

1. What are the possible cycle types of elements of  $S_5$ ? For each cycle type, determine how many elements have that cycle type, their order, and whether they are even or odd.
2. For  $n \geq 5$ , let  $\sigma \in S_n$  be the 5-cycle (12345). Find the centraliser of  $\sigma$  in  $S_n$ . By considering which members of the centraliser belong to  $A_n$ , give an alternative proof of the fact that the conjugacy class of  $\sigma$  in  $A_n$  is the same as that in  $S_n$  for  $n \geq 7$ , but is half its size for  $n = 5$  and  $n = 6$ .
3. Show that a group of order 441 cannot be simple. Show that a group of order 351 cannot be simple.
4. For a prime  $p$ , how many Sylow  $p$ -subgroups are there in  $S_p$ ? Check that your answer is consistent with Sylow's theorems. Deduce the size of the normaliser of such a subgroup, and write down the normaliser explicitly for each one.
5. Let  $p, q$  and  $r$  be (not necessarily distinct) primes. Show that no group of order  $pqr$  is simple.
6. Let  $G$  be a group of order 1001. Prove that  $G$  contains normal subgroups of orders 7, 11 and 13, generated by say  $g_1, g_2, g_3$  respectively. By considering expressions of the form  $g_i g_j g_i^{-1} g_j^{-1}$ , show that the  $g_i$  commute with each other. Deduce that  $G$  must be cyclic.
7. Let  $p$  and  $q$  be primes with  $q$  dividing  $p - 1$ . By considering a suitable subgroup of the group  $G$  of all maps from  $\mathbb{Z}_p$  to itself of the form  $x \mapsto ax + b$ , where  $a, b \in \mathbb{Z}_p$  with  $a \neq 0$ , show that there exists a non-abelian group of order  $pq$ . [Either use your knowledge of the multiplicative group of  $\mathbb{Z}_p$ , or else apply Cauchy's theorem to it.]
8. Let  $G$  be the group of rotational symmetries of the dodecahedron. Give two Orbit-Stabiliser proofs that  $G$  has order 60: one based on the action of  $G$  on the vertices and one based on the action of  $G$  on the faces. Without knowledge of what this group is, why is it obvious that the group of all symmetries of the dodecahedron cannot be  $S_5$ ?
9. By using the fact that a normal subgroup must be a union of conjugacy classes, prove directly that  $A_5$  and  $A_6$  are simple. Exhibit a subgroup of  $A_n$  of index  $n$ , and explain how the simplicity of  $A_n$  implies that there cannot be a proper subgroup of  $A_n$  of smaller index (for  $n \geq 5$ ).
10. Is there an infinite simple group?
- <sup>+</sup>11. For which natural numbers  $n$  is there a unique group of order  $n$ ?
- <sup>+</sup>12. Let  $G$  and  $H$  be groups such that  $G \times \mathbb{Z}$  is isomorphic to  $H \times \mathbb{Z}$ . Must  $G$  be isomorphic to  $H$ ?