COMMON PREPERIODIC POINTS FOR QUADRATIC POLYNOMIALS

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ABSTRACT. Let $f_c(z) = z^2 + c$ for $c \in \mathbb{C}$. We show there exists a uniform bound on the number of points in $\mathbb{P}^1(\mathbb{C})$ that can be preperiodic for both f_{c_1} and f_{c_2} with $c_1 \neq c_2$ in $\mathbb{C}$. The proof combines arithmetic ingredients with complex-analytic; we estimate an adelic energy pairing when the parameters lie in $\overline{\mathbb{Q}}$, building on the quantitative arithmetic equidistribution theorem in [FRL], and we use distortion theorems in complex analysis to control the size of the intersection of distinct Julia sets. The proof is effective, and we provide explicit constants for each of the results.

1. INTRODUCTION

Consider the family of quadratic polynomials

$$f_c(z) = z^2 + c$$

for c in $\mathbb{C}$, viewed as dynamical systems $f_c : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ on the Riemann sphere. Recall that a point $z \in \hat{\mathbb{C}}$ is said to be preperiodic if its forward orbit under f_c is finite. It is well known that the set of all preperiodic points for f_c will determine c. Indeed, we have

$$\mathrm{Preper}(f_{c_1}) = \mathrm{Preper}(f_{c_2}) \iff J(f_{c_1}) = J(f_{c_2}) \iff c_1 = c_2 \tag{1.1}$$

in this family, where $J(f_c)$ is the Julia set and $\mathrm{Preper}(f_c)$ the set of preperiodic points; a sketch of the argument is given in §2.3.

For any $c_1 \neq c_2$ in $\mathbb{C}$, the intersection of $\mathrm{Preper}(f_{c_1})$ and $\mathrm{Preper}(f_{c_2})$ is finite [BD, Corollary 1.3] [YZ, Theorem 1.3], even though their Julia sets can have complicated, infinite intersection. We investigate the question of how many preperiodic points are required to uniquely determine the polynomial, forgetting the information of the period or length of an orbit. We prove:

Theorem 1.1. *There exists a constant B so that*

$$|\mathrm{Preper}(f_{c_1}) \cap \mathrm{Preper}(f_{c_2})| \leq B$$

for any $c_1 \neq c_2$ in $\mathbb{C}$.

Remark 1.2. Our proof leads to an explicit value for B. Without making an effort to optimize our constants, we show that we can take $B = 10^{82}$. This bound is probably

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far from optimal. The largest intersection we know was found by Trevor Hyde: the set $\mathrm{Preper}(f_{-21/16}) \cap \mathrm{Preper}(f_{-29/16})$ consists of at least 27 points in $\hat{\mathbb{C}}$. These two polynomials also appear in [Po].

Remark 1.3. There is no uniform bound on the periods or orbit lengths of the elements of $\mathrm{Preper}(f_{c_1}) \cap \mathrm{Preper}(f_{c_2})$ as c_1 and c_2 vary. For example, taking c_1 and c_2 to be distinct centers of hyperbolic components within the Mandelbrot set, we will have $0 \in \mathrm{Preper}(f_{c_1}) \cap \mathrm{Preper}(f_{c_2})$ with periods as large as desired.

1.1. **Motivation and background.** For any pair of rational functions $f, g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree at least 2, it is known that a dichotomy holds: either the intersection $\mathrm{Preper}(f) \cap \mathrm{Preper}(g)$ is finite or $\mathrm{Preper}(f) = \mathrm{Preper}(g)$ [BD, YZ]. Moreover, except for maps conjugate to $z^{\pm d}$, the equality $\mathrm{Preper}(f) = \mathrm{Preper}(g)$ is equivalent to the statement that the measures of maximal entropy for f and g coincide; one implication is proved in [LP] and the other in [YZ, Theorem 1.5].

We suspect a much stronger result may hold, and we propose the following conjecture:

Conjecture 1.4. *For each degree $d \geq 2$, there exists a constant $B = B(d)$ so that either*

$$|\mathrm{Preper}(f) \cap \mathrm{Preper}(g)| \leq B$$

or

$$\mathrm{Preper}(f) = \mathrm{Preper}(g)$$

for any pair of rational functions f and g in $\mathbb{C}(z)$ of degree d.

Conjecture 1.4 would imply that a configuration of $B+1$ points on the Riemann sphere, if preperiodic for some map of degree $d \geq 2$, will almost uniquely determine the map among all maps of the same degree. A complete classification of all rational maps having the same measure of maximal entropy is still open, however, unless the maps are polynomial [BE, Bea]; see also [LP, Ye, Pa] for results about rational maps with the same maximal measure.

As discussed in [DKY], Conjecture 1.4 is analogous to a question posed by Mazur [Ma], proposing the existence of uniform bound – depending only on the genus g – on the number of torsion points on a compact Riemann surface of genus $g > 1$ inside its Jacobian. In fact, the special case of Conjecture 1.4 for the 1-parameter family of Lattès maps

$$f_t(z) = \frac{(z^2 - t)^2}{4z(z-1)(z-t)} \tag{1.2}$$

in degree 4, for $t \in \mathbb{C} \setminus \{0, 1\}$, was proved in [DKY]; it implies a positive answer to Mazur's question for a certain 2-parameter family of genus 2 Riemann surfaces.

Remark 1.5. The bound B in Conjecture 1.4, if it exists, must depend on the degree d. It is easy to find examples with growing degrees with growing numbers of common preperiodic points. For example, the sequences of polynomials

$$f_n(z) = z^2(z-1)\cdots(z-n) \quad \text{and} \quad g_n(z) = z(z-1)\cdots(z-n)(z-(n+1))$$

have degree $n+2$ with at least $n+1$ common preperiodic points, for all $n \geq 1$. Their sets of preperiodic points cannot be equal because their Julia sets are not the same: we have $0 \in J(g_n)$ but $0 \notin J(f_n)$ for all n.

1.2. **Further results and proof strategy.** The proof of Theorem 1.1 uses arithmetic techniques, and we first prove a version of Theorem 1.1 when the parameters c_1 and c_2 are algebraic numbers. The basic observation is that the set of preperiodic points of f_c is invariant under the action of the Galois group $\mathrm{Gal}(\overline{K}/K)$, for any number field K containing c. Finiteness of the intersection $\mathrm{Preper}(f_{c_1}) \cap \mathrm{Preper}(f_{c_2})$, when c_1 and c_2 are algebraic, is an immediate consequence of arithmetic equidistribution: large Galois orbits in the set $\mathrm{Preper}(f_c)$ are uniformly distributed with respect to the measure of maximal entropy μ_c [BR1, FRL, CL1], while $\mu_{c_1} = \mu_{c_2}$ if and only if $c_1 = c_2$. We provide a few simple examples in Section 2 to illustrate these ideas.

The uniform bound in Theorem 1.1 comes from controlling the rate of equidistribution, not just over $\mathbb{C}$ but at all places of the number field K simultaneously. To do so, we make use of an adelic energy pairing between the polynomials f_{c_1} and f_{c_2}. This is a sum of integrals, one for each of the primes associated to a number field K containing both c_1 and c_2, which we describe now. For any c in K and any place v of K, we let

$$\lambda_{c,v}(z) = \lim_{n\to\infty} \frac{1}{2^n} \log \max\{|f_c^n(z)|_v, 1\}$$

denote the v-adic escape-rate function of f, with z in the field of v-adic numbers $\mathbb{C}_v$. This is the usual escape-rate function on $\mathbb{C}$, for $v|\infty$, coinciding with the Green's function for the complement of the filled Julia set, with logarithmic pole at ∞. At every place v, the function $\lambda_{c,v}$ extends continuously and subharmonically to the Berkovich affine line $\mathbb{A}_v^{1,an}$, and its Laplacian is the canonical v-adic measure $\mu_{c,v}$ for f_c [BR2, FRL]. For archimedean places v, we recover the Brolin-Lyubich measure [Br, Ly]. The energy pairing is defined to be

$$\langle f_{c_1}, f_{c_2} \rangle := \sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \int_{\mathbb{A}_v^{1,an}} \lambda_{c_1,v}\, d\mu_{c_2,v}. \tag{1.3}$$

The pairing is symmetric, and each term in the sum is non-negative, vanishing if and only if $\mu_{c_1,v} = \mu_{c_2,v}$ [PST]. The integral thus provides a notion of distance between the two measures. In particular, we have

$$\langle f_{c_1}, f_{c_2} \rangle \geq 0 \text{ with equality if and only if } c_1 = c_2.$$

We prove:

Theorem 1.6. *There is a constant $\delta > 0$, such that*

$$\langle f_{c_1}, f_{c_2} \rangle \geq \delta$$

for all $c_1 \neq c_2 \in \overline{\mathbb{Q}}$.

In other words, two Julia sets cannot be too similar at *all* places of a given number field.

Theorem 1.7. *There are constants $\alpha_1, \alpha_2, C_1, C_2 > 0$ so that*

$$\alpha_1\, h(c_1, c_2) - C_1 \leq \langle f_{c_1}, f_{c_2} \rangle \leq \alpha_2\, h(c_1, c_2) + C_2,$$

for all $c_1 \neq c_2$ in $\overline{\mathbb{Q}}$, where h is the logarithmic Weil height on $\mathbb{A}^2(\overline{\mathbb{Q}})$.

Remark 1.8. The upper bound in Theorem 1.7 is straightforward to prove, and it is also fairly easy to obtain a weaker lower bound in terms of $h(c_1 - c_2)$, the Weil height of the difference, in place of the height $h(c_1, c_2)$; see Theorem 7.1. The lower bound of Theorem 1.7 is more delicate: see Section 8.

Finally, we relate the energy pairing to the number of common preperiodic points via a quantified version of the arithmetic equistribution theorems, building upon ideas of Favre, Rivera-Letelier, and Fili [FRL] [Fi]:

Theorem 1.9. *For all $0 < \varepsilon < 1$, there exists a constant $C(\varepsilon) > 0$ so that*

$$\langle f_{c_1}, f_{c_2} \rangle \leq \left(\varepsilon + \frac{C(\varepsilon)}{N(c_1, c_2) - 1} \right) (h(c_1, c_2) + 1)$$

for all $c_1 \neq c_2$ in $\overline{\mathbb{Q}}$ with

$$N(c_1, c_2) := |\mathrm{Preper}(f_{c_1}) \cap \mathrm{Preper}(f_{c_2})| > 1.$$

Remark 1.10. Note that $N(c_1, c_2) \geq 1$ for every c_1 and c_2, because ∞ is a fixed point for every f_c. Using standard distortion estimates in complex analysis to control the archimedean contributions to the pairing, our proof shows that we can take

$$C(\varepsilon) \asymp \log(1/\varepsilon)$$

in Theorem 1.9.

Theorems 1.6, 1.7, and 1.9 combine to give a uniform upper bound on the number $N(c_1, c_2)$ for all $c_1 \neq c_2$ in $\overline{\mathbb{Q}}$, thus proving Theorem 1.1 for c_1 and c_2 and $\overline{\mathbb{Q}}$. Once a uniform bound is obtained over $\overline{\mathbb{Q}}$, it is straightforward to show the same bound holds over $\mathbb{C}$, as we explain in §10.2, which completes the proof of Theorem 1.1.

This general strategy of proof was introduced in our earlier work [DKY], and the reader will recognize the similarities between Theorems 1.6, 1.7, and 1.9 here and Theorems 1.4, 1.5, and 1.6 of [DKY]. However, there are significant technical differences between these proofs. Most notably, in the setting of [DKY], the energy

integrals at non-archimedean places could be computed explicitly; here, we can only obtain estimates. For the computations at the archimedean places, the local heights (escape rates) are not smooth, and the shrinking Hölder exponents (as $c \to \infty$) leads to the loss of uniformity in rates of convergence in the equidistribution theorems. We make use of classical complex dynamical methods in this article, such as the Koebe 1/4 theorem and similar distortion statements; by contrast, in [DKY], we obtained the archimedean estimates through the use of degeneration theory and comparison to a limiting non-archimedean dynamical system associated to a function field, as carried out in [Fa] and [DF1, DF2]. The degeneration theory could be used here as well, at the expense of the effective bounds.

As in the setting of [DKY], our proofs are more about the associated canonical height functions $\hat{h}_c$ on $\mathbb{P}^1(\overline{\mathbb{Q}})$, for f_c with $c \in \overline{\mathbb{Q}}$, than about preperiodic points; the bound of Theorem 1.1 comes from the fact that $\hat{h}_c(x) = 0$ if and only if x is preperiodic for f_c [CS, Corollary 1.1.1]. Though we do not provide all the details, it is possible to prove a stronger statement about points of small height: there exist constants B and $b > 0$ so that $\left|\{x \in \mathbb{P}^1(\overline{\mathbb{Q}}) : \hat{h}_{c_1}(x) + \hat{h}_{c_2}(x) \le b\}\right| \le B$ for all $c_1 \neq c_2$ in $\overline{\mathbb{Q}}$. A version of this statement is proved for the Lattès family (1.2) in [DKY, Theorems 1.8 and 8.1].

1.3. **Effectiveness.** We illustrate the effectiveness of our method by providing explicit constants for each of the theorems stated above. The proof of Theorem 1.7 shows that we can take $\alpha_1 = 1/192$, $C_1 = 3/17$, $\alpha_2 = 1/2$ and $C_2 = 5/2$. The proof of Theorem 1.9 provides $C(\varepsilon) = 40\log(25/\varepsilon)$. The first proof of Theorem 1.6 that we present in §7.1 is not sufficient to provide an explicit value of δ, but further control on the archimedean energy pairing leads to $\delta = 10^{-75}$ in §11.1. This exceptionally small δ gives rise to the bound $B = 10^{82}$ in Theorem 1.1 that was stated in Remark 1.2.

1.4. **Height pairings.** The energy pairing $\langle f_{c_1}, f_{c_2}\rangle$ that we work with is a special case of a more general construction, the Arakelov-Zhang pairing, an arithmetic intersection number between adelically metrized line bundles; see [Zh], [PST], and [CL2]. In this case, each f_c with c in a number field K gives rise to a family of metrics on $O_{\mathbb{P}^1}(1)$, one for each place v of K, with non-negative curvature distribution equal to the canonical measure $\mu_{c,v}$ on the Berkovich projective line $\mathbf{P}_v^{1,an}$. Each such metric then gives rise to a height function $\hat{h}_c$ on $\mathbb{P}^1(\overline{\mathbb{Q}})$, recovering the dynamical canonical height for f_c of Call and Silverman [CS].

There are other natural height pairings that one could consider for $c_1, c_2 \in \overline{\mathbb{Q}}$. For example, Kawaguchi and Silverman study

$$[f,g]_{KS} := \sup_{x \in \mathbb{P}^1(\overline{\mathbb{Q}})} \left|\hat{h}_f(x) - \hat{h}_g(x)\right|$$

for any pair of maps $f, g : \mathbb{P}^1 \to \mathbb{P}^1$ defined over $\overline{\mathbb{Q}}$ [KS]. As a consequence of arithmetic equidistribution, we see that

$$\langle f, g\rangle \leq [f, g]_{KS}. \tag{1.4}$$

Indeed, along any infinite (non-repeating) sequence $x_n \in \mathbb{P}^1(\overline{\mathbb{Q}})$ for which $\hat{h}_f(x_n) \to 0$, we have by equidistribution that $\hat{h}_g(x_n) \to \langle f, g\rangle$ [PST, Theorem 1]. Such sequences always exist (the preperiodic points of f will have height 0), so we obtain (1.4). We do not know if a similar inequality holds in the reverse direction. However, as a corollary of Theorem 1.7, we have

Theorem 1.11. *There exist constants $\alpha, C > 0$ so that*

$$\alpha[f_{c_1}, f_{c_2}]_{KS} - C \;\leq\; \langle f_{c_1}, f_{c_2}\rangle \;\leq\; [f_{c_1}, f_{c_2}]_{KS}$$

for all $c_1, c_2 \in \overline{\mathbb{Q}}$.

Proof. From [KS, Theorem 1], we have $[f_{c_1}, f_{c_2}]_{KS} \leq \kappa_1(h(c_1) + h(c_2)) + \kappa_2$ for constants κ_1, κ_2 depending only on the degrees of the maps, and the definition of the Weil height shows that $h(c_1) + h(c_2) \leq 2h(c_1, c_2)$. The lower bound of the theorem then follows immediately from the lower bound in Theorem 1.7. □

A version of Theorem 1.11 also holds for the Lattès family $f_t(z) = (z^2 - t)^2/(4z(z - 1)(z - t))$, with $t_1, t_2 \in \overline{\mathbb{Q}} \setminus \{0, 1\}$, as a consequence of [DKY, Theorem 1.5].

Question 1.12. *Do we have*

$$\langle f, g\rangle \asymp [f, g]_{KS}$$

for all maps $f, g : \mathbb{P}^1 \to \mathbb{P}^1$, defined over $\overline{\mathbb{Q}}$, with constants depending only on the degrees of f and g?

1.5. **Outline.** Section 2 illustrates some basic examples towards understanding the content of Theorem 1.1. Local estimates on the pairing are carried out in Sections 3 – 6. In Section 7, we prove Theorem 1.6, and in Section 8 we prove Theorem 1.7. Theorem 1.9 is proved via quantitative equidistribution theory in Section 9, and Section 10 establishes our main result, Theorem 1.1. Finally, in Section 11 we make all bounds effective.

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2. Basic examples

Let $f_c(z) = z^2 + c$, for $c \in \mathbb{C}$. Note that

$$|\mathrm{Preper}(f_{c_1}) \cap \mathrm{Preper}(f_{c_2})| \geq 1$$

for every pair, because the sets always contain the point at ∞. Here we provide a few simple examples, illustrating some of the ideas that appear in our proof of Theorem 1.1. We also explain the known result (1.1).

2.1. **Disjoint filled Julia sets.** When two quadratic polynomials f_{c_1} and f_{c_2} have disjoint filled Julia sets, they have no common preperiodic points other than ∞. Sometimes the filled Julia sets have nontrivial intersection in $\mathbb{C}$, but – when the parameters are algebraic – the v-adic filled Julia sets are disjoint at some place v. Then, again, there can be no common preperiodic points other than ∞. Examples are shown in Figures 2.1 and 2.2. As we shall explain below, the filled Julia set of f_c at any non-archimedean place v (defined as the set of points with bounded orbit) with $|c|_v > 1$ is a subset of $\{z : |z| = |c|_v^{1/2}\}$, while it is the closed unit disk whenever $|c|_v \leq 1$.

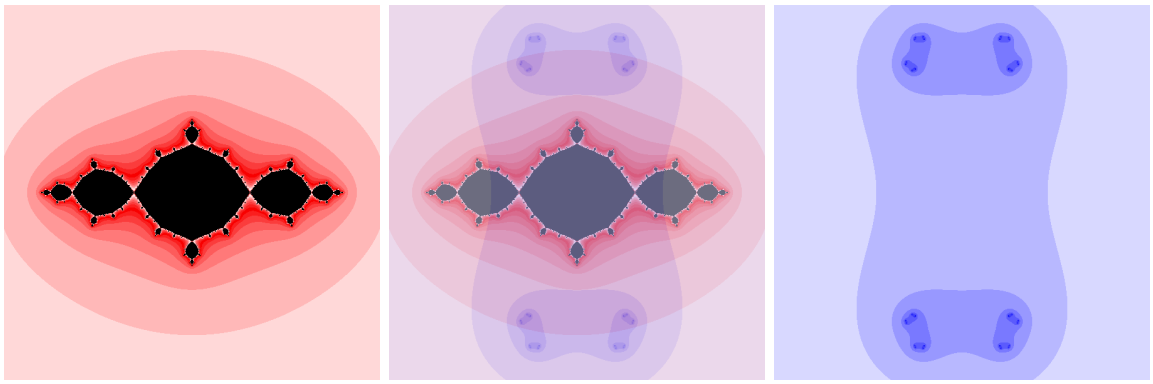

FIGURE 2.1. The filled Julia sets of $f(z) = z^2 - 1$ (left) and $g(z) = z^2 + 2$ (right) are disjoint; they have no common preperiodic points except for ∞.

2.2. **Galois orbits.** Let $f(z) = z^2$ and $g(z) = z^2 - 1$. Here we show that

$$\mathrm{Preper}(f) \cap \mathrm{Preper}(g) \;=\; \{0, 1, -1, \infty\}.$$

We know that the preperiodic points of f are the roots of unity, together with 0 and ∞. The preperiodic points of any f_c are roots of the polynomial equations given by $f_c^n(z) = f_c^m(z)$ for any $n > m \geq 0$; so the set of preperiodic points is invariant under the action of $\mathrm{Gal}(\overline{K}/K)$, whenever c lies in K. In this case, we can take $K = \mathbb{Q}$. So we need to show that for all $n \geq 3$, at least one of the primitive n-th roots of unity will have infinite forward orbit under the action of g.

The proof is elementary and has two steps:

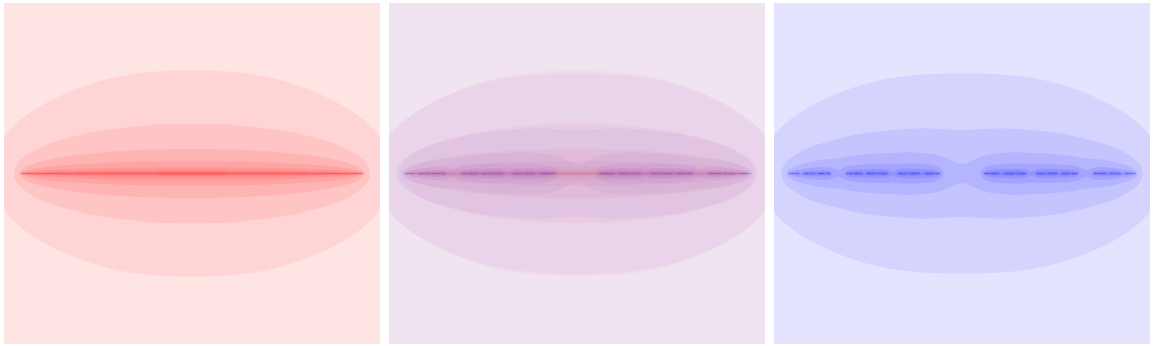

FIGURE 2.2. The filled Julia sets of $f(z) = z^2 - 2$ (left) and $g(z) = z^2 - 2.1$ (right) have significant overlap in $\mathbb{C}$, but there are no common preperiodic points except for ∞, because the filled Julia sets are disjoint at the primes 2 and 5.

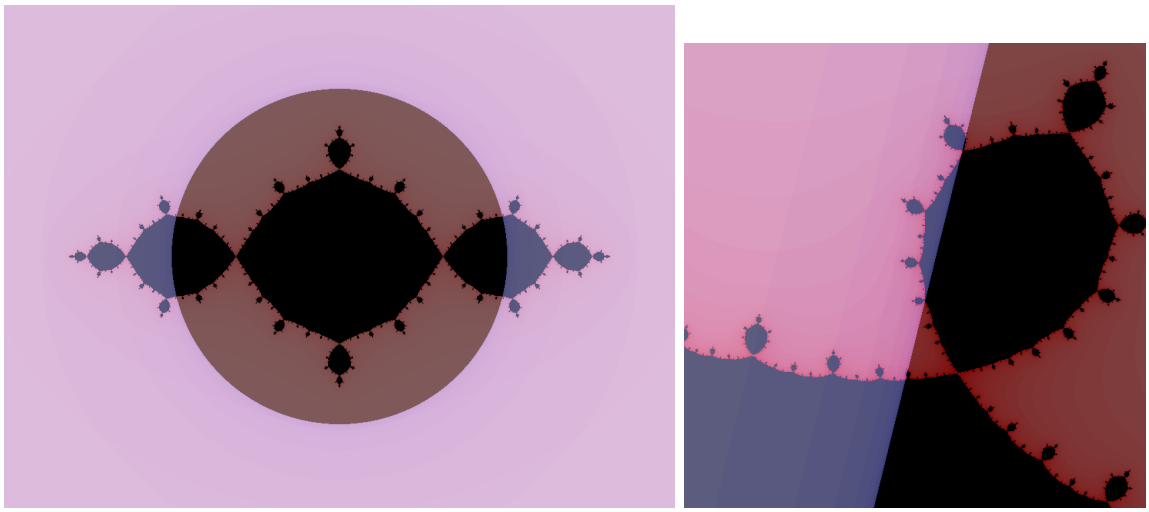

FIGURE 2.3. Filled Julia sets of $f(z) = z^2$ and $g(z) = z^2 - 1$, superimposed. At right, a zoom of the intersection of their boundaries, suggesting a possibly infinite overlap of Julia sets.

(1) Show that the subset of unit circle

$$S = \{e^{2\pi it} : t \in [0, 1/30] \cup [1/12, 5/12]\}$$

lies in the Fatou set for g; and

(2) for every $n \geq 3$, the set S contains at least one primitive n-th root of unity.

Step (2) can be checked by hand by observing that for each $12 < n < 30$, there is some k with $(k, n) = 1$ and $k/n \in [1/12, 5/12]$. Step (1) follows from a series of simple estimates, examining how g acts on arcs of the unit circle.

2.3. **The Julia sets are distinct.** It is well known that, for any polynomial, all but finitely many of the periodic points of f will be contained in its Julia set, their closure gives all of $J(f)$, and all the preperiodic points form a subset of the filled Julia set. Therefore

$$\mathrm{Preper}(f_{c_1}) = \mathrm{Preper}(f_{c_2}) \implies J(f_{c_1}) = J(f_{c_2})$$

for any $c_1, c_2 \in \mathbb{C}$. But it is also known that the Julia set determines c in this family $f_c(z) = z^2 + c$ [BE, Supplement to Theorem 1]; see also [Bea, Theorem 1]. This shows that (1.1) holds.

3. ARCHIMEDEAN ESTIMATES

In this section, we will carry out some archimedean estimates needed for the proofs of our main theorems. We work with $c \in \mathbb{C}$ and the Euclidean norm $|\cdot|$. We let $\lambda_c(z)$ denote the escape-rate function of $f_c(z) = z^2 + c$ and let μ_c denote the corresponding equilibrium measure supported on the Julia set J_c. Where possible, we provide explicit constants in our estimates, even if they are not optimal.

3.1. **Distortion.** We first recall some basic distortion statements for conformal maps.

Theorem 3.1 (Koebe 1/4 Theorem)**.** *Suppose $f : \mathbb{D} \to \mathbb{C}$ is univalent with $f(0) = 0$ and $f'(0) = 1$. Then $f(\mathbb{D}) \supset D(0, 1/4)$.*

Theorem 3.2. [BH, Corollary 3.3] *Suppose $f : U_R \to \hat{\mathbb{C}}$ is univalent and satisfies*

$$f(z) = z + \sum_{n \geq 1} \frac{a_n}{z^n}$$

near ∞. Then

$$f(U_R) \supset U_{2R},$$

for $U_R = \hat{\mathbb{C}} \setminus \overline{D}(0, R)$.

Applying these theorems to the Böttcher coordinate ϕ_c near ∞ for $f_c(z) = z^2 + c$ (see [Mi] for the definition of ϕ_c) and to the uniformizing map Φ for the complement of the Mandelbrot set $\mathcal{M}$, we get some simple inequalities.

Proposition 3.3. *For all c with $|c| > 2$ we have*

$$\log|c| - \log 2 \leq \lambda_c(c) \leq \log|c| + \log 2.$$

Proof. Let $\Phi(c) = \phi_c(c)$ be the uniformizing map from $\hat{\mathbb{C}} \setminus \mathcal{M}$ to $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ so that $\lambda_c(c) = \log|\Phi(c)|$. For the lower bound on $\lambda_c(c)$, applying Theorem 3.2 to Φ^{-1} gives

$$|c| \leq 2\, e^{\lambda_c(c)},$$

so that

$$\lambda_c(c) \geq \log|c| - \log 2.$$

For the upper bound on $\lambda_c(c)$, apply Theorem 3.2 to Φ and sets U_{2s}, $s \geq 1$, where $2s = |c|$. □

We can do similar things in the dynamical plane.

Proposition 3.4. *For each c with $|c| > 2$ and every z with $|z| > 2e^{\lambda_c(0)}$ (so in particular for all $|z| > 2^{3/2}|c|^{1/2}$), we have*

$$\log|z| - \log 2 \leq \lambda_c(z) \leq \log|z| + \log 2.$$

Proof. Let $R = e^{\lambda_c(0)}$. Then apply Theorem 3.2 to ϕ_c^{-1} and sets U_{sR} for all $s \geq 1$. Then for $sR = e^{\lambda_c(z)}$ so that $s^{-1} = e^{\lambda_c(0)-\lambda_c(z)}$, we find that

$$|z| \leq 2\, e^{\lambda_c(z)}.$$

This gives the lower bound of the proposition.

For the upper bound, set $R' = 2e^{\lambda_c(0)}$ and apply Theorem 3.2 to sets $U_{sR'}$ for all $s \geq 1$. Then for $|z| = sR'$ so that $s^{-1} = 2|z|^{-1}e^{\lambda_c(0)}$, we have

$$\lambda_c(z) \leq \log|z| + \log 2.$$

□

3.2. **Controlling escape rates from below.** We will need both upper and lower bounds on the escape rate λ_c near the Julia set J_c of $f_c(z) = z^2 + c$. We begin with an elementary observation.

Lemma 3.5. *Fix any c with $|c| \geq 25$. Let $\pm b$ be the two zeroes of f_c. Then*

$$\mu_c(D(b,1)) = \mu_c(D(-b,1)) = 1/2$$

and

$$\lambda_c(z) \geq \frac{1}{4}\log|c|$$

for all $z \notin D(b,1) \cup D(-b,1)$.

Proof. First observe that $b = i\sqrt{c}$, so that $|b| = |c|^{1/2}$. Suppose $b + t$ lies on the boundary of $D(b,1)$, so that $|t| = 1$. Then

$$f_c(b+t) = 2bt + t^2 = t(2b+t)$$

has absolute value $\geq 2|c|^{1/2} - 1 > |c|^{1/2} + 1$ for $|c| \geq 25$. In particular, f_c sends $D(b,1)$ with degree 1 over the union $D(b,1) \cup D(-b,1)$. Similarly for $D(-b,1)$, proving the first claim about the measure of each disk. As the Julia set of f_c is contained in these two disks, we know that λ_c is harmonic on the complement of their union. Under one further iterate, we have

$$|f_c^2(b+t)| \geq 4|c| - 4|c|^{1/2} + 1 - |c| = 3|c| - 4|c|^{1/2} + 1 \geq 3|c|$$

because $|c| \geq 25$. From Proposition 3.4, we conclude that

$$\lambda_c(b+t) = \frac{\lambda_c(f_c^2(b+t))}{4} \geq \frac{1}{4}(\log(3|c|) - \log 2) \geq \frac{1}{4}\log|c|$$

and similarly for $\lambda_c(-b+t)$ with $|t| = 1$. As λ_c is harmonic on $\mathbb{C}\setminus(D(b,1)\cup D(-b,1))$, this proves the lemma. □

We now extend the statement of Lemma 3.5 to two further preimages of 0 under f_c.

Lemma 3.6. *For $n = 1, 2, 3$, and for each $c \in \mathbb{C}$, we let $D_n(c)$ be the union of the 2^n disks of radius $\varepsilon_n = |2c|^{-(n-1)/2}$ centered at the solutions z to $f_c^n(z) = 0$. For each $|c| \geq 25$, the 2^n disks are disjoint, each has μ_c-measure $1/2^n$, and*

$$\lambda_c(z) \geq \frac{1}{2^{n+1}}\log|c|$$

for all $z \notin D_n(c)$ and $n = 1, 2, 3$.

Proof. Lemma 3.5 provides the result for $n = 1$ and for any $|c| \geq 25$. Note that the two disks of radius $2\varepsilon_1 = 2$ around the solutions to $f(z) = 0$ are disjoint.

For $n = 2, 3$, suppose that z is a solution to $f_c^n(z) = 0$. Note that

$$\lambda_c(z) = \frac{1}{2^{n+1}}\lambda_c(c) \leq \frac{1}{8}\lambda_c(c) \leq \frac{1}{8}(\log|c| + \log 2) < \frac{1}{4}\log|c|$$

by Proposition 3.3. Since $|c| \geq 25$, the point z must lie in the disks of radius 1 about $\pm i\sqrt{c}$ by Lemma 3.5. In particular, we know that $|z| > |c|^{1/2} - 1$, so that

$$\begin{aligned} |f_c(z+t) - f_c(z)| &\geq \frac{1}{|2c|^{(n-1)/2}}\left(2|c|^{1/2} - 2 - |2c|^{-(n-1)/2}\right) \\ &= \frac{1}{|2c|^{(n-2)/2}}\left(\sqrt{2}\left(1 - \frac{1}{|c|^{1/2}}\right) - \frac{1}{|2c|^{-n/2}}\right). \end{aligned}$$

As $|c| > 25$ and n is 2 or 3, we have

$$\sqrt{2}\left(1 - \frac{1}{|c|^{1/2}}\right) - \frac{1}{|2c|^{-n/2}} \geq \frac{4\sqrt{2}}{5} - \frac{1}{50} > 1,$$

and we conclude that

$$|f_c(z+t) - f_c(z)| > \frac{1}{|2c|^{(n-1)/2}} = \varepsilon_n.$$

By a similar argument, we also have

$$|f_c(z+t) - f_c(z)| \leq 2\varepsilon_{n-1}.$$

As the disks at level $n = 1$ and radius 2 are disjoint, this proves the lemma for $n = 2$.

It only remains to show that the disks at level $n = 2$ of radius $2\varepsilon_2 = \sqrt{2}/|c|^{1/2}$ are disjoint, and the proof will be complete also for $n = 3$. But this is clear for $|c|$

sufficiently large. Indeed, the points z satisfying $f_c^2(z) = 0$ have expansions in $c^{1/2}$ of the form $\pm\beta(c)$ and $\pm\beta'(c)$ where

$$\left|\beta(c) - \left(i\sqrt{c} + \frac{1}{2} + \frac{i}{8\sqrt{c}}\right)\right| = \left|\sum_{j=3}^{\infty} C^j_{\frac{1}{2}} \cdot \frac{1}{(i\sqrt{c})^{j-1}}\right| \leq \frac{5}{4|c|}, \tag{3.1}$$

with binomial coefficients

$$C^j_{1/2} = \begin{pmatrix} 1/2 \\ j \end{pmatrix},$$

and similarly

$$\left|\beta'(c) - \left(i\sqrt{c} - \frac{1}{2} + \frac{i}{8\sqrt{c}}\right)\right| \leq \frac{5}{4|c|}.$$

In particular, the distance between the two closest such roots satisfies

$$|\beta(c) - \beta'(c)| \geq 1 - \frac{5}{2|c|} \geq \frac{3}{4} \geq \sqrt{2}/|c|^{1/2}$$

for $|c| \geq 25$. □

3.3. **Controlling escape rates from above.** We now provide an upper bound, applying the Distortion Theorems stated above.

Lemma 3.7. *Fix any c with $|c| \geq 25$. For each $n \geq 1$ and for all $z \in \mathbb{C}$ with*

$$\operatorname{dist}(z, J_c) < \frac{1}{5 \cdot 3^n |c|^{(n-2)/2}}$$

we have

$$\lambda_c(z) \leq \frac{1}{2^n}(\log|c| + \log 2) < \frac{1}{2^{n-1}} \log|c|.$$

Proof. The two inverse branches of f_c are univalent on $D(0, |c|)$. Fix any point z_0 in J_c. From Lemma 3.5, we know that $|z_0| \leq |c|^{1/2} + 1$, so that f_c has two univalent branches of the inverse defined on the disk $D(z_0, |c| - |c|^{1/2} - 1)$ and $|(f_c^n)'(z_0)| \leq 2^n(|c|^{1/2} + 1)^n$. Applying Theorem 3.1 to the inverse branches of each iterate on these disks about points $z_0 \in J_c$, we find

$$f_c^{-n} D(z_0, |c| - |c|^{1/2} - 1) \supset D\left(f_c^{-n}(z_0), \frac{|c| - |c|^{1/2} - 1}{4 \cdot 2^n(|c|^{1/2} + 1)^n}\right).$$

From Proposition 3.4 (and the maximum principle for λ_c), we know that $\lambda_c(z) \leq \log|c| + \log 2$ on $D(0, c)$, and therefore

$$\lambda_c(z) \leq \frac{1}{2^n}(\log|c| + \log 2)$$

on each of these disks of radius $(|c| - |c|^{1/2} - 1)/(4 \cdot 2^n(|c|^{1/2} + 1)^n)$ about points in the Julia set. Finally, we observe that

$$\frac{|c| - |c|^{1/2} - 1}{4\,2^n(|c|^{1/2} + 1)^n} \geq \frac{|c|(1 - |c|^{-1/2} - |c|^{-1})}{4\,2^n|c|^{n/2}(1 + |c|^{-1/2})^n} \geq \frac{|c|(19/25)}{4\,2^n|c|^{n/2}(6/5)^n} \geq \frac{1}{5\,3^n\,|c|^{(n-2)/2}}$$

for all $|c| \geq 25$. □

Proposition 3.8. *Fix $L \geq 27$. For all $0 < r < 1/4$ and for all $c \in \mathbb{C}$, we have*

$$\lambda_c(z) \leq r \log \max\{|c|, L\}$$

for every z in a neighborhood of radius

$$\frac{1}{(\max\{|c|, L\})^{3\log(1/r)}}$$

around the filled Julia set K_c.

Proof. First assume that $|c| > L$. Note that $J_c = K_c$ in this case. Lemma 3.7 states that

$$\lambda_c(z) \leq \frac{1}{2^{n-1}} \log |c|$$

whenever $\operatorname{dist}(z, J_c) < (5 \cdot 3^n |c|^{(n-2)/2})^{-1}$. For $L \geq 27 = 3^3$, we have $5 \cdot 3^n = 15 \cdot 3^{n-1} < L^{1+(n-1)/3} = L^{(n+2)/3}$. Therefore,

$$5 \cdot 3^n |c|^{(n-2)/2} \leq L^{(n+2)/3} |c|^{(n-2)/2} < |c|^{5n/6}.$$

Taking $r = 1/2^{n-1}$, so that $n = \log(1/r)/(\log 2) + 1$, we can take any

$$\kappa(r) > 5n/6 = \frac{5}{6} \frac{1}{\log 2} \log(1/r) + \frac{5}{6} \approx 1.2 \log(1/r) + 5/6,$$

and then any z satisfying $\operatorname{dist}(z, J_c) \leq |c|^{-\kappa(r)}$ will also satisfy $\lambda_c(z) \leq r \log |c|$. In particular, for any $r < 1/4$, we can take $\kappa(r) = 3 \log(1/r)$. This proves the proposition for $|c| > L$.

Now assume $|c| \leq L$. For $|c| > 2$, Proposition 3.4 implies that if $|z| > 2^{3/2} |c|^{1/2}$, then

$$\lambda_c(z) \leq \log |z| + \log 2.$$

Consider the circle of radius L. For all $|c| \leq L$, we have $2^{3/2} |c|^{3/2} \leq 2^{3/2} L^{1/2} < L$, so that

$$\lambda_c(z) \leq \log L + \log 2, \tag{3.2}$$

for all $2 < |c| \leq L$ and for all $|z| = L$. But then, fixing z, and using the fact that $\lambda_c(z)$ is subharmonic in c, we obtain the inequality (3.2) for all $|c| \leq L$ and all $|z| = L$. Furthermore, for all $|c| > 2$ and $|z| \geq 2^{3/2} |c|^{1/2}$, we have the lower bound that

$$\lambda_c(z) \geq \log |z| - \log 2 \geq \frac{1}{2} \log(2|c|) > 0$$

so that the Julia set is contained in a disk of radius $2^{3/2} |c|^{1/2} \leq 2^{3/2} L^{1/2}$. On the other hand, for $|c| \leq 2$, it is easy to compute that the filled Julia set lies in a closed disk of radius 2, so we have

$$K_c \subset D(0, 2^{3/2} L^{1/2})$$

for all $|c| \leq L$.

We now fix any $z \notin K_c$ with $|z| < L$, and let $z_0 \in K_c$ be a closest point to z. Let $n = n(z)$ be the smallest $n \geq 0$ for which $|f^n(z)| \geq L$. Then

$$\lambda_c(z) = \frac{1}{2^{n-1}} \lambda_c(f_c^{n-1}(z)) \leq \frac{1}{2^{n-1}} (\log L + \log 2).$$

On the other hand, as $|f_c'(z)| = |2z| \leq 2L$ for all $|z| \leq L$, we have

$$12 < L - 2^{3/2} L^{1/2} \leq |f_c^n(z) - f_c^n(z_0)| \leq (2L)^n |z - z_0|,$$

so that

$$|z - z_0| \geq \frac{12}{(2L)^n}.$$

In other words, for all z within distance $12/(2L)^n$ of K_c, we have

$$\lambda_c(z) \leq \frac{1}{2^{n-1}} (\log L + \log 2) \leq \frac{1}{2^{n-2}} \log L.$$

Note that $2^8/12 < 27 \leq L$ and $2^4 < L$, and so

$$12/(2L)^n \geq 1/(2^{n-8} L^{n+1}) \geq 1/L^{(n-8)/4+n+1} = 1/L^{\frac{5}{4}n-1}$$

Writing $r = 1/2^{n-2}$, we have $n = \log(1/r)/\log 2 + 2$, so that

$$\frac{5}{4}n - 1 = \frac{5}{4 \log 2} \log(1/r) + \frac{3}{2} \approx 1.8 \log(1/r) + \frac{3}{2} \leq 3 \log(1/r)$$

for all $r < 1/4$. Consequently, for all $r < 1/4$, for all $|c| \leq L$, and for all z within distance $1/L^{3 \log(1/r)}$ of the filled Julia set K_c, we have that

$$\lambda_c(z) \leq r \log L.$$

□

4. Bounds on the archimedean pairing

In this section, we provide estimates on the archimedean contributions to the pairing $\langle f_{c_1}, f_{c_2} \rangle$, to obtain a local version of Theorem 1.7. As in the previous section, we work with $c \in \mathbb{C}$ and Euclidean absolute value $|\cdot|$. We let $\lambda_c(z)$ denote the escape-rate function of $f_c(z) = z^2 + c$, defined by

$$\lambda_c(z) = \lim_{n \to \infty} \frac{1}{2^n} \log^+ |f_c^n(z)|,$$

where $\log^+ = \max\{\log, 0\}$. We let $\mu_c = \frac{1}{2\pi} \Delta \lambda_c$ denote the equilibrium measure supported on the Julia set J_c. Where possible, we provide explicit constants, even if they are not optimal, for our estimates of the Euclidean energy

$$E_\infty(c_1, c_2) := \int \lambda_{c_1} \, d\mu_{c_2} = \int \lambda_{c_2} \, d\mu_{c_1}.$$

Theorem 4.1. *There exist constants $C, C' > 0$ so that*

$$\frac{1}{16}\log^+|c_1 - c_2| - C \le E_\infty(c_1, c_2) \le \frac{1}{2}\log^+\max\{|c_1|, |c_2|\} + C'$$

for all $c_1, c_2 \in \mathbb{C}$. Furthermore, there exists $L > 0$ so that if $r := \max\{|c_1|, |c_2|\} \ge L$ and

$$\frac{3}{r^{1/2}} \le |c_1 - c_2|,$$

then

$$\frac{1}{64}\log\max\{|c_1|, |c_2|\} \le E_\infty(c_1, c_2).$$

Remark 4.2. The proof shows that we can take $L = 1000$, $C = \frac{1}{16}\log 2L < 1/2$, and $C' = \log 8 < 5/2$ in Theorem 4.1.

4.1. **Proof of Theorem 4.1.** Throughout this proof, we will assume for notational convenience that

$$r = |c_1| \ge |c_2|.$$

We proceed by cases, determined by just how close the two parameters are. We then apply Proposition 3.6 to obtain the needed lower bounds on the escape rate of λ_{c_1} on the Julia set J_{c_2}.

Case 0. Suppose $|c_2| \le 25$. For $|c_2| \le 2$, it is straightforward to compute that the filled Julia set satisfies $K_{c_2} \subset \overline{D(0,2)}$. For $2 < |c_2| \le 25$, Proposition 3.4 provides a lower bound of

$$\lambda_{c_2}(z) \ge \log|z| - \log 2 \ge \frac{1}{2}\log(2|c_2|) > 0$$

for $|z| \ge 2^{3/2}|c_2|^{1/2}$. Therefore, the Julia set of f_{c_2} is contained in a disk of radius $2^{3/2}|c_2|^{1/2} \le 2^{3/2}5$. Thus, for all $|c_2| \le 25$ and $|c_1| > (2^{3/2}\cdot 5 + 1)^2 \approx 229.3$, Lemma 3.5 implies that $\lambda_{c_1}(z) \ge \frac{1}{4}\log|c_1|$ for all $z \in J_{c_2}$. This gives

$$\int \lambda_{c_1}\, d\mu_{c_2} \ge \frac{1}{4}\log|c_1|$$

for $|c_2| \le 25$ and $|c_1| \ge 230$.

In Cases 1-3, we may assume that $r = |c_1| \ge |c_2| \ge 25$.

Case 1. Suppose that for any choice of square roots, we have $|\sqrt{c_1} - \sqrt{c_2}| \ge 2$. By Lemma 3.5 we have $\lambda_{c_1}(z) \ge \frac{1}{4}\log|c_1|$ for all $z \in J_{c_2}$, so

$$\int \lambda_{c_1}(z)\, d\mu_{c_2} \ge \frac{1}{4}\log|c_1|$$

for $|c_1| \ge |c_2| \ge 25$.

Case 2. Suppose that there is a choice of square roots for which $\frac{2}{r^{1/2}} \leq |\sqrt{c_1} - \sqrt{c_2}| < 2$. With these choices of square roots, the solutions of $f_c^2(z) = 0$ are

$$\beta(c) = i\sqrt{c} + \frac{1}{2} + \frac{i}{8\sqrt{c}} + O\left(\frac{1}{|c|}\right)$$

and

$$\beta'(c) = i\sqrt{c} - \frac{1}{2} + \frac{i}{8\sqrt{c}} + O\left(\frac{1}{|c|}\right),$$

along with $-\beta(c)$ and $-\beta'(c)$. By Proposition 3.6, if the disk $D(\beta(c_2), 1/|2c_2|^{1/2})$ does not intersect any disk of radius $1/|2c_1|^{1/2}$ about a solution of $f_1^2(z) = 0$, then for all $z \in D(\beta(c_2), 1/|2c_2|^{1/2})$ we have

$$\lambda_{c_1}(z) \geq \frac{1}{8} \log |c_1|,$$

and since the same is true for the disk centered at $-\beta(c_2)$ by $\pm$ invariance, the inequality is satisfied for a set of μ_{c_2}-measure $1/2$. Therefore,

$$\int \lambda_{c_1} \, d\mu_{c_2} \geq \frac{1}{16} \log |c_1|.$$

On the other hand, as $|\sqrt{c_1} - \sqrt{c_2}| < 2$, if $D(\beta(c_2), 1/|2c_2|^{1/2})$ intersects any disk of radius $1/|2c_1|^{1/2}$ about a solution of $f_{c_1}^2(z) = 0$, that disk must be centered at either $\beta(c_1)$ or $\beta'(c_1)$, since $|\beta(c_2) + \beta(c_1)| \geq |c_1|^{1/2}$ and similarly for $\beta(c_2) + \beta'(c_1)$. We have

$$\beta(c_1) - \beta(c_2) = i(\sqrt{c_1} - \sqrt{c_2}) + \frac{i}{8}\left(\frac{1}{\sqrt{c_1}} - \frac{1}{\sqrt{c_2}}\right) + O\left(\frac{1}{|c_2|}\right),$$

so that using the assumed bounds, we have

$$|\beta(c_1) - \beta(c_2)| \geq \frac{2}{|c_1|^{1/2}} - \frac{1}{8}\left(\frac{4}{|c_1 c_2|^{1/2}}\right) + O\left(\frac{1}{|c_2|}\right) = \frac{2}{|c_1|^{1/2}} + O\left(\frac{1}{|c_2|}\right),$$

using for the middle term the crude bound $|c_1 - c_2| \leq 4|c_1|^{1/2}$ implied by $|\sqrt{c_1} - \sqrt{c_2}| < 2$. Then, exactly as in (3.1) in the proof of Lemma 3.6, we can take

$$|\beta(c_1) - \beta(c_2)| \geq \frac{2}{|c_1|^{1/2}} - \frac{5}{2}\frac{1}{|c_2|}$$

because $|c_1| \geq |c_2| \geq 25$. Since $|c_2|^{1/2} > |c_1|^{1/2} - 2$, taking $|c_1| \geq 230$ is enough to guarantee this distance will be larger than $2(1/|2c_2|^{1/2})$, and the disks $D(\beta(c_1), 1/|2c_1|^{1/2})$ and $D(\beta(c_2), 1/|2c_2|^{1/2})$ will be disjoint. Similarly we deduce that the disks $D(\beta'(c_1), 1/|2c_1|^{1/2})$ and $D(\beta'(c_2), 1/|2c_2|^{1/2})$ are disjoint.

But observe also that if

$$|\beta(c_2) - \beta'(c_1)| < \frac{2}{|2c_2|^{1/2}} = \frac{\sqrt{2}}{|c_2|^{1/2}},$$

then $\beta'(c_2)$ must be far from both $\beta'(c_1)$ and $\beta(c_1)$, because

$$\begin{aligned} |\beta'(c_2) - \beta(c_1)| &= |\beta'(c_2) - \beta(c_2) + \beta(c_2) - \beta'(c_1) + \beta'(c_1) - \beta(c_1)| \\ &\geq |\beta'(c_2) - \beta(c_2) + \beta'(c_1) - \beta(c_1)| - \frac{\sqrt{2}}{|c_2|^{1/2}} \\ &= 2 - \frac{\sqrt{2}}{|c_2|^{1/2}} - 4 \cdot \frac{5}{4}\frac{1}{|c_2|} \end{aligned}$$

We therefore have, for $r = |c_1| \geq 230$ and square roots satisfying $\frac{2}{r^{1/2}} \leq |\sqrt{c_1} - \sqrt{c_2}| < 2$, at least one of the four disks of radius $1/|2c_2|^{1/2}$ around a solution to $f_{c_2}^2(0)$ is disjoint from the four disks of radius $1/|2c_1|^{1/2}$ about the four solutions of $f_{c_1}^2(z) = 0$. By the $\pm$ symmetry, two of these disks must be disjoint. As these two disks carry $1/2$ of the measure μ_{c_2}, we have by Proposition 3.6 that

$$\int \lambda_{c_1} \, d\mu_{c_2} \geq \frac{1}{16} \log |c_1|.$$

Case 3. Suppose there is a choice of square roots for which

$$\frac{3}{2r} \leq |\sqrt{c_1} - \sqrt{c_2}| < \frac{2}{r^{1/2}}.$$

We will argue precisely as in Case 2, but with the third preimages of 0 rather than second. Two solutions of $f_c^3(z) = 0$ have the form

$$s(c) := i\sqrt{c} + \frac{1}{2} - \frac{i}{8\sqrt{c}} + \frac{1}{8c} + O\left(\frac{1}{|c|^{3/2}}\right)$$

and

$$s'(c) := i\sqrt{c} + \frac{1}{2} + \frac{3i}{8\sqrt{c}} - \frac{1}{8c} + O\left(\frac{1}{|c|^{3/2}}\right).$$

From the Taylor expansion, and the fact that $|c| > 100$, the above big-O's have the following estimate, to be proved below:

$$\left| s(c) - \left(i\sqrt{c} + \frac{1}{2} - \frac{i}{8\sqrt{c}} + \frac{1}{8c} \right) \right| \leq 5 \frac{1}{|c|^{\frac{3}{2}}} \tag{4.1}$$

and similarly for $s'(c)$. Notice that under the action of f_c, we have $s(c) \mapsto \beta(c)$ and $s'(c) \mapsto \beta'(c)$, and that both $s(c)$ and $s'(c)$ are distance at least $1/2$ from all other solutions of $f_c^3(z)$ (except each other).

If the disk of radius $1/|2c_2|$ about $s(c_2)$ intersects any disk of radius $1/|2c_1|$ about a solution of $f_{c_1}^3(z) = 0$, then that disk must be centered at either $s(c_1)$ or $s'(c_1)$, because of the form of the power series expansions of the various third preimages of 0. If this disk $D(s(c_2), 1/|2c_2|)$ is disjoint from both $D(s(c_1), 1/|2c_1|)$ and $D(s'(c_1), 1/|2c_1|)$, then from the $\pm$ symmetry and Proposition 3.6, we have

$$\int \lambda_{c_1} \, d\mu_{c_2} \geq \frac{1}{4 \cdot 16} \log |c_1| = \frac{1}{64} \log |c_1|.$$

Now, we have by our assumed bounds that $|\sqrt{c_1} - \sqrt{c_2}| < 2|c_1|^{-1/2}$, so that

$$|c_1 - c_2| = |(\sqrt{c_1} - \sqrt{c_2})(\sqrt{c_1} + \sqrt{c_2})| < \frac{4|c_1|^{1/2}}{|c_1|^{1/2}} = 4,$$

and therefore,

$$|s(c_1) - s(c_2)| \geq \frac{3}{2|c_1|} - \frac{2}{8|c_2|^{3/2}} - \frac{10}{|c_2|^{\frac{3}{2}}} > \frac{1}{|c_2|}$$

for $|c_1| \geq 1000$. So the disks $D(s(c_1), 1/|2c_1|)$ and $D(s(c_2), 1/|2c_2|)$ are disjoint. But if

$$|s'(c_1) - s(c_2)| < \frac{1}{|c_2|},$$

then

$$\begin{aligned}
|s(c_1) - s'(c_2)| &= |s(c_1) - s'(c_1) + s'(c_1) - s(c_2) + s(c_2) - s'(c_2)| \\
&\geq |s(c_1) - s'(c_1) + s(c_2) - s'(c_2)| - \frac{1}{|c_2|} \\
&\geq \left| \frac{-i}{2} \left(\frac{1}{\sqrt{c_1}} + \frac{1}{\sqrt{c_2}} \right) + 2O\left(\frac{1}{|c_2|^{3/2}} \right) \right| - \frac{1}{|c_2|} \\
&= \left| \frac{-i}{2} \left(\frac{\sqrt{c_1} + \sqrt{c_2}}{\sqrt{c_1 c_2}} \right) + 2O\left(\frac{1}{|c_2|^{3/2}} \right) \right| - \frac{1}{|c_2|} \\
&\geq \left| \frac{-i}{2} \left(\frac{\sqrt{c_1} + \sqrt{c_2}}{\sqrt{c_1 c_2}} \right) \right| - \frac{10}{|c_2|^{\frac{3}{2}}} - \frac{1}{|c_2|} \\
&\geq \frac{1}{2|c_1|^{1/2}} > \frac{1}{|c_2|}
\end{aligned}$$

for $|c_1| \geq 1000$. We conclude in this case that the disk $D(s'(c_2), 1/|2c_2|)$ is disjoint from the eight disks of radius $1/|2c_1|$ about solutions of $f_{c_1}^3(z) = 0$, and hence (again using symmetry and Proposition 3.6) we have

$$\int \lambda_{c_1} \, d\mu_{c_2} \geq \frac{1}{64} \log |c_1|.$$

Finally, since

$$|\sqrt{c_1} - \sqrt{c_2}| < \frac{3}{2|c_1|} \implies |c_1 - c_2| < \frac{3}{2|c_1|}(2|c_1|^{1/2}) \leq \frac{3}{|c_1|^{1/2}},$$

these three cases cover all possiblities for the stronger lower bound in Theorem 4.1.

Proof of estimate (4.1)**.** From the estimate (3.1), we have

$$\beta = i\sqrt{c} - \frac{1}{2} + \frac{i}{8\sqrt{c}} + a$$

with $|a| \le 5/4|c|$ whenever $|c| \ge 25$. Furthermore, let us assume that

$$s = \sqrt{-c+\beta} = i\sqrt{c}\left(1 + \frac{1}{i\sqrt{c}} + \frac{1}{2c} + \frac{1}{8i\sqrt{c}^3} - \frac{a}{c}\right)^{1/2}.$$

For convenience, we set

$$b = \left(1 + \frac{1}{i\sqrt{c}} + \frac{1}{2c} + \frac{1}{8i\sqrt{c}^3} - \frac{a}{c}\right)^{1/2} \quad \text{and} \quad e = \frac{1}{i\sqrt{c}} + \frac{1}{2c} + \frac{1}{8i\sqrt{c}^3} - \frac{a}{c}$$

and then one has

$$b = (1+e)^{1/2} = 1 + \frac{1}{2}e - \frac{1}{8}e^2 + \frac{1}{16}e^3 + \sum_{n\ge 4} C^n_{1/2} e^n \tag{4.2}$$

where $C^n_{1/2}$ are the binomial coefficients. In the following, we assume that $|c| \ge 100$, so that e can be estimated as $|e| \le \frac{11}{10}\frac{1}{\sqrt{|c|}}$. Consequently as $|C^n_{1/2}| < 1$, we have

$$\left|\sum_{n\ge 4} C^n_{1/2} e^n\right| \le 1.7\frac{1}{|c|^2} \text{ and } \frac{1}{2}\left|e - (\frac{1}{i\sqrt{c}} + \frac{1}{2c} + \frac{1}{8i\sqrt{c}^3})\right| = \left|\frac{a}{2c}\right| \le \frac{5}{8|c|^2},$$

and moreover

$$\frac{1}{8}\left|e^2 - (-\frac{1}{c} + \frac{1}{i\sqrt{c}^3})\right| \le \frac{1}{4}|c|^2 \text{ and } \frac{1}{16}\left|e^3 - (-\frac{1}{i\sqrt{c}^3})\right| \le \frac{1}{4|c|^2}.$$

Finally, we get an estimate of b using the expansion (4.2) and therefore the estimate (4.1) of s since $s = i\sqrt{c}\cdot b$.

□

We are now ready to complete the proof of the theorem. Choose any $L \ge 1000$. If $\max\{|c_1|, |c_2|\} \le L$ or $|c_1 - c_2| \le 2$, then $|c_1 - c_2| \le 2L$ and the lower bound holds trivially with the constant $\frac{1}{16}\log 2L$.

Otherwise, if $|c_1 - c_2| \ge \max\{|c_1|, |c_2|\}$, the hypotheses of either Case 0 or 1 hold, and we have

$$\frac{1}{8}\log^+|c_1 - c_2| \le \frac{1}{4}\log|c_1| \le \int \lambda_{c_1}(z)\, d\mu_{c_2},$$

as needed. On the other hand, if $\max\{|c_1|, |c_2|\} > L$ and $2 < |c_1 - c_2| < \max\{|c_1|, |c_2|\}$, then the hypotheses of either Case 0, 1, or 2 hold, and we have

$$\frac{1}{16}\log^+|c_1 - c_2| \le \frac{1}{16}\log^+\max\{|c_1|, |c_2|\} \le \int \lambda_{c_1}\, d\mu_{c_2}.$$

Thus in every case we have the first lower bound

$$\frac{1}{16}\log^+|c_1 - c_2| - C \le \int \lambda_{c_1}\, d\mu_{c_2}$$

for $C \ge \frac{1}{16}\log 2L$.

To prove the upper bounds, suppose first that $|c_1| = \max\{|c_1|, |c_2|\} > 2$. If $|c_2| \geq 2$, then by Proposition 3.4, the Julia set of f_{c_2} is contained in the disk $D(0, 2^{3/2}|c_2|^{1/2})$; this also holds for $|c_2| \leq 2$ as in that case, $J_{c_2} \subset \overline{D(0,2)}$. Also by Proposition 3.4, we have for all $z \in D(0, 2^{3/2}|c_1|^{1/2})$ (which contains $D(0, 2^{3/2}|c_2|^{1/2})$) that

$$\lambda_{c_1}(z) \leq \frac{3}{2}\log 2 + \frac{1}{2}\log|c_1| + \log 2. \tag{4.3}$$

On the other hand, for $|c_1| = \max\{|c_1|, |c_2|\} \leq 2$, we use the fact that $\lambda_{c_1}(z)$ is subharmonic in both z and c_1, so that the inequality (4.3) holds on the circle $\{|z| = 4\}$, replacing $|c_1|$ with 2, for all $|c_1| \leq 2$.

Applying this inequality to $z \in J_{c_2}$ we see that

$$\int \lambda_{c_1}\, d\mu_{c_2} \leq \frac{1}{2}\log^+ \max\{|c_1|, |c_2|\} + \log 8$$

for all $c_1, c_2 \in \mathbb{C}$. This completes the proof of Theorem 4.1.

5. Nonarchimedean bounds for prime $p \neq 2$

Let $c_1 \neq c_2$ be two elements of $\overline{\mathbb{Q}}$. Fix a number field containing c_1 and c_2, and fix a non-archimedean place v which does *not* divide $p = 2$. In this section, we provide estimates on the local energy

$$E_v := \int \lambda_{c_1,v}\, d\mu_{2,v} = \int \lambda_{c_2,v}\, d\mu_{1,v}.$$

Because the place v is fixed throughout this section, we will drop the dependence on v in the absolute value $|\cdot|_v$, denote the local Julia set of f_c by J_c, its escape rate by λ_c, and the equilibrium measure by μ_c.

Theorem 5.1. *Fix a number field K and place v of K that does not divide $p = 2$. For all $c_1, c_2 \in K$, we have*

$$\frac{1}{4}\log^+|c_1 - c_2| \leq E_v \leq \frac{1}{2}\log^+ \max\{|c_1|, |c_2|\}.$$

Furthermore, if $r := |c_1| = |c_2| > 1$ and

$$|c_1 - c_2| > \frac{1}{r^{1/2}},$$

then

$$E_v \geq \frac{1}{16}\log r.$$

We also prove an estimate on λ_c from above, at points near the v-adic Julia set of f_c, that will be needed for the proof of Theorem 1.9.

5.1. **Structure of the Julia set.** We work with the dynamics of f_c on the Berkovich affine line $\mathbf{A}_v^{1,an}$, associated to the complete and algebraically closed field $\mathbb{C}_v$, and we denote by $\zeta_{x,r}$ the Type II point corresponding to the disk of radius $r \in \mathbb{Q}_{>0}$ about x.

For $|c| \leq 1$, the map f_c has good reduction, and $J_c = \zeta_{0,1}$ is the Gauss point. For $|c| > 1$, the Julia set of f_c is a Cantor set of Type I points, lying in the union of the two open disks $D(\pm b, |c|^{1/2})$ with $f_c(\pm b) = 0$. In particular, all points $z \in J_{c,v}$ will satisfy $|z| = |c|^{1/2}$. For any point z with absolute value $|z| > |c|^{1/2}$, we have $|f^n(z)| = |z|^{2^n}$ for all $n \geq 1$, so that

$$\lambda_c(z) = \log |z|. \tag{5.1}$$

It is also the case that

$$\lambda_c(z) \leq \frac{1}{2} \log |c| \tag{5.2}$$

for all $|z| \leq |c|^{1/2}$.

Taking one further preimage of 0, we may choose β and β' so that

$$f_c(\beta) = b, \quad f_c(\beta') = -b, \quad |\beta - b| = |\beta' - b| = |\beta - \beta'| = 1, \tag{5.3}$$

and the Julia set will lie in the union of the four disks $D(\pm\beta, 1)$ and $D(\pm\beta', 1)$. See Figure 6.1.

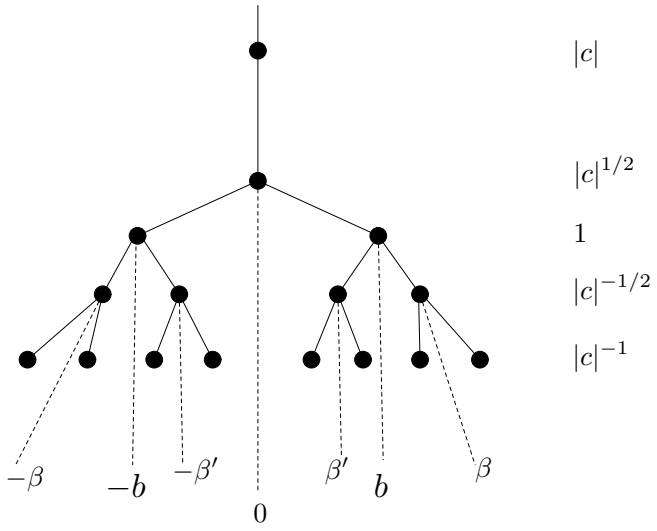

FIGURE 5.1. The tree structure of the non-archimedean Julia set, with $|c|_v > 1$ and $v \nmid 2$.

Note: identifying the branches from the Type II point $\zeta_{b,1}$ with the elements of $\mathbb{P}^1(\overline{\mathbb{F}}_p)$, and denoting the class of $z \in \mathbb{C}_v$ by $\tilde{z}$, we have

$$\tilde{\beta} = \tilde{b} + \alpha \quad \text{and } \tilde{\beta}' = \tilde{b} - \alpha$$

for some $\alpha \in \overline{\mathbb{F}}_p$. In other words, the disks containing the Julia set are centered around the preimages of 0. This is because the transformation from $\zeta_{b,1}$ to its image $f_c(\zeta_{b,1}) = \zeta_{0,|c|^{1/2}}$, acting on these branches, is linear in local coordinates. Similarly at the Type II points $\zeta_{\beta,|c|^{-1/2}}$ and $\zeta_{\beta',|c|^{-1/2}}$, the branches containing the Julia set will be symmetric about β and β'. We will exploit this symmetry in our proof.

5.2. **Proof of Theorem 5.1.** If $|c_1|$ or $|c_2|$ is ≤ 1, then because of good reduction, we have

$$E_v = \frac{1}{2}\max\{\log^+|c_1|, \log^+|c_2|\} = \frac{1}{2}\log^+|c_1 - c_2|.$$

If $|c_1|$ and $|c_2|$ are both > 1, then we can split into further cases. For $|c_1| > |c_2|$, we have

$$\lambda_{c_2}(z_1) = \frac{1}{2}\log|c_1|$$

from 6.1 for all points z_1 in Julia set J_{c_1}. Similarly for $|c_1| < |c_2|$, and therefore,

$$E_v = \frac{1}{2}\max\{\log^+|c_1|, \log^+|c_2|\} = \frac{1}{2}\log^+|c_1 - c_1|.$$

For the remainder of the proof we assume that

$$r := |c_1| = |c_2| > 1.$$

From (6.2), we will have

$$\lambda_{c_2}(z_1) \leq \frac{1}{2}\log|c_2|$$

at all points z_1 of the Julia set $J_{c_1,v}$. Therefore,

$$E_v \leq \frac{1}{2}\log|c_2| = \frac{1}{2}\log r,$$

proving the upper bound in the theorem.

For the lower bound on E_v, we now break the proof into cases, depending on how close the two parameters are to one another.

Case 1. Assume that

$$1 < r^{1/2} < s := |c_1 - c_2| \leq r = |c_1| = |c_2|.$$

Let z_1 be any point in the Julia set J_{c_1}. Then its image $z_1^2 + c_1$ must lie in the disks $D(\pm b_1, r^{1/2})$ and have absolute value $r^{1/2}$, so that $f_{c_2}(z_1) = z_1^2 + c_2 = (z_1^2 + c_1) + (c_2 - c_1)$ satisfies

$$|f_{c_2}(z_1)| = s > r^{1/2}.$$

It follows that $|f_{c_2}^n(z_1)| = s^{2^{n-1}}$ for all n. This gives

$$\lambda_{c_2}(z_1) = \frac{1}{2}\log s = \frac{1}{2}\log|c_1 - c_2|$$

for all z_1 in the Julia set of f_{c_1}. Therefore,

$$E_v = \frac{1}{2}\log|c_1 - c_2| > \frac{1}{4}\log r.$$

Case 2. Now suppose $|c_1 - c_2| = r^{1/2}$, and recall that $b_i^2 = -c_i$. Note that

$$(b_1 + b_2)(b_1 - b_2) = b_1^2 - b_2^2 = c_2 - c_1 \tag{5.4}$$

and at least one of the factors on the left hand side has norm $r^{1/2}$ so the other must have norm 1. Let's assume the second has norm 1, so that

$$|b_1 - b_2| = 1.$$

If the two branches from $\zeta_{b_1,1} = \zeta_{b_2,1}$ containing J_{c_1} are disjoint from those containing J_{c_2}, then for any element $z_2 \in J_{c_2}$ we have

$$|f_{c_1}(z_2)| = r^{1/2} \text{ and } |f_{c_1}^n(z_2)| = (r^{1/2})^{2^{n-1}} \text{ for all } n \geq 2$$

so that

$$\lambda_{c_1}(z_2) = \frac{1}{4}\log r = \frac{1}{2}\log|c_1 - c_2|$$

for all $z_2 \in J_{c_2}$, and

$$E_v = \frac{1}{4}\log r = \frac{1}{2}\log|c_1 - c_2|.$$

Note that from (6.3), we have

$$(\beta_1 - \beta_2)(\beta_1 + \beta_2) = \beta_1^2 - \beta_2^2 = b_1 - c_1 - (b_2 - c_2) = (b_1 - b_2) + (c_2 - c_1), \tag{5.5}$$

and the right-hand-side has absolute value $|c_1 - c_2| = r^{1/2}$, so that

$$|\beta_1 - \beta_2| = 1.$$

However, it can happen that one of the branches from $\zeta_{b_1,1}$ containing J_{c_1} does coincide with a branch containing J_{c_2}, so that, for example, $D(\beta_1, 1) = D(\beta_2', 1)$. Indeed,

$$(\beta_1 - \beta_2')(\beta_1 + \beta_2') = (b_1 + b_2) + (c_2 - c_1)$$

and the terms on the right-hand-side might cancel to give absolute value smaller than $r^{1/2}$. But by the symmetry of the disks around the points b_i, if $D(\beta_1, 1) = D(\beta_2', 1)$, then the other disks $D(\beta_1', 1)$ and $D(\beta_2, 1)$ must be disjoint. Indeed, if $\tilde{b}_1 + \alpha_1 = \tilde{\beta}_1 = \tilde{\beta}_2{}' = \tilde{b}_2 - \alpha_2$ and $\tilde{b}_1 - \alpha_1 = \tilde{\beta}_1{}' = \tilde{\beta}_2 = \tilde{b}_2 + \alpha_2$ in $\overline{\mathbb{F}}_p$, then

$$2\alpha_1 = -2\alpha_2 \implies \alpha_1 = -\alpha_2 \text{ because } p \neq 2,$$

so we must have $\tilde{b}_1 = \tilde{b}_2$, which contradicts the fact that $|b_1 - b_2| = 1$.

It follows that for all $z_2 \in D(\beta_2', 1)$, we can compute

$$\lambda_{c_1}(z_2) = \frac{1}{4}\log r.$$

By the symmetry of the Julia sets, this will also hold for points in the disk $D(-\beta_2', 1)$, and together they make up half (w.r.t. the measure μ_{c_2}) of J_{c_2}. Therefore,

$$E_v \geq \frac{1}{8}\log r = \frac{1}{4}\log|c_1 - c_2|.$$

Case 3. Assume that

$$1 < s := |c_1 - c_2| < r^{1/2}.$$

Then from (5.4), we have

$$\frac{1}{r^{1/2}} < |b_1 - b_2| = \frac{s}{r^{1/2}} < 1.$$

Also, from (5.5), we see that

$$\frac{1}{r^{1/2}} < |\beta_1 - \beta_2| = \frac{s}{r^{1/2}} < 1$$

and similarly for β_1' and β_2'. Consequently, the four disks $D(\pm\beta_1, s/r^{1/2})$ and $D(\pm\beta_1', s/r^{1/2})$ are disjoint from the corresponding disks around $\pm\beta_2$ and $\pm\beta_2'$. Furthermore, for any $z_2 \in J_{c_2}$, we have

$$|f_{c_1}(z_2)| = s, \quad |f_{c_1}^2(z_2)| = sr^{1/2}, \text{ and } |f_{c_1}^n(z_2)| = (sr^{1/2})^{2^{n-2}} \text{ for all } n > 2,$$

so that

$$\lambda_{c_1}(z_2) = \frac{1}{4}\log(sr^{1/2})$$

for all $z_2 \in J_{c_2}$. Therefore,

$$E_v = \frac{1}{4}\log(sr^{1/2}) = \frac{1}{8}\log r + \frac{1}{4}\log|c_1 - c_2|.$$

Case 4. Now suppose $|c_1 - c_2| = 1$. The proof here is similar to Case 2, but we work with the disks around β and β'. From (5.4) and (5.5) we have

$$|b_1 - b_2| = |\beta_1 - \beta_2| = |\beta_1' - \beta_2'| = \frac{1}{r^{1/2}}.$$

Because of the symmetry of the Julia set around β and β', if for example the disks $D(\beta_1, 1/r^{1/2})$ and $D(\beta_2', 1/r^{1/2})$ coincide, then the disks $D(\beta_1', 1/r^{1/2})$ and $D(\beta_2, 1/r^{1/2})$ must be disjoint, so that

$$|f_{c_1}(z)| = 1, \quad |f_{c_1}^2(z)| = r^{1/2}, \quad \text{and } |f_{c_1}^n(z)| = (r^{1/2})^{2^{n-2}} \text{ for all } n > 2,$$

for all $z \in D(\beta_2', 1/r^{1/2})$. Therefore,

$$\lambda_{c_1}(z) = \frac{1}{8}\log r$$

for half of J_{c_2}, and consequently,

$$E_v \geq \frac{1}{16}\log r$$

in all cases with $|c_1 - c_2| = 1$.

Case 5. The final case to treat is with

$$1/r^{1/2} < s := |c_1 - c_2| < 1.$$

We have

$$\frac{1}{r} < |b_1 - b_2| = |\beta_1 - \beta_2| = |\beta_1' - \beta_2'| = \frac{s}{r^{1/2}} < \frac{1}{r^{1/2}}$$

from (5.4) and (5.5). All points $z_2 \in J_{c_2}$ will satisfy

$$|f_{c_1}^3(z_2)| = r^{3/2}\frac{s}{r^{1/2}} = rs \quad \text{and } |f_{c_1}^n(z_2)| = (rs)^{2^{n-3}} \text{ for all } n > 3,$$

so that

$$\lambda_{c_1}(z_2) = \frac{1}{8}\log(rs)$$

and

$$E_v = \frac{1}{8}\log(rs) > \frac{1}{16}\log r.$$

This completes the proof of the theorem.

5.3. **An upper bound on the local height near the Julia set.** We will use the following proposition in the proof of Theorem 1.9. This is a non-archimedean analog to the distortion estimate provided in Proposition 3.8.

Proposition 5.2. *Suppose v is a non-archimedean place of K, not dividing 2. For each c with $|c|_v > 1$ and all $0 < r < 1$, we have*

$$\lambda_{c,v}(z) \le r \log |c|_v$$

for all z within distance

$$\frac{1}{|c|_v^{\log(1/r)}}$$

of the Julia set J_c in $\mathbf{P}^1_v$. For $|c|_v \le 1$, we have $\lambda_{c,v}(z) = 0$ for all $|z|_v \le 1$.

Proof. Recall that all points x of the Julia set J_c satisfy $|x| = |c|_v^{1/2}$. For all $x \in J_c$ and all $z = x + y$ with $|y| < |c|^{1/2}$, we have

$$|f_c(z) - f_c(x)|_v = |2xy + y^2|_v = |y||c|^{1/2}.$$

Recall that $\lambda_c(z) = \log|z|$ for all $|z| \ge |c|^{1/2}$.

In particular, for any $n \ge 1$ and any point z within distance $|c|/|c|^{n/2}$ of the Julia set will have $\lambda_c(z) = 2^{-n}\lambda_c(f^n(z)) \le 2^{-n}\log|c|$. In other words, setting $r = 1/2^n$, then

$$\log(1/r) = n\log 2 > \frac{n}{2} - 1$$

so that any point z within distance $|c|^{-\log(1/r)}$ of the Julia set will satisfy

$$\lambda_c(z) \le r \log |c|.$$

The proof of the last statement of the proposition is immediate, because f_c has good reduction with $J_c = \zeta_{0,1}$ and $\lambda_{c,v}(z) = \log^+|z|_v$. □

6. Nonarchimedean bounds for prime $p = 2$

Let $c_1 \neq c_2$ be two elements of $\overline{\mathbb{Q}}$. Fix a number field containing c_1 and c_2, and fix a non-archimedean place v which divides the prime $p = 2$. In this section, we provide estimates on the local energy

$$E_v := \int \lambda_{c_1,v}\, d\mu_{c_2,v} = \int \lambda_{c_2,v}\, d\mu_{c_1,v}.$$

Because the place v is fixed throughout this section, we will drop the dependence on v in the absolute value $|\cdot|_v$, denote the local Julia set of f_c by J_c, its escape rate by λ_c, and the equilibrium measure by μ_c.

Theorem 6.1. *Suppose c_1 and c_2 lie in a number field K, and v is a non-archimedean place of K with $v \mid 2$. For all $c_1, c_2 \in K$, we have*

$$\frac{1}{16}\log^+|c_1 - c_2| \leq E_v \leq \frac{1}{2}\log^+\max\{|c_1|, |c_2|\}.$$

Furthermore, if $r := |c_1| = |c_2| > 16$ and

$$|c_1 - c_2| > \frac{2}{r^{1/2}},$$

then

$$E_v \geq \frac{1}{16}\log r - \frac{3}{16}\log 2.$$

We also prove an estimate on λ_c from above, at points near the v-adic Julia set of f_c, that will be needed for the proof of Theorem 1.9.

6.1. **Structure of the Julia set.** As in the previous section, we work with the dynamics of f_c on the Berkovich affine line $\mathbf{A}^{1,an}_v$, associated to the complete and algebraically closed field $\mathbb{C}_v$, and we denote by $\zeta_{x,r}$ the Type II point corresponding to the disk of radius $r \in \mathbb{Q}_{>0}$ about x.

And as before, for $|c| \leq 1$, the map f_c has good reduction, and $J_c = \zeta_{0,1}$ is the Gauss point. For $|c| > 1$ and for any point z with absolute value $|z| > |c|^{1/2}$, we have $|f^n(z)| = |z|^{2^n}$ for all $n \geq 1$, so that

$$\lambda_c(z) = \log|z|. \tag{6.1}$$

It is also the case that

$$\lambda_c(z) \leq \frac{1}{2}\log|c| \tag{6.2}$$

for all $|z| \leq |c|^{1/2}$.

But unlike the setting of the previous section, the geometry of the Julia set and the dynamics on the associated tree is not constant for all $|c| > 1$. First, for $1 < |c| \leq 4$, the map f_c has *potential good reduction*, so its Julia set is a single Type II point. For all $|c| > 4$, the Julia set will be a Cantor set of Type I points. As in the previous section, the Julia set is contained in $\{z \in \mathbb{C}_v : |z| = |c|^{1/2}\}$ for all $|c| > 4$.

We let $\pm b$ satisfy $f_c(\pm b) = 0$. Note that

$$|b - (-b)| = |2b| = |b|/2 = |c|^{1/2}/2.$$

We let β and β' be further preimages of 0, so that

$$f_c(\beta) = b, \quad f_c(\beta') = -b, \quad |\beta - b| = |\beta' - b| = |\beta - \beta'| = 1, \tag{6.3}$$

and the Julia set will lie in the union of the four disks $D(\pm\beta, 1)$ and $D(\pm\beta', 1)$.

Distances between points scale as follows:

Lemma 6.2. *Suppose $|c| > 4$ and z is in the Julia set of f_c. For any $|y| > |c|^{1/2}/2$, we have*

$$|f_c(z + y) - f_c(z)| = |y|^2.$$

For $|y| < |c|^{1/2}/2$, we have

$$|f_c(z + y) - f_c(z)| = |y||c|^{1/2}/2.$$

Proof. Computing the image of $z + y$, we have

$$f_c(z + y) = (z + y)^2 + c = (z^2 + c) + (y^2 + 2yz).$$

Because z lies in the Julia set, we know that $|z| = |c|^{1/2}$, and the result follows. □

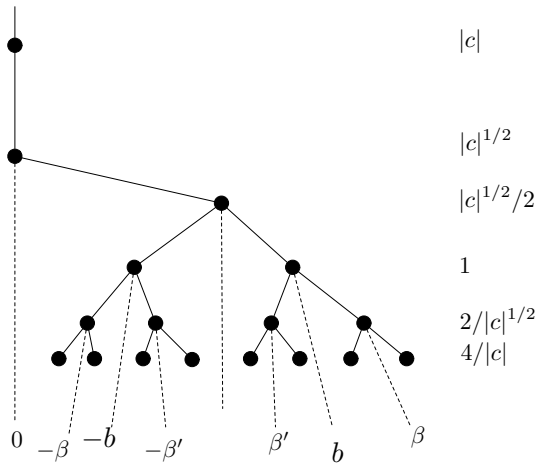

FIGURE 6.1. The tree structure of the non-archimedean Julia set, with $|c|_v > 4$ and $v \mid 2$, vertically ordered by $|\cdot|_v$ as noted on the right.

Note that for $|c| > 16$, the Type II point

$$\zeta_{0,|c|/4} = f_c(\zeta_{b,|c|^{1/2}/2})$$

will lie above $\zeta_{0,|c|^{1/2}}$. This simplifies computations and is the reason for taking $r > 16$ in the statement of Theorem 6.1.

6.2. **Proof of Theorem 6.1.** Suppose first that $|c_1| \neq |c_2|$. If $|c_i| \leq 1$ for at least one i, then

$$E_v = \frac{1}{2}\max\{\log^+|c_1|, \log^+|c_2|\} = \frac{1}{2}\log^+|c_1 - c_2|.$$

If $1 < |c_1| < |c_2|$ (or vice versa), then all points $z \in J_{c_2}$ satisfy $|z| = |c_2|^{1/2} > |c_1|^{1/2}$, so that $\lambda_{c_1}(z) = \log|z| = \frac{1}{2}\log|c_2|$, giving

$$E_v = \frac{1}{2}\max\{\log^+|c_1|, \log^+|c_2|\} = \frac{1}{2}\log^+|c_2 - c_1|.$$

Note that whenever $1 < |c_2| \leq |c_1|$, we have $\lambda_1(z) \leq \log|c_1|$ for all $z \in J_{c_2}$. It follows that

$$E_v \leq \frac{1}{2}\max\{\log^+|c_1|, \log^+|c_2|\}$$

in every case, proving the upper bound of the theorem.

If $1 < |c_1| = |c_2| \leq 16$, then $|c_1 - c_2| \leq 16$, so that $\frac{1}{4}\log|c_1 - c_2| \leq \log 2$.

For the remainder of the proof, we assume that

$$r := |c_1| = |c_2| > 16.$$

Exactly as in the proofs of Theorems 4.1 and 5.1, we break the proof into cases, according to how close the two parameters are. Recall that $\pm b_i$ denotes the preimages of 0 by f_{c_i}.

Case 1. Assume that the preimages b_1 and b_2 are chosen so that

$$s := |b_1 - b_2| \leq |b_1 + b_2|$$

and

$$r^{1/2}/2 < s \leq r^{1/2}.$$

Since $|b_i + b_i| = |b_i|/2 = r^{1/2}/2$, and because

$$(b_1 - b_2)(b_1 + b_2) = b_1^2 - b_2^2 = c_2 - c_1 \tag{6.4}$$

it follows that

$$|c_1 - c_2| = s^2.$$

For all $z \in J_{c_2}$, we have

$$|f_{c_1}^n(z)| = s^{2^n}$$

so that

$$\lambda_1(z) = \log s$$

and

$$E_v = \log s = \frac{1}{2}\log|c_1 - c_2| > \frac{1}{2}\log r - \log 2 \geq \frac{1}{4}\log 4$$

for all $r > 16$.

Case 2. Assume that the preimages b_1 and b_2 are chosen so that

$$s := |b_1 - b_2| \leq |b_1 + b_2|$$

and

$$1 < s \le r^{1/2}/2.$$

Then $|b_1 + b_2| = r^{1/2}/2$, so that

$$|c_1 - c_2| = sr^{1/2}/2$$

from (6.4). For all $z \in J_{c_2}$, we have

$$|f_{c_1}(z)| = sr^{1/2}/2 \quad \text{and } |f_{c_1}^n(z)| = (sr^{1/2}/2)^{2^{n-1}} \text{ for all } n \ge 2$$

so that

$$\lambda_1(z) = \frac{1}{2}\log(sr^{1/2}/2)$$

and

$$E_v = \frac{1}{2}\log(sr^{1/2}/2) = \frac{1}{2}\log|c_1 - c_2| > \frac{1}{2}\log r - \log 2 \ge \frac{1}{4}\log r$$

for all $r > 16$.

Case 3. Assume that the preimages b_1 and b_2 are chosen so that

$$1 = |b_1 - b_2| < |b_1 + b_2| = r^{1/2}/2.$$

Then

$$|c_1 - c_2| = r^{1/2}/2$$

from (6.4).

Recall that $\pm\beta_i$ and $\pm\beta_i'$ are further preimages of 0, and

$$(\beta_1 - \beta_2)(\beta_1 + \beta_2) = \beta_1^2 - \beta_2^2 = b_1 - c_1 - (b_2 - c_2) = (b_1 - b_2) + (c_2 - c_1). \quad (6.5)$$

Similarly for β_i'. It follows that

$$|\beta_1 - \beta_2| = |\beta_1' - \beta_2'| = 1.$$

We also have

$$(\beta_1 - \beta_2')(\beta_1 + \beta_2') = \beta_1^2 - (\beta_2')^2 = b_1 - c_1 - (-b_2 - c_2) = (b_1 + b_2) + (c_2 - c_1). \quad (6.6)$$

The right-hand-side is the sum of two terms with the same absolute value and may lead to cancellation, so it could happen that $D(\beta_1, 1) = D(\beta_2', 1)$. On the other hand, we also have

$$(\beta_1' - \beta_2)(\beta_1' + \beta_2) = (\beta_1')^2 - \beta_2^2 = -b_1 - c_1 - (b_2 - c_2) = -(b_1 + b_2) + (c_2 - c_1), \quad (6.7)$$

and $|(b_1 + b_2) - (-(b_1 + b_2))| = |2||b_1 + b_2| = r^{1/2}/4$. In other words, the cancellation on the right-hand-sides of (6.6) and (6.7) cannot bring us smaller than $r^{1/2}/4$ in both equations. Consequently, we have

$$|\beta_1 - \beta_2'| \text{ or } |\beta_1' - \beta_2| \ge (r^{1/2}/4)/(r^{1/2}/2) = \frac{1}{2}.$$

Consequently, at least half of the Julia set J_{c_2} by symmetry must be at distance at least $1/2$ from the Julia set J_{c_1}. Note that $r > 16$ implies that $1/2 > 2/r^{1/2}$. So, for half of the points $z \in J_{c_2}$, we have

$$|f_{c_1}^2(z)| = \frac{1}{2}\left(\frac{r^{1/2}}{2}\right)^2 = \frac{r}{8},$$

and thus

$$\lambda_1(z) = \frac{1}{4}\log(r/8) = \frac{1}{4}(\log r - \log 8)$$

for these z values. We conclude that

$$E_v \geq \frac{1}{8}\log(r/8) = \frac{1}{4}\log|c_1 - c_2| - \frac{1}{8}\log 2 > \frac{1}{16}\log|c_1 - c_2|$$

and

$$E_v \geq \frac{1}{16}\log r - \frac{1}{8}\log 2$$

for all $r > 16$.

Case 4. Assume that the preimages b_1 and b_2 are chosen so that

$$s := |b_1 - b_2| < |b_1 + b_2| = r^{1/2}/2$$

and

$$2/r^{1/2} < s < 1.$$

Then (6.4) implies that

$$1 < |c_1 - c_2| = sr^{1/2}/2 < r^{1/2}/2.$$

It follows that all points $z \in J_{c_2}$ are distance s from J_{c_1}, so that

$$|f_{c_1}^2(z)| = (r^{1/2}/2)^2 s = rs/4 \text{ and } |f_{c_1}(z)^n| = (rs/4)^{2^{n-2}} \text{ for all } n \geq 3$$

and

$$\lambda_{c_1}(z) = \frac{1}{4}\log(rs/4).$$

Therefore,

$$E_v = \frac{1}{4}\log(rs/4) = \frac{1}{4}\log|c_1 - c_2| + \frac{1}{4}\log(r^{1/2}/2) > \frac{1}{8}\log r - \frac{1}{4}\log 2 \geq \frac{1}{16}\log r$$

for all $r > 16$.

Case 5. Assume that the preimages b_1 and b_2 are chosen so that

$$2/r^{1/2} = |b_1 - b_2| < |b_1 + b_2| = r^{1/2}/2.$$

Then

$$|c_1 - c_2| = 1$$

from (6.4). Equation (6.5) implies that

$$|\beta_1 - \beta_2| = |\beta_1' - \beta_2'| = 2/r^{1/2},$$

and (6.6) and (6.7) imply that

$$|\beta_1 - \beta_2'| = |\beta_1' - \beta_2| = 1.$$

To determine how the Julia sets might overlap, we take pass to third preimages of 0, defining γ_i and γ_i' so that

$$f_{c_i}(\gamma_i) = \beta_i,\ f_{c_i}(\gamma_i') = \beta_i',\ \text{and } |\gamma_i - \gamma_i'| = |\gamma_i - \beta_i| = |\gamma_i' - \beta_i| = 2/r^{1/2}.$$

But here, as in Case 3 above, we may have overlap. We compute that

$$(\gamma_1 - \gamma_2)(\gamma_1 + \gamma_2) = \gamma_1^2 - \gamma_2^2 = (\beta_1 - \beta_2) + (c_2 - c_1) \tag{6.8}$$

so that

$$|\gamma_1 - \gamma_2| = 2/r^{1/2}.$$

But

$$(\gamma_1 - \gamma_2')(\gamma_1 + \gamma_2') = \gamma_1^2 - (\gamma_2')^2 = (\beta_1 - \beta_2') + (c_2 - c_1)$$

and both terms on the right-hand-size have absolute value 1. So it can happen that $D(\gamma_1, 2/r^{1/2}) = D(\gamma_2', 2/r^{1/2})$. Similarly for γ_1' with γ_2. But both pairs cannot be too close, because

$$(\beta_1 - \beta_2') - (\beta_1' - \beta_2) = (\beta_1 - \beta_1') + (\beta_2 - \beta_2') = (\beta_1 - \beta_1') + (\beta_1 - \beta_2') + \varepsilon$$

for some $|\varepsilon| \le 2/r^{1/2}$. It follows that

$$|(\beta_1 - \beta_2') - (\beta_1' - \beta_2)| = |(\beta_1 - \beta_1') + (\beta_1 - \beta_2')| = |2(\beta_1 - \beta_1')| = \frac{1}{2}$$

so that

$$|\gamma_1 - \gamma_2'| \text{ or } |\gamma_1' - \gamma_2| \ge (1/2)/(r^{1/2}/2) = \frac{1}{r^{1/2}}.$$

The same estimates will hold for the third preimages of 0 near β_i', as well as those near $-\beta_i$ and $-\beta'$. Consequently, at least half of the Julia set J_{c_2} must be at distance at least $1/r^{1/2}$ from the Julia set J_{c_1}. Note that $r > 16$ implies that $1/r^{1/2} > 4/r$. So, for half of the points $z \in J_{c_2}$, we have

$$|f_{c_1}^3(z)| = \left(\frac{r^{1/2}}{2}\right)^3 \frac{1}{r^{1/2}} = \frac{r}{8},$$

and thus

$$\lambda_{c_1}(z) = \frac{1}{8}\log(r/8) = \frac{1}{8}\log r - \frac{1}{8}\log 8$$

for these z values. We conclude that

$$E_v \ge \frac{1}{16}\log(r/8) = \frac{1}{16}\log r - \frac{1}{16}\log 8.$$

Case 6. Assume that the preimages b_1 and b_2 are chosen so that

$$s := |b_1 - b_2| \le |b_1 + b_2|$$

and

$$4/r < s < 2/r^{1/2} < s.$$

Then

$$2/r^{1/2} < |c_1 - c_2| = sr^{1/2}/2 < 1.$$

We also compute

$$|\gamma_1 - \gamma_2| = |\beta_1 - \beta_2| = s$$

from (6.5) and (6.8), and

$$|f_{c_1}(z)^3| = s(r^{1/2}/2)^3 \text{ for all } z \in J_{c_2}$$

so that

$$\lambda_{c_1}(z) = \frac{1}{8}\log(sr^{3/2}/8)$$

and

$$E_v = \frac{1}{8}\log(sr^{3/2}/8) \geq \frac{1}{8}\log(r^{1/2}/2) = \frac{1}{16}\log r - \frac{1}{8}\log 2.$$

Finally, note that if $|b_1 - b_2| \leq 4/r$, then $|c_1 - c_2| \leq 2/r^{1/2}$, so Case 6 completes the proof of the theorem. The first statement of the theorem is covered by Cases 1 through 4.

6.3. **An upper bound on the local height near the Julia set.** We will use the following proposition in the proof of Theorem 1.9. This is a non-archimedean analog to the distortion estimate provided in Proposition 3.8.

Proposition 6.3. *Suppose v is a non-archimedean place of K dividing 2. For any $0 < r < 1/4$, we have*

$$\lambda_{c,v}(z) \leq r \log \max\{|c|_v, 16\}$$

for all z within distance

$$\frac{1}{\max\{|c|_v, 16\}^{\log(1/r)}}$$

of the filled Julia set within $\mathbb{C}_v$.

Proof. First assume that $|c|_v > 4$. Recall that all points x of the Julia set J_c (which agrees with the filled Julia set in this setting) satisfy $|x| = |c|_v^{1/2}$. From Lemma 6.2, we know that for all $x \in J_c$ and all $z = x + y$ with $|y| < |c|^{1/2}/2$, we have

$$|f_c(z) - f_c(x)|_v = |2xy + y^2|_v = |y||c|^{1/2}/2.$$

Recall also that $\lambda_c(z) = \log|z|$ for all $|z| > |c|^{1/2}$.

In particular, for any $n \geq 2$, any point z within distance

$$\frac{|c|}{4}\left(\frac{2}{|c|^{1/2}}\right)^n \geq \frac{1}{|c|^{n/2-1}}$$

of the Julia set will have $\lambda_c(z) = 2^{-n}\lambda_c(f^n(z)) \le 2^{-n}\log|c|$. In other words, setting $r = 1/2^n$, then

$$\log(1/r) = n\log 2 > \frac{n}{2} - 1$$

so that any point z within distance $|c|^{-\log(1/r)}$ of the Julia set will satisfy

$$\lambda_c(z) \le r\log|c| \le r\log\max\{|c|_v, 16\}.$$

Now assume $|c|_v \le 4$. Then f_c has potential good reduction with $J_c = \zeta_{b,1}$, where b is any element of the filled Julia set. Consequently, all points z within distance 1 of the filled Julia set are in the filled Julia set and thus satisfy $\lambda_c(z) = 0$. □

7. Bounds on the energy pairing

In this section, we use the estimates of the previous sections to prove a weak version of Theorem 1.7, and we use it to deduce Theorem 1.6. We let $h(x)$ denote the logarithmic Weil height of $x \in \overline{\mathbb{Q}}$ and $h(x_1, x_2)$ the Weil height on $\mathbb{A}^2(\overline{\mathbb{Q}})$.

Theorem 7.1. *We have*

$$\frac{1}{16}h(c_1 - c_2) - \frac{\log 2000}{16} \le \langle f_{c_1}, f_{c_2}\rangle \le \frac{1}{2}h(c_1, c_2) + \frac{5}{2}$$

for all $c_1 \ne c_2$ in $\overline{\mathbb{Q}}$.

7.1. **Proof of Theorem 7.1.** Fix $c_1 \ne c_2$ in $\overline{\mathbb{Q}}$, and let K be any number field containing them. Summing over all places of K, we have by Theorem 4.1, Theorem 5.1, and Theorem 6.1 that

$$\begin{aligned}\frac{1}{16}\sum_{v\in M_K}\frac{[K_v:\mathbb{Q}_v]}{[K:\mathbb{Q}]}\log^+|c_1 - c_2| - \frac{1}{16}\log 2000 \quad &\le \quad \langle f_{c_1}, f_{c_2}\rangle \\ \le \frac{1}{2}\sum_{v\in M_K}\frac{[K_v:\mathbb{Q}_v]}{[K:\mathbb{Q}]}\log^+\max\{|c_1|_v, |c_2|_v\} + \frac{5}{2},\end{aligned}$$

where the added constants come from the archimedean places (Remark 4.2). This completes the proof of the theorem.

7.2. **Proof of Theorem 1.6.** We will assume towards contradiction that there is a sequence of triples $c_{1,n} \ne c_{2,n} \in \overline{\mathbb{Q}}$ and $\varepsilon_n > 0$ such that

$$\langle f_{c_{1,n}}, f_{c_{2,n}}\rangle < \varepsilon_n,$$

where $\varepsilon_n \to 0$ as n tends to infinity. Let K_n be a number field containing $c_{1,n}$ and $c_{2,n}$. We will show that this forces the pairing at a (proportionally) large number of archimedean places of K_n to be close to 0; as a consequence we will deduce that the height $h(c_{1,n} - c_{2,n})$ must get large. This in turn would contradict Theorem 7.1.

Let M_n^∞ denote the set of all archimedean places of K_n. For each $v \in M_n^\infty$, we let

$$E_v(c_{1,n}, c_{2,n}) = \int \lambda_{c_1,v}\, d\mu_{c_2,v}$$

denote the local contribution to the energy pairing. We let $S_n \subset M_n^\infty$ be the set of archimedean places with

$$E_v(c_{1,n}, c_{2,n}) < 2\varepsilon_n.$$

Since $\sum_{v\in M_n^\infty}[K_{n,v} : \mathbb{Q}_v] = [K_n : \mathbb{Q}]$ and $\langle f_{c_{1,n}}, f_{c_{2,n}}\rangle < \varepsilon_n$, we see that $\sum_{v\in M_n^\infty\setminus S_n}[K_{n,v} : \mathbb{Q}_v] \le [K_n : \mathbb{Q}]/2$. Therefore,

$$\sum_{v\in S_n}[K_{n,v} : \mathbb{Q}_v] \ge \frac{[K_n : \mathbb{Q}]}{2}.$$

Take $L = 1000$ as in Remark 4.2, and choose any $M > L$. From the continuity and positivity of E_v, there is a sequence $\delta_n \to 0^+$ as $n \to \infty$ such that

$$E_v(c_1, c_2) \ge 2\varepsilon_n$$

whenever $|c_1 - c_2|_v \ge \delta_n$ and $|c_1|_v, |c_2|_v \le M$ for $c_1, c_2 \in \mathbb{C}_v = \mathbb{C}$.

If one of the c_i, say c_1, has absolute value bigger than M and if $|c_1 - c_2|_v > 3/|c_1|_v^{1/2}$, then

$$E_v(c_1, c_2) \ge \frac{1}{64}\log|c_1|_v \ge \frac{1}{64}\log M$$

from Theorem 4.1. When n is big, we have $2\varepsilon_n < \frac{1}{64}\log M$, and so for any $v \in S_n$ as $E_v(c_{1,n}, c_{2,n}) < 2\varepsilon_n$, we must have

$$|c_{1,n} - c_{2,n}|_v \le \max\{\delta_n, \frac{3}{M^{1/2}}\}.$$

Hence for any n with $2\varepsilon_n < \frac{1}{64}\log M$ and $\delta_n < 3/M^{1/2}$, we conclude that

$$|c_{1,n} - c_{2,n}|_v \le 3/M^{1/2} < 1$$

for all $v \in S_n$. Consequently,

$$\begin{aligned} h(c_{1,n} - c_{2,n}) &\ge \sum_{v\in M_{K_n}\setminus S_n} \frac{[K_{n,v} : \mathbb{Q}_v]}{[K_n : \mathbb{Q}]}\log|c_{1,n} - c_{2,n}|_v \\ &= \sum_{v\in M_n} \frac{[K_{n,v} : \mathbb{Q}_v]}{[K_n : \mathbb{Q}]}\log\frac{1}{|c_{1,n} - c_{2,n}|_v} \\ &\ge \left(\sum_{v\in M_n} \frac{[K_{n,v} : \mathbb{Q}_v]}{[K_n : \mathbb{Q}]}\right)\log\frac{M^{1/2}}{3} \ \ge\ \frac{1}{2}\log\frac{M^{1/2}}{3}. \end{aligned}$$

We thus have by Theorem 7.1 that

$$\frac{1}{32}\log\frac{M^{1/2}}{3} - \frac{\log 2000}{16} \le \langle f_{c_{1,n}}, f_{c_{2,n}}\rangle < \epsilon_n,$$

a contradiction for M sufficiently large. □

8. Strong lower bound on the energy pairing

Throughout this section, we assume that c_1 and c_2 are distinct elements of $\overline{\mathbb{Q}}$. We prove Theorem 1.7, which gives bounds on the energy pairing $\langle f_{c_1}, f_{c_2}\rangle$ in terms of the heights of the parameters.

The upper bound in Theorem 1.7 is easy and was stated as part of Theorem 7.1. The lower bound is a balancing act between "helpful" primes and the other primes of a given number field K containing the pair c_1 and c_2. A place v of K will be helpful if at least one absolute value $|c_i|_v$ is large and the two parameters are not too close in the v-adic distance. In this good setting, we can apply the stronger lower bounds on the local energy pairing, as in the second statement of Theorem 4.1. By showing that a significant proportion of primes are helpful, we obtain the lower bound of Theorem 1.7.

8.1. **An auxiliary height.** Fix some constant $L > 1$ and consider the following function h_L on $\mathbb{A}^2(\overline{\mathbb{Q}})$. For c_1, c_2 in a number field K, we put

$$r_v = [K_v : \mathbb{Q}_v]/[K : \mathbb{Q}],$$

and set

$$\ell_v = \begin{cases} \log\max\{|c_1|_v, |c_2|_v, L\} & \text{for } v \text{ archimedean} \\ \log\max\{|c_1|_v, |c_2|_v, 16\} & \text{for } v|2 \\ \log\max\{|c_1|_v, |c_2|_v, 1\} & \text{otherwise} \end{cases}$$

and define

$$h_L(c_1, c_2) := \sum_{v \in M_K} r_v \ell_v.$$

Note that

$$h(c_1, c_2) \le h_L(c_1, c_2) \le h(c_1, c_2) + \log L + \log 16,$$

where $h(c_1, c_2)$ is the usual logarithmic Weil height on $\mathbb{A}^2(\overline{\mathbb{Q}})$.

8.2. **Helpful places.** With $L > 1$ fixed, and elements c_1 and c_2 in the number field K, we say that the quantity ℓ_v is large if

$$\ell_v > \begin{cases} \log L & \text{for } v \text{ archimedean} \\ \log 16 & \text{for } v \mid 2 \\ 0 & \text{otherwise} \end{cases}$$

We define M_{help} to be the subset of M_K for which ℓ_v is large and

$$|c_1 - c_2|_v > \kappa_v e^{-\ell_v/2},$$

where

$$\kappa_v = \begin{cases} 3 & \text{for } v \text{ archimedean} \\ 2 & \text{for } v \mid 2 \\ 1 & \text{otherwise} \end{cases}$$

and we call these places "helpful". We define M_{close} to be the subset of M_K for which ℓ_v is large and

$$|c_1 - c_2| \leq \kappa_v e^{-\ell_v/2}$$

and call these places "close". We will say that a place v is in M_{bounded} if ℓ_v fails to be large.

The helpful places constitute a significant portion of the contribution to the height:

Lemma 8.1. *For any $c_1, c_2 \in \overline{\mathbb{Q}}$ and any $L \geq 1$, we have*

$$\sum_{v \in M_K \setminus M_{\text{close}}} r_v \ell_v \;\geq\; \frac{1}{3} h_L(c_1, c_2) - \frac{2}{3}\log 6.$$

Or, we can rearrange the terms to obtain

$$\sum_{v \in M_{\text{help}}} r_v \ell_v \;\geq\; \frac{1}{3} h_L(c_1, c_2) - \log(16 \cdot 6^{2/3} \cdot L).$$

for any $c_1, c_2 \in \overline{\mathbb{Q}}$ and any $L \geq 1$.

Proof. We use the product formula on $c_1 - c_2$, so that

$$1 = \prod_v |c_1 - c_2|_v^{r_v}.$$

At the close places, we know that $|c_1 - c_2|$ is bounded from above by $\kappa_v e^{-\ell_v/2}$. At all other places, we have $|c_1 - c_2|_v \leq e^{\ell_v}$ if non-archimedean, and $|c_1 - c_2|_v \leq 2e^{\ell_v} \leq \kappa_v e^{\ell_v}$ if archimedean. Therefore, we have

$$\begin{aligned} 1 \;&\leq\; \prod_{v \in M_{\text{close}}} (\kappa_v e^{-\ell_v/2})^{r_v} \prod_{v \in M_\infty \setminus M_{\text{close}}} (\kappa_v e^{\ell_v})^{r_v} \prod_{v \in M_K \setminus (M_\infty \cup M_{\text{close}})} (e^{\ell_v})^{r_v} \\ &\leq\; 6 \prod_{v \in M_{\text{close}}} (e^{-\ell_v/2})^{r_v} \prod_{v \in M_K \setminus M_{\text{close}}} (e^{\ell_v})^{r_v}. \end{aligned}$$

Taking logarithms gives

$$\frac{1}{2} \sum_{v \in M_{\text{close}}} r_v \ell_v \;\leq\; \sum_{v \in M_K \setminus M_{\text{close}}} r_v \ell_v + \log 6. \tag{8.1}$$

Adding $\frac{1}{2}\sum_{v \in M_K \setminus M_{\text{close}}} r_v \ell_v$ to both sides yields

$$\frac{1}{2} h_L(c_1, c_2) \;\leq \frac{3}{2} \sum_{v \in M_K \setminus M_{\text{close}}} r_v \ell_v + \log 6,$$

proving the first statement of the lemma.

Expanding the right-hand-side of (8.1), we see that

$$\frac{1}{2} \sum_{v \in M_{\text{close}}} r_v \ell_v \leq \sum_{v \in M_{\text{help}}} r_v \ell_v + \sum_{v \in M_{\text{bounded}}} r_v \ell_v + \log 6$$

so that

$$\sum_{v\in M_{\text{help}}} r_v\ell_v \geq \frac{1}{2}\sum_{v\in M_{\text{close}}} r_v\ell_v - \sum_{v\in M_{\text{bounded}}} r_v\ell_v - \log 6.$$

Adding $\frac{1}{2}\sum_{v\in M_{\text{help}}} r_v\ell_v$ to both sides, we obtain

$$\begin{aligned}\frac{3}{2}\sum_{v\in M_{\text{help}}} r_v\ell_v &\geq \frac{1}{2}h_L(c_1,c_2) - \frac{3}{2}\sum_{v\in M_{\text{bounded}}} r_v\ell_v - \log 6\\ &\geq \frac{1}{2}h_L(c_1,c_2) - \frac{3}{2}(\log L + \log 16) - \log 6\\ &= \frac{1}{2}h_L(c_1,c_2) - \frac{3}{2}\log(16\cdot 6^{2/3}\cdot L),\end{aligned}$$

which proves the lemma. □

8.3. **Proof of Theorem 1.7.** Fix c_1, c_2 and choose any number field K containing both. Fix any $L \geq 1000$ so that Theorem 4.1 is satisfied. Decompose M_K into $M_{\text{help}} \cup M_{\text{close}} \cup M_{\text{bounded}}$ as in §8.2. Note that $\frac{1}{16}\log r - \frac{3}{16}\log 2 \geq \frac{1}{64}\log r$ for any $r \geq 16$. Then Theorems 5.1, 6.1, and 4.1 applied in the helpful places combine to say

$$\begin{aligned}\langle f_{c_1}, f_{c_2}\rangle &= \sum_{v\in M_K} r_v E_v \geq \sum_{v\in M_{\text{help}}} r_v E_v\\ &\geq \frac{1}{64}\sum_{v\in M_{\text{help}}} r_v \log\max\{|c_1|_v, |c_2|_v\}\\ &= \frac{1}{64}\sum_{v\in M_{\text{help}}} r_v\ell_v. \qquad (8.2)\end{aligned}$$

Combined with Lemma 8.1, this proves that for all c_1 and c_2 in $\overline{\mathbb{Q}}$, we have

$$\begin{aligned}\langle f_{c_1}, f_{c_2}\rangle &\geq \frac{1}{3\cdot 64}h_L(c_1,c_2) - \frac{1}{64}\log(16\cdot 6^{2/3}\cdot L)\\ &\geq \frac{1}{192}h(c_1,c_2) - \frac{1}{64}\log(16\cdot 6^{2/3}\cdot L)\end{aligned}$$

This proves the lower bound of the theorem with $\alpha_1 = 1/192$ and $C_1 = \frac{1}{64}\log(16\cdot 6^{2/3}\cdot L) < 0.17 < \frac{3}{17}$ for $L = 1000$. The upper bound of the theorem was proved already as Theorem 7.1 with $\alpha_2 = 1/2$ and $C_2 = 5/2$.

9. Quantitative equidistribution

Our goal in this section is to prove Theorem 1.9, providing an upper bound on the energy pairing $\langle f_{c_1}, f_{c_2}\rangle$, in terms of the number of common preperiodic points, for $c_1 \neq c_2$ in $\overline{\mathbb{Q}}$, assuming f_{c_1} and f_{c_2} share at least one preperiodic point other than ∞. We build upon the ideas developed in the proof of [FRL, Theorem 3] and [Fi, Theorem 4].

9.1. **Adelic measures and heights on $\mathbb{P}^1(\overline{\mathbb{Q}})$.** Following Favre and Rivera-Letelier [FRL], we define the mutual energy of measures ρ and ρ' on $\mathbb{P}^1(\mathbb{C})$ by

$$(\rho,\rho') := -\iint_{\mathbb{C}\times\mathbb{C}\setminus\mathrm{Diag}} \log|z-w|\, d\rho(z)\, d\rho'(w),$$

where Diag is the diagonal, assuming $\log|z-w|$ is in $L^1(\rho\otimes\rho')$. If the measures have total mass 0 with continuous potentials on $\mathbb{P}^1$, we have $(\rho,\rho)\geq 0$ with equality if and only if $\rho=0$. Similarly, one defines

$$(\rho,\rho')_v := -\iint_{\mathbf{A}^1_v\times\mathbf{A}^1_v\setminus\mathrm{Diag}} \delta_v(z,w)\, d\rho(z)\, d\rho'(w) \tag{9.1}$$

on the Berkovich line over $\mathbb{C}_v$, with respect to a non-archimedean valuation, with the appropriate kernel $\delta_v(z,w)$ in place of $\log|z-w|_v$. More information can be found in [BR2].

Now let K be a number field. An adelic measure is a collection $\mu=\{\mu_v\}_{v\in M_K}$ of probability measures on the Berkovich $\mathbf{P}^{1,an}_v$, with continuous potentials at all places v and for which all but finitely many are trivial (meaning that they are supported at the Gauss point). For any adelic measure μ, a height function is defined on $\mathbb{P}^1(\overline{\mathbb{Q}})$ by

$$h_\mu(F) := \frac{1}{2}\sum_{v\in M_K}\frac{[K_v:\mathbb{Q}_v]}{[K:\mathbb{Q}]}\,([F]-\mu_v,[F]-\mu_v)_v,$$

where F is any finite, $\mathrm{Gal}(\overline{K}/K)$-invariant subset of $\overline{K}$, and $[F]$ is the probability measure supported equally on the elements of F. We put

$$h_\mu(\infty) := \frac{1}{2}\sum_{v}\frac{[K_v:\mathbb{Q}_v]}{[K:\mathbb{Q}]}(\mu_v,\mu_v)_v.$$

The equidistribution theorem of [FRL, BR1, CL1] states that if F_n is a seqence of $\mathrm{Gal}(\overline{K}/K)$-invariant finite sets with $h_\mu(F_n)\to 0$ and $|F_n|\to\infty$ as $n\to\infty$, the discrete probability measures

$$\mu_n := \frac{1}{|F_n|}\sum_{x\in F_n}\delta_x$$

converge weakly to the measure μ_v on $\mathbf{P}^{1,an}_v$ at each place v of K.

There is a pairing between any two such heights, h_μ and h_ν, associated to adelic measures μ and ν, as

$$\langle h_\mu, h_\nu\rangle = \frac{1}{2}\sum_{v\in M_K}\frac{[K_v:\mathbb{Q}_v]}{[K:\mathbb{Q}]}(\mu_v-\nu_v,\mu_v-\nu_v)_v. \tag{9.2}$$

It satisfies $\langle h_\mu,h_\nu\rangle=0 \iff h_\mu=h_\nu \iff \mu=\nu$. The energy pairing (1.3) between two quadratic polynomials is a special case, taking the dynamical canonical heights $\hat{h}_{c_1}$ and $\hat{h}_{c_2}$ associated to their adelic equilibrium measures.

Remark 9.1. The height h_μ can be defined for arbitrary adelic measures, but small sequences (meaning the sequences $\{F_n\}$ of Galois-invariant sets with $h_\mu(F_n) \to 0$ and $|F_n| \to \infty$) do not always exist.

9.2. **Height pairing as a distance.** Following [Fi], we consider a distance between two adelic measures $\mu = \{\mu_v\}$ and $\nu = \{\nu_v\}$ on $\mathbb{P}^1$ over a number field K, defined by

$$d(\mu, \nu) := \langle h_\mu, h_\nu \rangle^{1/2},$$

where $\langle h_\mu, h_\nu \rangle$ was defined in (9.2); see [Fi, Theorem 1].

Suppose that c_1 and c_2 are elements of a number field K. Let $\mu_1 := \{\mu_{c_1,v}\}$ and $\mu_2 := \{\mu_{c_2,v}\}$ be the equilibrium measures of f_{c_1} and f_{c_2}, respectively. Let F be any finite, nonempty, $\mathrm{Gal}(\overline{K}/K)$-invariant subset of $\mathbb{P}^1(\overline{\mathbb{Q}})$. Let $[F]$ denote the probability measure supported equally on the elements of F. For each place v of K, choose a positive real $\varepsilon_v > 0$, with $\varepsilon_v = 1$ for all but finitely many v. The collection $\varepsilon := \{\varepsilon_v\}_{v \in M_K}$ will be called an adelic radius. As in [FRL], we consider the adelic measure $[F]_\varepsilon$, defined as a regularization of the probability measure $[F]$: it is supported on the circles of radius ε_v about each point of F. At a non-archimedean place, this means the Type II or III point associated to the disk of radius ε_v. The triangle inequality implies that

$$\langle f_{c_1}, f_{c_2} \rangle^{1/2} = d(\mu_1, \mu_2) \le d(\mu_1, [F]_\varepsilon) + d(\mu_2, [F]_\varepsilon) \tag{9.3}$$

for any choices of F and ε.

It is worth noting that, if the radius $\varepsilon_v \to 0$ at some place, then the right-hand-side of (9.3) will tend to ∞. This is because the potential of the measure $[F]_\varepsilon$ at v will blow up near the points of F. On the other hand, for ε_v too large, the measure $[F]_\varepsilon$ is not a good approximation of $[F]$. Thus, for (9.3) to be useful in our proof of Theorem 1.1, we will need to choose ε well. This general strategy also appears in the proofs of [FRL, Theorem 3] and in [Fi, Proposition 13]. In our case, the choice of $\varepsilon = \{\varepsilon_v\}_{v \in M_K}$ will be governed by Proposition 3.8 and its non-archimedean counterparts, and this leads to Theorem 1.9.

Lemma 9.2. *Let K be a number field and fix $c_1 \neq c_2$ in K. We have*

$$\langle f_{c_1}, f_{c_2} \rangle^{1/2} \le \sum_{i=1}^{2} \left(\sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \left(-(\mu_i, [F]_\varepsilon)_v + \frac{\log(1/\varepsilon_v)}{2|F|} \right) \right)^{1/2}$$

for any choice of finite, non-empty, $\mathrm{Gal}(\overline{K}/K)$-invariant subset F of $\overline{\mathbb{Q}}$ and any adelic radius $\varepsilon = \{\varepsilon_v\}_{v \in M_K}$.

Proof. We first observe that

$$\begin{aligned} d(\mu_i, [F]_\varepsilon)^2 &= \frac{1}{2}\sum_v \frac{[K_v:\mathbb{Q}_v]}{[K:\mathbb{Q}]}\,(\mu_i - [F]_\varepsilon, \mu_i - [F]_\varepsilon)_v \\ &= \sum_v \frac{[K_v:\mathbb{Q}_v]}{[K:\mathbb{Q}]}\left(-(\mu_i, [F]_\varepsilon)_v + \frac{1}{2}([F]_\varepsilon, [F]_\varepsilon)_v\right) \end{aligned}$$

because $(\mu_i, \mu_i)_v = 0$ at every place. The self-pairing of $[F]_\varepsilon$ can be estimated in terms of the self-pairing of $[F]$ ([Fi, Lemma 12] and [FRL, Lemma 4.11]), as

$$([F]_\varepsilon, [F]_\varepsilon)_v \le ([F],[F])_v + \frac{\log(1/\varepsilon_v)}{|F|}.$$

But observe that

$$\sum_v \frac{[K_v:\mathbb{Q}_v]}{[K:\mathbb{Q}]}([F],[F])_v = 0$$

by the product formula on K. So the triangle inequality (9.3) completes the proof of the proposition. □

9.3. **Proof of Theorem 1.9.** Fix any $L \ge 27$, and recall the definition of the auxiliary height h_L on $\mathbb{A}^2(\overline{\mathbb{Q}})$ from §8.1. An appropriate choice of $\varepsilon = \{\varepsilon_v\}$ in Lemma 9.2 gives:

Proposition 9.3. *Fix any $L \ge 27$. Fix c_1 and c_2 in $\overline{\mathbb{Q}}$, and assume f_{c_1} and f_{c_2} have $N > 1$ preperiodic points in common in $\mathbb{P}^1(\overline{\mathbb{Q}})$. Then for all $0 < \delta < 1/4$, we have*

$$\langle f_{c_1}, f_{c_2}\rangle \le 4\left(\delta + \frac{3\log(1/\delta)}{2(N-1)}\right) h_L(c_1, c_2)$$

Proof. Fix a number field K containing c_1 and c_2. Let F be the $\mathrm{Gal}(\overline{K}/K)$-invariant set of common preperiodic points for f_{c_1} and f_{c_2} in $\overline{\mathbb{Q}}$, so that $|F| = N-1$. For each place $v \in M_K$, recall the definition of ℓ_v from §8.1. Fix $0 < \delta < 1/4$ and set

$$\varepsilon_v = \delta^{3\,\ell_v}.$$

Note that $\varepsilon_v = 1$ for all but finitely many places $v \in M_K$.

For each archimedean place v, note that

$$\varepsilon_v = e^{-3\,\ell_v \log(1/\delta)} = \max\{|c_1|_v, |c_2|_v, L\}^{-3\log(1/\delta)},$$

so Proposition 3.8 implies that

$$\lambda_{c_i,v}(z) \le \delta\,\ell_v$$

for any point z within a neighborhood of radius ε_v of the filled Julia set K_{c_i}. As all points of F lie in K_{c_i}, this implies that

$$-(\mu_i, [F]_\varepsilon)_v \le \delta\,\ell_v$$

for this choice of ε_v and each i.

Similarly for each non-archimedean place $v \nmid 2$, we apply Proposition 5.2, and for each non-archimedean $v \mid 2$, we apply Proposition 6.3.

Summing over all places, we find that

$$\begin{aligned}\sum_{v\in M_K} \frac{[K_v:\mathbb{Q}_v]}{[K:\mathbb{Q}]}\left(-(\mu_i,[F]_\varepsilon)_v + \frac{\log(1/\varepsilon_v)}{2|F|}\right) &\le \sum_v \frac{[K_v:\mathbb{Q}_v]}{[K:\mathbb{Q}]}\left(\delta\,\ell_v + \frac{3\log(1/\delta)}{2|F|}\,\ell_v\right)\\ &= \left(\delta + \frac{3\log(1/\delta)}{2|F|}\right) h_L(c_1,c_2).\end{aligned}$$

Lemma 9.2 then implies

$$\begin{aligned}\langle f_{c_1}, f_{c_2}\rangle^{1/2} &\le \sum_{i=1}^{2}\left(\sum_{v\in M_K}\frac{[K_v:\mathbb{Q}_v]}{[K:\mathbb{Q}]}\left(-(\mu_i,[F]_\varepsilon)_v + \frac{\log(1/\varepsilon_v)}{2|F|}\right)\right)^{1/2}\\ &\le 2\left(\left(\delta + \frac{3\log(1/\delta)}{2|F|}\right) h_L(c_1,c_2)\right)^{1/2}\end{aligned}$$

Squaring both sides yields the proposition. □

Now fix any ε between 0 and 1, and let $\delta = \varepsilon/25$. Applying Proposition 9.3 with $L = 27$, we have

$$\begin{aligned}\langle f_{c_1}, f_{c_2}\rangle &\le 4\left(\delta + \frac{3\log(1/\delta)}{2(N-1)}\right) h_L(c_1,c_2)\\ &\le 4\left(\delta + \frac{3\log(1/\delta)}{2(N-1)}\right)(h(c_1,c_2) + \log 16 + \log 27)\\ &\le \left(\varepsilon + \frac{C(\varepsilon)}{N-1}\right)(h(c_1,c_2)+1)\end{aligned}$$

with $C(\varepsilon) = 40\log(25/\varepsilon)$. This completes the proof of Theorem 1.9.

10. Proof of Theorem 1.1

In this section, we prove Theorem 1.1, providing a uniform bound on the number of common preperiodic points for any pair f_{c_1} and f_{c_2} with $c_1 \neq c_2$ in $\mathbb{C}$.

10.1. **Proof over $\overline{\mathbb{Q}}$.** Assume that c_1 and c_2 are in $\overline{\mathbb{Q}}$.

We first use Theorem 1.7 and 1.9 to provide a bound on

$$N := N(c_1,c_2) = |\mathrm{Preper}(f_{c_1}) \cap \mathrm{Preper}(f_{c_2})|$$

when the height $h(c_1,c_2)$ is large. The two theorems combined show that, if $N > 1$, then it must satisfy

$$\alpha_1\, h(c_1,c_2) - C_1 \le \left(\varepsilon + \frac{C(\varepsilon)}{N-1}\right)(h(c_1,c_2)+1)$$

for every choice of $0 < \varepsilon < 1$, and thus,

$$\left(\alpha_1 - \varepsilon - \frac{C(\varepsilon)}{N-1}\right)(h(c_1,c_2)+1) \leq C_1 + \alpha_1.$$

Taking $\varepsilon = \alpha_1/2$, we have

$$\frac{\alpha_1}{2} - \frac{C(\varepsilon)}{N-1} \leq \frac{C_1+\alpha_1}{h(c_1,c_2)+1}.$$

If we assume that

$$h(c_1,c_2)+1 > \frac{4(C_1+\alpha_1)}{\alpha_1},$$

then the inequality becomes

$$N - 1 < \frac{4C(\alpha_1/2)}{\alpha_1}, \tag{10.1}$$

providing a uniform bound on N for all pairs (c_1, c_2) of sufficiently large height.

Now suppose that $h(c_1,c_2)+1 \leq 4(C_1+\alpha_1)/\alpha_1$. We combine the uniform lower bound of Theorem 1.6 with the upper bound of Theorem 1.9 to obtain

$$\delta \leq \left(\varepsilon + \frac{C(\varepsilon)}{N-1}\right)(h(c_1,c_2)+1) \leq \left(\varepsilon + \frac{C(\varepsilon)}{N-1}\right)\frac{4(C_1+\alpha_1)}{\alpha_1}$$

for any choice of $0 < \varepsilon < 1$. This unwinds to give

$$N - 1 \leq \frac{C(\varepsilon)}{\frac{\alpha\delta}{4(C_1+\alpha_1)} - \varepsilon}. \tag{10.2}$$

Choosing any $\varepsilon < \alpha_1\delta/4(C_1+\alpha_1)$ gives a uniform bound on N.

10.2. **Proof over $\mathbb{C}$.** Let B denote a uniform bound on the number of common preperiodic points over all $c_1 \neq c_2$ in $\overline{\mathbb{Q}}$. Now fix c_1 in $\mathbb{C} \setminus \overline{\mathbb{Q}}$. For any $c_2 \in \mathbb{C}$, if f_{c_1} and f_{c_2} have at least one preperiodic point in common, then the field $\mathbb{Q}(c_1,c_2)$ must have transcendence degree 1 over $\mathbb{Q}$. Moreover, if $x_1, x_2, \ldots, x_{B+1}$ denote distinct common preperiodic points for f_{c_1} and f_{c_2}, then $k = \mathbb{Q}(c_1, c_2, x_1, \ldots, x_{B+1})$ will also be of transcendence degree 1, as each x_i satisfies relations of the form

$$f_{c_1}^{n_i}(x_i) = f_{c_1}^{m_i}(x_i) \text{ for } n_i > m_i \geq 0 \quad \text{and} \quad f_{c_2}^{k_i}(x_i) = f_{c_2}^{l_i}(x_i) \text{ for } k_i > l_i \geq 0. \tag{10.3}$$

We may view k as the function field $K(T)$ of an algebraic curve T defined over a number field K. In this way, the maps f_{c_1} and f_{c_2} are viewed as families of maps, parameterized by $t \in T(\mathbb{C})$, and the relations (10.3) hold persistently in t.

Now assume $c_2 \neq c_1$, so that the specializations $f_{c_1(t)}$ and $f_{c_2(t)}$ are distinct for all but finitely many $t \in T(\mathbb{C})$. As the elements $\{x_1, \ldots, x_{B+1}\}$ are distinct in k, their specializations $\{x_1(t), \ldots, x_{B+1}(t)\}$ are also distinct for all but finitely many t in $T(\mathbb{C})$. In particular, this implies that we can find $t \in T(\overline{\mathbb{Q}})$ so that $c_1(t) \neq c_2(t)$ in $\overline{\mathbb{Q}}$ and $f_{c_1(t)}$ and $f_{c_2(t)}$ share at least $B+1$ preperiodic points; this is a contradiction.

Thus, the theorem is proved for all pairs $c_1 \neq c_2$ in $\mathbb{C}$, with the same bound as for pairs $c_1 \neq c_2$ in $\overline{\mathbb{Q}}$.

11. Effective bounds on common preperiodic points

In this section, we make effective Theorems 1.6, 1.7, and 1.9, to produce an explicit value for the bound B of Theorem 1.1:

Theorem 11.1. *For all $c_1 \neq c_2 \in \mathbb{C}$, we have*

$$|\mathrm{Preper}(f_{c_1}) \cap \mathrm{Preper}(f_{c_2})| \leq 10^{82}.$$

11.1. **An explicit lower bound in Theorem 1.6.** In order to provide an effective lower bound δ for Theorem 1.6, we need to improve our estimates on the energy pairing $E_v(c_1, c_2)$ when $|c_1 - c_2|_v$ is small at an archimedean place v. Here we compute that we can take $\delta = 10^{-75}$.

Let $H = 2001^{100/99}$. Suppose that c_1 and c_2 lie in a number field K, and suppose that for at least $99/100$ of the archimedean places of K, we have

$$|c_1 - c_2|_v \leq 1/H.$$

Then $h(c_1 - c_2) \geq \frac{99}{100} \log H$, and the proof of Theorem 7.1 implies that

$$\langle f_{c_1}, f_{c_2} \rangle \geq \frac{1}{16} h(c_1 - c_2) - \frac{1}{16} \log(2000) \geq \frac{\log(2001/2000)}{16} > 3.12 \cdot 10^{-5} > 10^{-75}.$$

Now suppose that we have $|c_1 - c_2|_v > 1/H$ for at least $1/100$ of the archimedean places of K. Let $M = 9H^2$ so that

$$|c_1 - c_2|_v > \frac{1}{H} = \frac{3}{M^{1/2}}$$

at all of these places. If $\max\{|c_1|_v, |c_2|_v\} > M$, then Theorem 4.1 implies that

$$E_v(c_1, c_2) \geq \frac{1}{64} \log M > 0.14$$

at this place v. On the other hand, if $\max\{|c_1|_v, |c_2|_v\} \leq M$, we have the following bound:

Proposition 11.2. *Fix any $M \geq 1000$. Then for all $s \geq M^2$, we have*

$$E_\infty(c_1, c_2) \geq \frac{|c_1 - c_2|^2}{32 s^4} - \frac{117}{100} \frac{M^3}{s^6},$$

provided $\max\{|c_1|, |c_2|\} \leq M$.

Assuming Proposition 11.2, we complete our computations. With $M = 9H^2$, we have

$$E_v(c_1, c_2) \;\geq\; \frac{|c_1 - c_2|_v^2}{32 s^4} - \frac{117 \cdot 9^3 H^6}{100 s^6} \;\geq\; \frac{1}{32 s^4 H^2} \left(1 - \frac{117 \cdot 2^5 9^3 H^8}{100 s^2}\right)$$

for all archimedean places v with $|c_1 - c_2|_v > 1/H$, $\max\{|c_1|_v, |c_2|_v\} \leq 9H^2$, and $s > 9^2H^4$. Choosing s satisfying $s^2 = 117 \cdot 2^6 9^3 H^8/100$, we conclude that

$$E_v(c_1, c_2) \;\geq\; \frac{100^2}{2^{18}9^6 117^2 H^{18}}$$

for all such places v. This shows that, summing only over the archimedean places, we have

$$\begin{aligned}\langle f_{c_1}, f_{c_2}\rangle \;&\geq\; \sum_{v \in M_K^\infty} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} E_v(c_1, c_2) \\ &\geq\; \frac{1}{100}\, \max\left\{0.14, \frac{100^2}{2^{18}9^6 117^2 H^{18}}\right\} \;>\; 10^{-75}\end{aligned}$$

whenever $|c_1 - c_2|_v > 1/H$ for at least $1/100$ of the archimedean places of K. This completes the computation of δ, and it remains only to prove Proposition 11.2.

Proof of Proposition 11.2. The result will follow from a series of elementary estimates on the values of the escape-rate functions outside the filled Julia set. Let φ_c be the Böttcher function for $f_c(z) = z^2 + c$, so that $\varphi_c(f_c(z)) = \varphi_c^2(z)$ for all z large enough, and therefore φ_c has expansion

$$\varphi_c(z) = z + \frac{c}{2z} + \cdots \tag{11.1}$$

for z near ∞. We set

$$\lambda(z) := \lambda_{c_1}(z) - \lambda_{c_2}(z),$$

the difference of two escape-rate functions. The energy pairing satisfies

$$2E_\infty(c_1, c_2) = 2\int_{\mathbb{C}} \lambda_{c_1} dd^c \lambda_{c_2} = -\int_{\mathbb{C}} \lambda dd^c \lambda = \int_{\mathbb{C}} d\lambda \wedge d^c\lambda.$$

Now fix any large $s > 0$, and define $D_s^c := \{z \in \mathbb{C} : |z| > s\}$. By Green's formula,

$$2E_\infty(c_1, c_2) \geq \int_{D_s^c} d\lambda \wedge d^c\lambda = -\int_{\partial D_s^c} \lambda d^c\lambda = -\frac{1}{2\pi i}\int_{\partial D_s^c} \lambda\left(\frac{\partial\lambda}{\partial z}dz - \frac{\partial\lambda}{\partial\bar z}d\bar z\right)$$

We will estimate the latter integral.

Note that λ satisfies

$$\lambda(z) = \log|\varphi_{c_1}| - \log|\varphi_{c_2}|$$

near ∞. For simplicity, write $\varepsilon := c_1 - c_2$. By the expansion (11.1) of φ_c

$$2\lambda(z) = \frac{\varepsilon}{2z^2} + \frac{\bar\varepsilon}{2\bar z^2} + O\left(\frac{1}{|z|^3}\right).$$

Similarly, by using the Taylor expansion and letting $z = se^{i\theta}$ on the boundary ∂D_s^c,

$$2\left(\frac{\partial\lambda}{\partial z}dz - \frac{\partial\lambda}{\partial\bar z}d\bar z\right) = \left[-\left(\frac{\varepsilon}{4s^3e^{2i\theta}} + \frac{\bar\varepsilon}{4s^3e^{-2i\theta}}\right) + O\left(\frac{1}{s^4}\right)\right] is d\theta.$$

Consequently

$$-4\frac{1}{2\pi i}\int_{\partial D_s^c}\lambda\left(\frac{\partial\lambda}{\partial z}dz-\frac{\partial\lambda}{\partial\bar z}d\bar z\right)=\frac{\varepsilon\bar\varepsilon}{4s^4}+O\left(\frac{1}{s^5}\right)$$

This gives

$$2E_\infty(c_1,c_2)\geq-\frac{1}{2\pi i}\int_{\partial D_s^c}\lambda\left(\frac{\partial\lambda}{\partial z}dz-\frac{\partial\lambda}{\partial\bar z}d\bar z\right)=\frac{\varepsilon\bar\varepsilon}{16s^4}+O\left(\frac{1}{s^5}\right)\tag{11.2}$$

where $\varepsilon=c_1-c_2$. To prove the proposition, we need control on the big-O term.

In the rest of this section, we fix an $M\geq 1000$.

Lemma 11.3. *Let $z,c_i\in\mathbb{C}$ with $|z|\geq M^2$, $|c_i|\leq M$ for $i=1,2$ and $\epsilon=c_1-c_2$. Then*

$$\left|4\lambda(z)-\left(\frac{\epsilon}{z^2}+\frac{\bar\epsilon}{\bar z^2}\right)\right|\leq\sum_{i=1,2}\left(\frac{202}{100}\frac{|c_i|}{|z|^4}+\frac{101}{100}\cdot\frac{|c_i|^2}{|z|^4}\right)$$

Proof. First note that

$$|\log(1+x)-x|=\left|-\frac{x^2}{2}+\frac{x^3}{3}+\cdots\right|\leq\frac{|x|^2}{2(1-|x|)}.\tag{11.3}$$

For any $|z|\geq|c|$ and $|z|>4$, inductively it is easy to check that for each $n\geq1$

$$(|z|-|c/z|)^{2^n}\leq|f_c^n(z)|\leq(|z|+|c/z|)^{2^n},\tag{11.4}$$

hence

$$\log(|z|-|c/z|)\leq\lambda_c(z)\leq\log(|z|+|c/z|)$$

and

$$\log(|z^2+c|-|c|/|z^2+c|)\leq\lambda_c(z^2+c)=2\lambda_c(z)\leq\log(|z^2+c|+|c|/|z^2+c|).$$

Consequently for any $|z|\geq M^2$ and $|c|\leq M$, by (11.3) one has

$$\begin{aligned}\left|2\lambda_c(z)-\log|z^2+c|\right|&\leq\left|\log\left(1\pm\frac{|c|}{|z^2+c|^2}\right)\right|\leq\frac{|c|}{|z^2+c|^2}+\frac{|c|^2}{|z^2+c|^4}\frac{1}{2(1-\frac{|c|}{|z^2+c|^2})}\\&\leq\frac{101}{100}\frac{|c|}{|z|^4}.\end{aligned}$$

Now, by the triangle inequality and (11.3) we have

$$\begin{aligned}\left|4\lambda(z)-\left(\frac{\epsilon}{z^2}+\frac{\bar\epsilon}{\bar z^2}\right)\right|&\leq\sum_{i=1,2}\left(\left|4\lambda_{c_i}(z)-2\log|z^2+c_i|\right|+\left|\log(z^2+c_i)-\log z^2-\frac{c_i}{z^2}\right|\right)\\&\quad+\sum_{i=1,2}\left|\log(\bar z^2+\bar c_i)-\log\bar z^2-\frac{\bar c_i}{\bar z^2}\right|\\&\leq\sum_{i=1,2}\left(\frac{202}{100}\frac{|c_i|}{|z|^4}+\frac{|c_i|^2/|z|^4}{1-\left|\frac{c_i}{z^2}\right|}\right)\leq\sum_{i=1,2}\left(\frac{202}{100}\frac{|c_i|}{|z|^4}+\frac{101}{100}\cdot\frac{|c_i|^2}{|z|^4}\right).\end{aligned}$$

□

Lemma 11.4. *For any $z, c \in \mathbb{C}$ with $|z| \geq M^2$ and $|c| \leq M$, we have*

$$\left| \prod_{i=1}^{n} \frac{(f_c^{i-1}(z))^2}{f_c^i(z)} - 1 + \frac{c}{z^2} \right| \leq \frac{102}{100} \cdot \frac{|c|^2}{|z|^4} + \frac{104}{100} \cdot \frac{|c|}{|z|^4}.$$

Proof. For $\alpha \in \mathbb{C}$ with $|\alpha| < 1$,

$$|e^\alpha - 1| = \left| \alpha + \frac{\alpha^2}{2!} + \cdots \right| \leq |\alpha| + \frac{|\alpha|^2}{2!} + \cdots \leq \frac{|\alpha|}{1 - |\alpha|}.$$

From (11.3) and (11.4), for each $i \geq 2$, it has

$$\begin{aligned} \left| \log \frac{(f_c^{i-1}(z))^2}{f_c^i(z)} \right| &= \left| \log \frac{1}{1 - c/(f_c^{i-1}(z))^2} \right| = \left| \log \left(1 - \frac{c}{(f_c^{i-1}(z))^2} \right) \right| \\ &\leq \left| \frac{c}{(f_c^{i-1}(z))^2} \right| \left(1 + \frac{\left| \frac{c}{(f_c^{i-1}(z))^2} \right|}{2 \left(1 - \left| \frac{c}{(f_c^{i-1}(z))^2} \right| \right)} \right) \\ &\leq \frac{101}{100} \frac{|c|}{(|z| - |c/z|)^{2^i}}, \end{aligned}$$

here for the last inequality we use the fact that $|c|/|f_c^{i-1}(z)|^2 \leq |c|/(|z| - |c/z|)^{2^i} \leq 1/1000$. Therefore, since $(|z| - |c/z|)^2 \geq M^2/2$, we conclude

$$\left| \log \prod_{i=2}^{n} \frac{(f_c^{i-1}(z))^2}{f_c^i(z)} \right| \leq \sum_{i=2}^{n} \frac{101}{100} \cdot \frac{|c|}{(|z| - |c/z|)^{2^i}} \leq \frac{102}{100} \cdot \frac{|c|}{(|z| - |c/z|)^4} \leq \frac{103}{100} \cdot \frac{|c|}{|z|^4}.$$

For $i = 1$,

$$\left| \frac{(f_c^{i-1}(z))^2}{f_c^i(z)} - 1 + \frac{c}{z^2} \right| = \left| \frac{1}{1 + \frac{c}{z^2}} - 1 + \frac{c}{z^2} \right| \leq \frac{|c|^2}{|z|^4} \cdot \frac{1}{1 - |c/z^2|} \leq \frac{101}{100} \cdot \frac{|c|^2}{|z|^4}.$$

Finally, let

$$\alpha = \log \prod_{i=2}^{n} \frac{(f_c^{i-1}(z))^2}{f_c^i(z)} \text{ and } \beta = \frac{z^2}{z^2 + c} - 1 + \frac{c}{z^2}$$

and then

$$\begin{aligned} \left| \prod_{i=1}^{n} \frac{(f_c^{i-1}(z))^2}{f_c^i(z)} - 1 + \frac{c}{z^2} \right| &= \left| e^\alpha \left(\beta + 1 - \frac{c}{z^2} \right) - \left(1 - \frac{c}{z^2} \right) \right| \\ &\leq |e^\alpha \beta| + \left| (e^\alpha - 1) \left(1 - \frac{c}{z^2} \right) \right| \\ &\leq \left(1 + \frac{|\alpha|}{1 - |\alpha|} \right) |\beta| + \frac{|\alpha|}{1 - |\alpha|} \left(1 + \frac{|c|}{|z|^2} \right). \end{aligned}$$

The inequalities of α and β give

$$\left|\prod_{i=1}^{n}\frac{(f_c^{i-1}(z))^2}{f_c^i(z)}-1+\frac{c}{z^2}\right| \le \left(1+\frac{\frac{103}{100}\cdot\frac{|c|}{|z|^4}}{1-\frac{103}{100}\cdot\frac{|c|}{|z|^4}}\right)\cdot\frac{101}{100}\cdot\frac{|c|^2}{|z|^4}+\frac{\frac{103}{100}\cdot\frac{|c|}{|z|^4}}{1-\frac{103}{100}\cdot\frac{|c|}{|z|^4}}\left(1+\frac{|c|}{|z|^2}\right)$$
$$\le \frac{102}{100}\cdot\frac{|c|^2}{|z|^4}+\frac{104}{100}\cdot\frac{|c|}{|z|^4}.$$

□

Lemma 11.5. *With the same hypotheses as Lemma 11.4, we have that*

$$\left|\frac{2\partial\lambda_c(z)}{\partial z}-\frac{1}{z}+\frac{c}{z^3}\right| \le \frac{102}{100}\cdot\frac{|c|^2}{|z|^5}+\frac{104}{100}\cdot\frac{|c|}{|z|^5}$$

Proof. Consider

$$2\lambda_c(z)=\lim_{n\to\infty}\frac{2\log^+|f_c^n(z)|}{2^n}=\lim_{n\to\infty}\left(\frac{\log f_c^n(z)}{2^n}+\frac{\log f_{\bar{c}}^n(\bar{z})}{2^n}\right).$$

Taking partial derivatives of both sides and by Lemma 11.4, we have

$$\left|\frac{2\partial\lambda_c(z)}{\partial z}-\frac{1}{z}+\frac{c}{z^3}\right| = \left|\lim_{n\to\infty}\frac{1}{2^n}\frac{\prod_{i=1}^n f_c'(f_c^{i-1}(z))}{f^n(z)}-\frac{1}{z}+\frac{c}{z^3}\right|$$
$$= \left|\frac{1}{z}\lim_{n\to\infty}\left(\prod_{i=1}^{n}\frac{(f_c^{i-1}(z))^2}{f_c^i(z)}-1+\frac{c}{z^2}\right)\right|$$
$$\le \frac{102}{100}\cdot\frac{|c|^2}{|z|^5}+\frac{104}{100}\cdot\frac{|c|}{|z|^5}.$$

□

Similarly

$$\left|\frac{2\partial\lambda_c(z)}{\partial\bar{z}}-\frac{1}{\bar{z}}+\frac{\bar{c}}{\bar{z}^3}\right| \le \frac{102}{100}\cdot\frac{|c|^2}{|z|^5}+\frac{104}{100}\cdot\frac{|c|}{|z|^5}. \tag{11.5}$$

Now we are ready to control the big-O term in (11.2). Write

$$\lambda=\left(\frac{\varepsilon}{4z^2}+\frac{\bar{\varepsilon}}{4\bar{z}^2}\right)+\left[\lambda-\left(\frac{\varepsilon}{4z^2}+\frac{\bar{\varepsilon}}{4\bar{z}^2}\right)\right],$$
$$\frac{\partial\lambda}{\partial z}dz=\left[\left(\frac{\partial\lambda}{\partial z}+\frac{\varepsilon}{2z^3}\right)-\frac{\varepsilon}{2z^3}\right]dz \text{ and } \frac{\partial\lambda}{\partial\bar{z}}d\bar{z}=\left[\left(\frac{\partial\lambda}{\partial\bar{z}}+\frac{\bar{\varepsilon}}{2\bar{z}^3}\right)-\frac{\bar{\varepsilon}}{2\bar{z}^3}\right]d\bar{z}.$$

We set

$$I_1=\frac{|\varepsilon|}{2s}\max_{i=1,2}\left[\frac{102}{100}\cdot\frac{|c_i|^2}{|s|^5}+\frac{104}{100}\cdot\frac{|c_i|}{|s|^5}\right],$$
$$I_2=\frac{1}{4}\sum_{i=1,2}\left(\frac{202}{100}\frac{|c_i|}{|s|^4}+\frac{101}{100}\cdot\frac{|c_i|^2}{|s|^4}\right)\max_{i=1,2}\left(\frac{102}{100}\cdot\frac{|c_i|^2}{|s|^5}+\frac{104}{100}\cdot\frac{|c_i|}{|s|^5}\right)\cdot s,$$
$$I_3=\frac{1}{4}\sum_{i=1,2}\left(\frac{202}{100}\frac{|c_i|}{|s|^4}+\frac{101}{100}\cdot\frac{|c_i|^2}{|s|^4}\right)\frac{|\varepsilon|}{2s^2}.$$

Lemmas 11.3 and 11.5 along with inequalities (11.5) and (11.2) give

$$2\,E_\infty(c_1,c_2) \geq \frac{\varepsilon\bar{\varepsilon}}{16s^4} - 2(I_1+I_2+I_3).$$

Now since the fixed $M \geq 1000$, assume that $|c_i| \leq M$ for $i = 1, 2$, and suppose that $s \geq M^2$. Then $|\varepsilon| \leq 2M$, so that

$$I_1 \leq \frac{103}{100}\cdot\frac{M^3}{s^6},\ I_2 \leq \frac{1}{1000}\cdot\frac{M^3}{s^6},\ \text{and } I_3 \leq \frac{102}{800}\cdot\frac{M^3}{s^6}.$$

Therefore,

$$2(I_1+I_2+I_3) \leq \frac{234}{100}\frac{M^3}{s^6}.$$

This completes the proof of the proposition. □

11.2. **Explicit bound.** As shown in §8.3, we have $\alpha_1 = 1/192$ and $C_1 = 3/17$ in Theorem 1.7, and we may take and $C(\varepsilon) = 40\log(25/\varepsilon)$ in Theorem 1.9 as shown in §9.3. Therefore, $C(\alpha_1/2) = 40\log(50/\alpha_1) < 367$, and whenever $c_1 \neq c_2 \in \overline{\mathbb{Q}}$ so that f_{c_1} and f_{c_2} have $N(c_1,c_2) > 1$ common preperiodic points and $h(c_1,c_2) > 139$, we have

$$N(c_1,c_2) < 281857 < 10^6$$

from (10.1). For the set of parameters with $h(c_1,c_2) \leq 139$, the bound we obtain is much larger, as it depends on the small δ from Theorem 1.6. We can take $\delta = 10^{-75}$, as explained in §11.1. Taking $\varepsilon = \alpha_1\delta/(8(C_1+\alpha_1))$ in (10.2), we find that

$$\begin{aligned} N(c_1,c_2) - 1 &\leq \frac{8(C_1+\alpha_1)\cdot 40\log(25/\varepsilon)}{\alpha_1\delta} \\ &= \frac{320(C_1+\alpha_1)}{\alpha_1\delta}\ \log\frac{200(C_1+\alpha)}{\alpha_1\delta} \\ &\leq \frac{320\cdot 35}{\delta}\ \log\frac{200\cdot 35}{\delta} \\ &\leq 75\cdot 320\cdot 35\cdot 10^{75}\ \log(200\cdot 35\cdot 10), \end{aligned}$$

so that

$$N(c_1,c_2) = |\mathrm{Preper}(f_{c_1}) \cap \mathrm{Preper}(f_{c_2})| < 10^{82}.$$

The same bound holds for all $c_1 \neq c_2$ in $\mathbb{C}$, as explained in §10.2.

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