_{\tilde{L}} \cong K_{\tilde{L}} \cong \prod GL_{n_i}$.

We also suppose given a rational structure such that G, $\tilde{L}$, $\tilde{P}$, K are all split groups defined over F_q, and write F for the corresponding Frobenius morphism.

We summarise the known structure theory for (G, K), for simplicity stating it only when $(G, K) = (GL_{2n}, Sp_{2n})$. Conjugacy classes in G intersect $X = \{g \in G \mid \theta g = g^{-1}\}$ in a single K-orbit (or not at all), and every K-orbit in X intersects $X_{\tilde{L}}$ in a single $K_{\tilde{L}}$-orbit. Likewise, conjugacy classes in G^F intersect X^F in at most a single K^F-orbit, and every K^F-orbit in X^F intersects $X_{\tilde{L}}^F$ in a single $K_{\tilde{L}}^F$-orbit. [Kl]. For every $x \in X$, $Z_K(x)$ is connected. In particular, unipotent orbits in X or X^F are parametrised by the partitions of n, $\mathcal{P}_n$; $x \in X_{\mathrm{uni}}$ corresponds to the partition λ if it has Jordan blocks of size $\lambda^2 = (\lambda_1 \geq \lambda_1 \geq \lambda_2 \geq \lambda_2 \geq \cdots)$.

There is only one K-conjugacy class of θ-stable maximal tori in G. If A is a maximal θ-split torus of $\tilde{L}$, then it is a maximal θ-split torus of G also, and the small Weyl groups coincide: $W(A, K_{\tilde{L}}) = W(A, K) = S_n$.

There are $1.3.5\dots(2n-1)$ K-orbits on the flag variety $\mathcal{B}_G$. The cells in $K\backslash\mathcal{B}_G$ are in 1-1 correspondence with the cells of $K_{\tilde{L}}\backslash\mathcal{B}_{\tilde{L}}$, i.e. with the two sided cells of GL_n. Recall such cells are parametrised by partitions of n, where a cell has partition λ if its W-graph is the irreducible representation of S_n with partition λ (normalised so the trivial representation has partition (n)). If this is so the Springer correspondence associates to this representation the unipotent class in GL_n with partition λ.

A cell in $K\backslash\mathcal{B}_G$ corresponds to the partition λ if its W-graph carries the irreducible representation of S_{2n} with partition λ^2. If this is so the Springer correspondence in G associates to this representation the unipotent class in X with partition λ. Moreover, under the functor $D_{K_{\tilde{P}}}(\tilde{L}/B) \to D_K(K\tilde{P}/B) \to D_K(G/B)$ (for B a Borel of $\tilde{P}$, see (2.1)) the perverse sheaves in the cell in $K_{\tilde{L}}\backslash\mathcal{B}_{\tilde{L}}$ corresponding to λ become (some of) the perverse sheaves in the cell in $K\backslash\mathcal{B}_G$ corresponding to λ. This follows from [T].

With this background, we can state the following theorem, inspired by [BKS], which is joint work with Lusztig. We will consider the character sheaves in $(G,K)^\wedge_{\mathcal{L}}$ as living on X via $D_K(X)\tilde{\to}D_{K\times K}(G)$, and those in $(\tilde{L},K_{\tilde{L}})^\wedge_{\mathcal{L}}$ as living on $X_{\tilde{L}}$.

7.4.1 Theorem. *i) The functor $\Phi = {}^pH^0 Ind^G_{\tilde{L},\tilde{P}} : \mathcal{M}(\tilde{L},K_{\tilde{L}})^\wedge_{\mathcal{L}} \to \mathcal{M}(G,K)^\wedge_{\mathcal{L}}$ induces a bijection between the simple character sheaves of $(\tilde{L},K_{\tilde{L}})$ and those of (G,K). This correspondence commutes with the ones described above (for example, if $\mathcal{A}$ is a unipotent character sheaf of $X_{\tilde{L}}$ associated to a cell* **c***, then $\Phi\mathcal{A}$ is associated to the cell corresponding to* **c***). ii) Moreover, if $\mathcal{A} \in (\tilde{L},K_{\tilde{L}})^\wedge_{\mathcal{L}}$, $\phi : F^*\mathcal{A}\tilde{\to}\mathcal{A}$ an isomorphism, then we can define $\phi' : F^*\Phi\mathcal{A}\tilde{\to}\Phi\mathcal{A}$ so that for $x \in X^F_{\tilde{L}}$*

$$\chi_{\Phi\mathcal{A},\phi'}(x) = \chi_{\mathcal{A},\phi}(x)|_{q\mapsto q^2}$$

In particular, the character sheaves of (G,K) have support all of X, and the cohomology sheaves of a character sheaf, when restricted to the Lusztig stratification, are local systems with finite monodromy.

7.4.2 Corollary. *Let C be a unipotent conjugacy class in $X_{\tilde{L}}$, $\mathcal{E}_0$ the intersection cohomology extension of the constant sheaf on C, and $\mathcal{E}$ the intersection cohomology extension of the constant sheaf on ΦC, where ΦC is the unipotent conjugacy class in X corresponding to the class C in $X_{\tilde{L}}$. Then we have, for unipotent classes C' in $X_{\tilde{L}}$*

$$\dim(\mathcal{H}^i\mathcal{E} \mid \Phi C') = \begin{cases} 0, & \text{if } i \text{ is odd} \\ \dim(\mathcal{H}^{i/2}\mathcal{E}_0 \mid C'), & \text{if } i \text{ is even.} \end{cases}$$

Remark. 1) In [BKS], "basic functions" are defined. These are precisely the characteristic functions of $\Phi Ind^{\tilde{L}}_{T,B\cap\tilde{L}}\mathcal{L}$, for $\mathcal{L}$ a local system on T. 2) Though the characteristic functions of the character sheaves of $X_{\tilde{L}}$ are the characters of $X^F_{\tilde{L}}$, those of (G,K) are not always the characters of $K^F\backslash G^F/K^F$—indeed for unipotent character sheaves they need not even be orthogonal. For the precise relation, see [BKS] and (7.4.8), (7.4.9) below. One may also use the functor Av of section 5 to investigate this relation; I will expand on this in the expository paper mentioned above.

7.4.3 Proof.The proof is by induction on $\dim G$, and will occupy the rest of this section. We may assume the central character $\mathcal{L}$ comes from a local system on T/T_K, for otherwise $(\tilde{L},K_{\tilde{L}})^\wedge_{\mathcal{L}} = (G,K)^\wedge_{\mathcal{L}} = \emptyset$.

We first suppose $\mathcal{L}$ is such that $W'_{\mathcal{L}} \neq W$. Then $W'_{\mathcal{L}} = W_M$, the Weyl group of a proper Levi subgroup M which is the centraliser of a θ-split torus. Then $(M,K_M) = (\prod GL_{2n'_i}, \prod Sp_{2n'_i})$ with $\dim M < \dim G$. Consider the diagram

$$\begin{tikzcd} (M,K_M)^\wedge_{\mathcal{L}} \arrow[r, "Ind^G_{M,Q}"] & (G,K)^\wedge_{\mathcal{L}} \\ (M\cap\tilde{L},K\cap M\cap\tilde{L})^\wedge_{\mathcal{L}} \arrow[u, "\Phi_M"] \arrow[r, "Ind^{\tilde{L}}_{M\cap\tilde{L},Q'\cap\tilde{L}}"] & (\tilde{L},K_{\tilde{L}})^\wedge_{\mathcal{L}} \arrow[u, "\Phi"] \end{tikzcd}$$

where $Q = MU_Q$ is any parabolic of G with Levi subgroup M, and $Q' \subset \tilde{P}$ is a parabolic subgroup of G with Levi subgroup $M\cap\tilde{L}$. We do not know this diagram commutes. However, the horizontal arrows are equivalences of categories, as $W'_{\mathcal{L}} \subseteq W_M$, $W'_{\mathcal{L}}\cap\tilde{L} \subseteq W_{M\cap\tilde{L}} = W_M\cap\tilde{L}$; and

$\Phi_M = {}^pH^0 Ind^M_{\tilde{L}\cap M, \tilde{P}\cap M}$ induces a bijection between the simple perverse sheaves, by our induction hypothesis. Hence there are the same number of simple character sheaves in $(G,K)^\wedge_{\mathcal{L}}$ as in $(\tilde{L}, K_{\tilde{L}})^\wedge_{\mathcal{L}}$, and so to show Φ induces a bijection between the simple perverse sheaves of $(\tilde{L}, K_{\tilde{L}})^\wedge_{\mathcal{L}}$ and those of $(G,K)^\wedge_{\mathcal{L}}$ it is enough to show $\Phi|X^{\mathrm{rss}}$ takes irreducible local systems on $X^{\mathrm{rss}}_{\tilde{L}}$ to irreducible local systems on X^{rss} bijectively. But is is clear from (4.3) applied to $\tilde{P}$ that for a local system $\mathcal{A}$ on $X^{\mathrm{rss}}_{\tilde{L}}$, $\Phi\mathcal{A}|X^{\mathrm{rss}}_{\tilde{L}} = \mathcal{A}$. This proves (i) in case $W'_{\mathcal{L}} \neq W$. (Note that $Ind^G_{\tilde{L},\tilde{P}}\mathcal{A}|X^{\mathrm{rss}}_{\tilde{L}} \neq H^*(\mathcal{B}_L)\otimes\mathcal{A}$ in general—$W(A,K)$ monodromy can creep in.)

If $W'_{\mathcal{L}} = W$, then $\mathcal{L}$ arises from a local system on G/G^{der}, and by (2.4) applied to (G,K) and $(\tilde{L}, K_{\tilde{L}})$ we reduce to the case when $\mathcal{L}$ is the constant local system; i.e. unipotent central character. We may further assume G is simple, $(G,K) = (GL_{2n}, Sp_{2n})$.

Pick a θ-stable maximal torus $T \subset \tilde{L}$, and a θ-stable Borel subgroup B of G, with $T \subset B \subset \tilde{P}$. We show all unipotent characters are summands of $Ind^G_{T,B}\bar{\mathcal{Q}}_l$. The unipotent character sheaves are precisely the constituents of $ch_G(\mathcal{O}_{\mathbf{c}} \boxtimes \mathcal{O}_{\mathbf{c}})$, as $\mathbf{c}$ runs through the cells of $K\backslash\mathcal{B}$, where for each cell $\mathbf{c}$, $\mathcal{O}_{\mathbf{c}}$ is some (arbitarily chosen) perverse sheaf in $\mathbf{c}$ (3.3). We can always choose $\mathcal{O}_{\mathbf{c}}$ so that it comes from a perverse sheaf on $K_{\tilde{L}}\backslash\mathcal{B}_{\tilde{L}}$ via $D_{K_{\tilde{P}}}(\tilde{P}/B) \to D_K(G/B)$. Then $Ind^G_{\tilde{L},\tilde{P}} \circ ch_{\tilde{L}} = ch_G$, so the character sheaves in $(G,K)^\wedge_{\bar{\mathcal{Q}}_l}$ occur as the constituents of $Ind^G_{\tilde{L},\tilde{P}}\mathcal{A}$, as $\mathcal{A}$ runs through the character sheaves of $(\tilde{L}, K_{\tilde{L}})^\wedge_{\bar{\mathcal{Q}}_l}$. Finally, every unipotent character sheaf of $(\tilde{L}, K_{\tilde{L}})^\wedge_{\bar{\mathcal{Q}}_l}$ is a constituent of $Ind^{\tilde{L}}_{T,B\cap\tilde{L}}\bar{\mathcal{Q}}_l$ [L1], so the unipotent character sheaves of (G,K) are the constituents of $Ind^G_{T,B}\bar{\mathcal{Q}}_l = Ind^G_{\tilde{L},\tilde{P}} \circ Ind^{\tilde{L}}_{T,B\cap\tilde{L}}\bar{\mathcal{Q}}_l$.

From this we further see there are at least $|W(A,K)^\wedge|$ unipotent character sheaves, namely the perverse extensions of $\Phi\mathcal{A}|X^{\mathrm{rss}}$, as $\mathcal{A}$ runs through the character sheaves in $(\tilde{L}, K_{\tilde{L}})^\wedge_{\bar{\mathcal{Q}}_l}$. These sheaves have support all of G, and are summands of ${}^pH^0 Ind^G_{T,B}\bar{\mathcal{Q}}_l$. So to finish the proof of (i), we need only show there are no other constituents of $Ind^G_{T,B}\bar{\mathcal{Q}}_l$.

Put $Z' = \{(x, kB) \in X \times \mathcal{B}^\theta \mid k^{-1}xk \in B\}$, $Z = \{(x,kB) \in \mathfrak{p} \times \mathcal{B}^\theta \mid x \in \mathrm{Lie}^k B\}$, where $\mathfrak{p}$ is the (-1)-eigenspace of $d\theta$ on $\mathfrak{g}$, and let $\pi' : Z' \to X$, $\pi : Z \to \mathfrak{p}$ be the projections onto the first factor. Then Z' is an open subset of Z via the embedding $G = GL_{2n} \hookrightarrow \mathfrak{g} = \mathfrak{gl}_{2n}$, $g \mapsto g - 1$, and so to prove $Ind^G_{T,B}\bar{\mathcal{Q}}_l \cong \pi'_!\bar{\mathcal{Q}}_l$ has only $|W(A,K)^\wedge|$ distinct constituents, it suffices to prove $\pi_!\bar{\mathcal{Q}}_l$ has at most $|W(A,K)^\wedge|$ distinct constituents.

We deduce this from the following lemma, due to Lusztig, whose proof is a variant of results in, for example, [L6] or [S]. Let $N(\mathfrak{p})$ denote the variety of nilpotent elements of $\mathfrak{p}$, and $\mathcal{F} : D_K(\mathfrak{p}) \to D_K(\mathfrak{p})$ the Deligne-Fourier transform, an equivalence of categories preserving t-structures. (We identify $\mathfrak{p}$ and $\mathfrak{p}^*$ by the Killing form.)

Lemma. *$\mathcal{F}(\pi_!\bar{\mathcal{Q}}_l|N(\mathfrak{p})) \cong \pi_!\bar{\mathcal{Q}}_l[-2r](-r)$, where $r = \dim(U_B \cap X)$.*

As a consequence of the lemma, $\mathcal{F}(\pi_!\bar{\mathcal{Q}}_l)$ is supported on the nilpotent cone $N(\mathfrak{p})$. But $N(\mathfrak{p})$ is a union of $|\mathcal{P}_n| = |W(A,K)^\wedge|$ K-orbits, each of which supports only the trivial K-equivariant local system. Hence $\mathcal{F}(\pi_!\bar{\mathcal{Q}}_l)$, and so $\pi_!\bar{\mathcal{Q}}_l$, can have at most $|W(A,K)^\wedge|$ irreducible constituents.

This completes the proof of (i).

7.4.4 We now know the character sheaves are the constituents of $Ind^G_{T,B}\mathcal{L}$, as $\mathcal{L}$ runs through the local systems on A. To prove (ii), we give another model for $Ind^G_{T,B}\mathcal{L}$.

Put $Y = \{(x, kL) \in X^{\mathrm{rss}} \times K/(K\cap L) \mid k^{-1}xk \in L\}$, $pr_1 : Y \to X^{\mathrm{rss}}$, $(x,kL) \mapsto x$, a $W(A,K)$-Galois cover, and $q : Y \to A$, $(x,kL) \mapsto k^{-1}xk$. Here A is a maximal θ-split torus, as always, and $L = Z_G(A)$. Then $(pr_1)_! q^*\mathcal{L}$ is a semisimple local system on X^{rss}, and we have:

Lemma. *$Ind^G_{T,B}\mathcal{L} \cong IC((pr_1)_! q^*\mathcal{L}) \otimes H^*(\mathcal{B}_L)$, where IC denotes the intersection cohomology extension, here normalised so as to give a perverse sheaf.*

Proof.By the results of (i), to prove the lemma it is enough to show $Ind^G_{T,B}\mathcal{L}|X^{\mathrm{rss}} \cong (pr_1)_! q^*\mathcal{L} \otimes H^*(\mathcal{B}_L)$ (up to shift). This follows from the isomorphism $\{(x,kB) \in X^{\mathrm{rss}} \times \mathcal{B}^\theta \mid k^{-1}xk \in B\} \cong Y \times \mathcal{B}_L$, whose proof we omit.

So to define $Ind_{T,B}^G\mathcal{L}$ we need only the maximal θ-split torus $A \subset T$. We continue to denote this sheaf $Ind_{T,B}^G\mathcal{L}$, even when we have not chosen a maximal torus T or Borel B containing A.

Now suppose $FA = A$, and there exists an isomorphism of local systems $\phi_0 : F^*\mathcal{L}\tilde{\to}\mathcal{L}$. We normalise ϕ_0 so that it induces the identity map on the stalk $\mathcal{L}_e$ at $e \in A$. Then the varieties L, $\mathcal{B}_L$, Y are all defined over F_q, and by the functorality of IC-extension we can define isomorphisms $\phi : F^*Ind_{T,B}^G\mathcal{L}\tilde{\to}Ind_{T,B}^G\mathcal{L}$, $\phi' : F^*\,{}^pH^0Ind_{T,B}^G\mathcal{L}\tilde{\to}{}^pH^0Ind_{T,B}^G\mathcal{L}$. We have $\chi_{Ind_{T,B}^G\mathcal{L},\phi}(x) = |\mathcal{B}_L^F|\chi_{{}^pH^0Ind_{T,B}^G\mathcal{L},\phi'}(x)$, for all $x \in X^F$.

The following sequence of results (and their proofs) are exact analogues of those of [L1,8–10] in the case of a torus. This method of proof (of 7.4(ii)) is very similar to that of [BKS].

7.4.5 Lemma. *If $x \in X^F$, x unipotent, then $\chi_{Ind_{T,B}^G\mathcal{L},\phi}(x)$ is independent of the choice of $\mathcal{L}$ (having normalised ϕ_0 as above). Denote this function $Q_A = Q_A^G : X_{\text{uni}}^F \to \bar{\mathcal{Q}}_l$.*

We also denote the function $\chi_{Ind_{T,B\cap\tilde{L}}^{\tilde{L}},\phi} : (X_{\tilde{L}})_{\text{uni}}^F \to \bar{\mathcal{Q}}_l$ by $\tilde{Q}_A$; it too is independent of the choice of $\mathcal{L}$ [L1,8.3.2].

7.4.6 Proposition. *If $x \in X^F$, $x = su$ is its Jordan decomposition, then*

$$\chi_{Ind_{T,B}^G\mathcal{L},\phi}(x) = |Z_K^0(s)^F|^{-1} \sum_{\substack{k\in K^F\\ k^{-1}sk\in A}} Q_{kAk^{-1}}^{Z_G^0(s)}(u)\chi_{\mathcal{L},\phi_0}(k^{-1}sk)$$

7.4.7 Proposition. *Suppose we have two maximal θ-split F-stable tori A_1, A_2; and local systems $\mathcal{L}_i$ on A_i with given isomorphisms $\phi_{0,i} : F^*\mathcal{L}_i\tilde{\to}\mathcal{L}_i$ inducing the identity at the stalks over $e \in A_i$. Put $L_i = Z_G(A_i)$, $W(A_i,K) = N_K(A_i)/Z_K(A_i)$. Pick T_i an F_q-split maximal torus of L_i. Then*

$$i)\ \sum_{u\in X_{\text{uni}}^F} Q_{A_1}(u)Q_{A_2}(u) = \begin{cases} |W(A_1,K)||T_1{}_K^F|^{-1}(|K^F||L_1^F|_p^{-1})|\mathcal{B}_{L_1}^F|, & \textit{if } A_1 \textit{ is } K^F\textit{-conjugate to } A_2\\ 0, & \textit{otherwise.}\end{cases}$$

$$ii)\ \sum_{x\in X^F} \chi_{Ind_{T_1,B_1}^G\mathcal{L},\phi_1}(x)\chi_{Ind_{T_2,B_2}^G\mathcal{L},\phi_2}(x)$$
$$= (|K^F||L_1^F|_p^{-1})|\mathcal{B}_{L_1}^F|\Big(\sum_{\substack{n\in N_K(L_2,L_1)/(L_1\cap K)\\ Fn=n}} |T_1{}_K^F|^{-1}\sum_{a\in A_1^F}\chi_{\mathcal{L}_1,\phi_{0,1}}(a)\chi_{\mathcal{L}_2,\phi_{0,2}}(nan^{-1})\Big)$$

where $N_K(L_2,L_1) = \{n \in K \mid {}^nA_1 = A_2\}$, and $|a|_p$ denotes the highest power of p to divide $|a|$.
Further, $|\mathcal{B}_L^F| = |A^F|_{q\mapsto q^2}/|A^F|$, $|T_K^F| = |A^F|$, and $|K^F||L^F|_p^{-1} = |X_{\tilde{L}}^F|_{q\mapsto q^2}$.

We can also apply the discussion of [L1,10.1–10.6.1], word for word, to the complex ${}^pH^0Ind_{T,B}^G\mathcal{L}$ and to its endomorphism algebra. This algebra, isomorphic to the group algebra of $\mathcal{W} = \{w \in W(A,K) \mid w^*\mathcal{L} \simeq \mathcal{L}\}$, is also naturally isomorphic to the endomorphism algebra of $Ind_{T,B\cap\tilde{L}}^{\tilde{L}}\mathcal{L}$.

As a consequence, if $\mathcal{A}$ is an irreducible summand of $Ind_{T,B\cap\tilde{L}}^{\tilde{L}}\mathcal{L}$, and $\Phi\mathcal{A}$ the corresponding character sheaf on (G,K) then $\text{Hom}(\mathcal{A}, Ind_{T,B\cap\tilde{L}}^{\tilde{L}}\mathcal{L})\tilde{\to}\text{Hom}(\Phi\mathcal{A}, Ind_{T,B}^G\mathcal{L})$. Denote this vector space $V_\mathcal{A}$. Then if $\phi_\mathcal{A} : F^*\mathcal{A}\tilde{\to}\mathcal{A}$ is an isomorphism, we can define an isomorphism $\phi_{\Phi\mathcal{A}} : F^*\Phi\mathcal{A}\tilde{\to}\Phi\mathcal{A}$, and

$$\begin{aligned}\chi_{\mathcal{A},\phi_\mathcal{A}} &= |\mathcal{W}|^{-1}\sum_{w\in\mathcal{W}} tr((\theta_w\sigma_\mathcal{A})^{-1}, V_\mathcal{A})\chi_{Ind_{T,B\cap\tilde{L}}^{\tilde{L}}\mathcal{L},\theta_w\tilde{\phi}}\\ \chi_{\Phi\mathcal{A},\phi_{\Phi\mathcal{A}}} &= |\mathcal{W}|^{-1}\sum_{w\in\mathcal{W}} tr((\theta_w\sigma_\mathcal{A})^{-1}, V_\mathcal{A})\chi_{{}^pH^0Ind_{T,B}^G\mathcal{L},\theta_w\phi'}\end{aligned} \tag{7.4.8}$$

where $\sigma_w : V_{\mathcal{A}} \to V_{\mathcal{A}}$ are certain maps defined in [L1,10.4], and θ_w are the basis of the endomorphism algebra of $Ind_{T,B\cap\tilde{L}}^{\tilde{L}}\mathcal{L}$ defined in [L1,10.2].

Now, by the character formula (7.4.6), its analogue for $Ind_{T,B\cap\tilde{L}}^{\tilde{L}}\mathcal{L}$, and the identities (7.4.8), it suffices to show (7.4ii) for unipotent elements; i.e. to show $\chi_{\Phi\mathcal{A},\phi_{\Phi\mathcal{A}}}(x) = \chi_{\mathcal{A},\phi_{\mathcal{A}}}(x)|_{q\mapsto q^2}$ for $x \in (X_{\tilde{L}})^F_{\rm uni}$. So we can assume $\mathcal{L}$ is $\bar{\mathcal{Q}}_l$.

Arguing as in [L1,10.9] we get from (7.4.7i) that

$$(7.4.9) \quad \sum_{u\in X^F_{\rm uni}} \chi_{\Phi\mathcal{A}_1,\phi_{\Phi\mathcal{A}_1}}(u)\chi_{\Phi\mathcal{A}_2,\phi_{\Phi\mathcal{A}_2}}(u)$$

$$= |W(A,K)|^{-1} \sum_{w\in W(A,K)} tr((\theta_w\sigma_{\mathcal{A}_1})^{-1}, V_{\mathcal{A}_1}) tr((\theta_w\sigma_{D\mathcal{A}_2})^{-1}, V_{D\mathcal{A}_2})(|A_w^F|^{-1}|G^F|)_{q\mapsto q^2}$$

$$= \left(\sum_{u\in(X_{\tilde{L}})^F_{\rm uni}} \chi_{\mathcal{A}_1,\phi_{\mathcal{A}_1}}(u)\chi_{\mathcal{A}_2,\phi_{\mathcal{A}_2}}(u)\right)_{q\mapsto q^2}$$

where $D\mathcal{A}_2$ is the Verdier dual of $\mathcal{A}_2$, and A_w is an F-stable maximal θ-split torus on which F acts as $t \mapsto {}^w t^q$. (The second equality is a consequence of [L1,10.9].)

To finish the proof we proceed as in [L1,24]. We need to know $\Phi\mathcal{A}|X_{\rm uni}$ is a perverse sheaf, the IC-extension of the constant sheaf on the orbit corresponding to the support of $\mathcal{A}|X_{\tilde{L}}^{\rm uni}$. This follows from what we have done above.

The orthogonality relations (7.4.9) completely determine the characteristic functions of $\Phi\mathcal{A}|X_{\rm uni}$ (analogue of [L1,24.4]) and $\mathcal{A}|X_{\tilde{L}}^{\rm uni}$ ([L1,24.4]); and it is clear that with the identifications of orbits and sheaves as above that $\chi_{\Phi\mathcal{A},\phi_{\Phi\mathcal{A}}}(x) = \chi_{\mathcal{A},\phi_{\mathcal{A}}}(x)|_{q\mapsto q^2}$ for $x \in (X_{\tilde{L}})^F_{\rm uni}$, as desired.

The corollary (7.4.2) follows immediately from the above discussion and the analogue of [L1,24.8].

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