

All rings are commutative with 1 unless otherwise stated

1. Let $\omega = (1 + \sqrt{-3})/2$, and let $R = \{a + b\omega : a, b \in \mathbb{Z}\}$. Show that R is a ring. What are the units of R ?
2. Show that a (non-trivial) ring R is a field if and only if the only ideals of R are $\{0\}$ and R .
3. An element r of a ring R is *nilpotent* if $r^n = 0$ for some n . What are the nilpotent elements of \mathbb{Z}_6 ? Of \mathbb{Z}_8 ? Of \mathbb{Z}_{24} ? Show that if r is nilpotent then $1 + r$ is a unit. Show also that the nilpotent elements form an ideal.
4. Let r be an element of a ring R . Show that, in $R[X]$, the polynomial $1 + rX$ is a unit if and only if r is nilpotent. Is it possible for the polynomial $1 + X$ to be a product of two non-units?
5. Show that if p and q are coprime then the ring $\mathbb{Z}_p \oplus \mathbb{Z}_q$ is isomorphic to \mathbb{Z}_{pq} .
6. Let A and B be elements of the power-set ring $\mathbb{P}(S)$, for some set S . What is the principal ideal (A) ? What is the ideal (A, B) ?
7. Let $I_1 \subset I_2 \subset I_3 \subset \dots$ be ideals in a ring R . Show that the union $I = \bigcup_{n=1}^{\infty} I_n$ is also an ideal. If each I_n is proper, explain why I must be proper.
8. Let I and J be ideals in a ring R . Show that $I \cap J$ is also an ideal, and that $R/(I \cap J)$ is isomorphic to a subring of $R/I \oplus R/J$.
9. Let R be the ring $C[0, 1]$ of continuous real-valued functions on $[0, 1]$, and let $I = \{f \in R : f(x) = 0 \text{ for } 0 \leq x \leq \frac{1}{2}\}$. Show that I is an ideal. What is R/I ?
10. An element r of a ring R is *idempotent* if $r^2 = r$. What are the idempotent elements of \mathbb{Z}_6 ? Of \mathbb{Z}_8 ? Of \mathbb{Z}_{24} ? Show that if r is idempotent then so is $1 - r$. Show also that if r is idempotent then (r) is naturally a ring, and that R is isomorphic to $(r) \oplus (1 - r)$.
11. A ring R is *Boolean* if every element of R is idempotent. For each n , exhibit a Boolean ring of order 2^n . Prove that every finite Boolean ring is isomorphic to a power-set ring $\mathbb{P}(S)$, for some set S . Give an example to show that this need not remain true for infinite Boolean rings.
12. Show that if an element of a (not necessarily commutative) ring has two right-inverses then it has infinitely many.
13. For each n , write down an ideal of $\mathbb{Z}[X]$ that is generated by n elements but not by $n - 1$ elements.
14. Is every abelian group the additive group of some ring?