# UNIFORM MANIN-MUMFORD FOR A FAMILY OF GENUS 2 CURVES

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ABSTRACT. We introduce a general strategy for proving quantitative and uniform bounds on the number of common points of height zero for a pair of inequivalent height functions on $\mathbb{P}^1(\overline{\mathbb{Q}})$. We apply this strategy to prove a conjecture of Bogomolov, Fu, and Tschinkel asserting uniform bounds on the number of common torsion points of elliptic curves in the case of two Legendre curves over $\mathbb{C}$. As a consequence, we obtain two uniform bounds for a two-dimensional family of genus 2 curves: a uniform Manin-Mumford bound for the family over $\mathbb{C}$, and a uniform Bogomolov bound for the family over $\overline{\mathbb{Q}}$.

## 1. INTRODUCTION

In this article, we use the arithmetic Arakelov-Zhang intersection of heights on $\mathbb{P}^1(\overline{\mathbb{Q}})$ to prove a uniform Manin-Mumford bound for a two-dimensional family of genus 2 curves over $\mathbb{C}$. The Manin-Mumford Conjecture, proved by Raynaud [Ra], asserts

$$|j_P(X) \cap J(X)^{tors}| < \infty \tag{1.1}$$

for any smooth complex projective curve $X$ of genus $g \geq 2$ and any point $P \in X(\mathbb{C})$, where $j_P : X \hookrightarrow J(X)$ is the Abel-Jacobi embedding of $X$ into its Jacobian $J(X)$ based at $P$, and $J(X)^{tors}$ is the set of torsion points of the Jacobian. In the case of genus $g = 2$, the curve is hyperelliptic, and the fixed points of the hyperelliptic involution provide geometrically natural choices of base point for the Abel-Jacobi map. We provide a uniform bound on the torsion images under such a map, provided the curve admits an elliptic involution.

**Theorem 1.1.** *There exists a uniform constant $B$ such that*

$$|j_P(X) \cap J(X)^{tors}| \leq B$$

*for all smooth curves $X$ over $\mathbb{C}$ of genus 2 admitting a degree-two map to an elliptic curve and all Weierstrass points $P$ on $X$.*

The curves satisfying the hypothesis of Theorem 1.1 form a complex surface $\mathcal{L}_2$ in the moduli space $\mathcal{M}_2$ of genus 2 curves. These $X$ are also characterized by the

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property that their Jacobians admit real multiplication by the real quadratic order of discriminant 4. Further details on $\mathcal{L}_2$ are given in Section 9.

**Remark.** We do not give bounds on $B$, but this bound can be made effective by providing explicit estimates for the continuity constants of Section 4. Regarding this constant, it follows from Theorem 1 of [Po] that there are infinitely many curves $X \in \mathcal{L}_2$ for which $|j_P(X) \cap J(X)^{tors}|$ is at least 22 for a Weierstrass point $P$, and we know of no curve $X \in \mathcal{M}_2$ and point $P \in X$ satisfying $|j_P(X) \cap J(X)^{tors}| > 22$.

The question of uniformity in (1.1) was raised by Mazur in [Ma] who asked if a bound could be given that depends only on the genus $g$ of the curve $X$. Quantitative bounds on torsion points on curves have been obtained when the curve is defined over a number field, notably by Coleman [Co], Buium [Bu], Hrushovski [Hr], and more recently by Katz, Rabinoff, and Zureick-Brown [KRZG]. By quantifying the $p$-adic approach to (1.1), these authors achieve bounds for general families of curves; however, these bounds all involve dependence on field of definition or the choice of a prime for the family of curves, so are not uniform for families over $\overline{\mathbb{Q}}$ or $\mathbb{C}$.

Our new technique which yields Theorem 1.1 is a quantification of the approach of Szpiro, Ullmo, and Zhang [SUZ, Ul, Zh1] to proving (1.1), utilizing adelic equidistribution theory. We first reduce to the setting where the curve is defined over $\overline{\mathbb{Q}}$. Over $\overline{\mathbb{Q}}$, we build on the proof of the quantitative equidistribution theorem for height functions on $\mathbb{P}^1(\overline{\mathbb{Q}})$ of Favre and Rivera-Letelier [FRL1].

In fact, we deduce Theorem 1.1 from a case of the following conjecture, discussed by Bogomolov and Tschinkel [BT] and stated formally as [BFT, Conjectures 2 and 12], which asserts uniform bounds on common torsion points for pairs of elliptic curves. By a *standard projection* $\pi : E \to \mathbb{P}^1$ of an elliptic curve $E$ over $\mathbb{C}$, we mean any degree-two quotient that identifies a point $P$ and its inverse $-P$ for some choice of identity point for $E$.

**Conjecture 1.2.** [BFT] *There exists a uniform constant $B$ such that*

$$|\pi_1(E_1^{tors}) \cap \pi_2(E_2^{tors})| \leq B$$

*for any pair of elliptic curves $E_i$ over $\mathbb{C}$ and any pair of standard projections $\pi_i$ for which $\pi_1(E_1[2]) \neq \pi_2(E_2[2])$.*

Note that if the four 2-torsion points of $E_1$ and of $E_2$ satisfy $\pi_1(E_1[2]) = \pi_2(E_2[2])$, then $E_1$ is isomorphic to $E_2$ and $\pi_1(E_1^{tors}) = \pi_2(E_2^{tors})$. The finiteness of $\pi_1(E_1^{tors}) \cap \pi_2(E_2^{tors})$, under the assumption that $\pi_1(E_1[2]) \neq \pi_2(E_2[2])$, follows from the main theorem of Raynaud in [Ra]; indeed, the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$ lifts to a (singular) curve $C \subset E_1 \times E_2$ via $\pi_1 \times \pi_2$ with normalization of genus $g \geq 2$ [BT].

We prove Conjecture 1.2 in the case of maximal overlap of the 2-torsion points; i.e., when

$$|\pi_1(E_1[2]) \cap \pi_2(E_2[2])| = 3.$$

This setting corresponds to the case where the (normalization of the) curve $C$ in $E_1 \times E_2$ has minimal genus 2. By fixing coordinates on $\mathbb{P}^1$, we may work with the Legendre family of elliptic curves

$$E_t \ : \ y^2 = x(x-1)(x-t)$$

with $t \in \mathbb{C} \setminus \{0,1\}$ and the standard projection $\pi(x,y) = x$ on $E_t$.

**Theorem 1.3.** *There exists a uniform constant $B$ such that*

$$|\pi(E_{t_1}^{tors}) \cap \pi(E_{t_2}^{tors})| \leq B,$$

*for all $t_1 \neq t_2$ in $\mathbb{C} \setminus \{0,1\}$.*

To prove Theorem 1.3, we introduce a general strategy for bounding the number of common height-zero points for any pair of distinct height functions $h_1, h_2 : \mathbb{P}^1(\overline{\mathbb{Q}}) \to \mathbb{R}$ that arise from continuous, semipositive, adelic metrics on the line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$. Our heights are normalized so that their Arakelov-Zhang intersection number, which we denote by $h_1 \cdot h_2$, will satisfy

$$h_1 \cdot h_2 \geq 0 \text{ with equality if and only if } h_1 = h_2.$$

Details on these heights and the pairing are given in Section 2. The value of $h_1 \cdot h_2$ provides a notion of distance between the two heights (as was observed by Fili in [Fi]). Large numbers of common zeroes between $h_1$ and $h_2$ will imply that $h_1$ and $h_2$ are close. Indeed, it follows from equidistribution [CL1, FRL1, BR] that

$$\lim_{n\to\infty} h_2(x_n) = h_1 \cdot h_2$$

for any infinite sequence of distinct points $x_n \in \mathbb{P}^1(\overline{\mathbb{Q}})$ such that $h_1(x_n) \to 0$ as $n \to \infty$. While this measure of closeness between two heights is not generally uniform in families of heights, by bounding the height pairing $h_1 \cdot h_2$ from below, we obtain an upper bound on the number of common zeroes.

In the context of Theorem 1.3, we consider the family of height functions $\hat{h}_t$ on $\mathbb{P}^1(\overline{\mathbb{Q}})$ induced from the Néron-Tate canonical height on the elliptic curve $E_t$, for $t \in \overline{\mathbb{Q}} \setminus \{0,1\}$; its zeroes are precisely the elements of $\pi(E_t^{tors})$. We implement this general strategy by proving three bounds on the intersection pairing $\hat{h}_{t_1} \cdot \hat{h}_{t_2}$. We prove a uniform lower bound on the pairing:

**Theorem 1.4.** *There exists $\delta > 0$ such that*

$$\hat{h}_{t_1} \cdot \hat{h}_{t_2} \geq \delta$$

*for all $t_1 \neq t_2 \in \overline{\mathbb{Q}} \setminus \{0,1\}$.*

We also prove an asymptotic lower bound for parameters $t_1$ and $t_2$ with large height:

**Theorem 1.5.** *There exist constants $\alpha, \beta > 0$ such that*

$$\hat{h}_{t_1} \cdot \hat{h}_{t_2} \;\geq\; \alpha\, h(t_1, t_2) - \beta$$

*for all $t_1 \neq t_2$ in $\overline{\mathbb{Q}} \setminus \{0, 1\}$. Here $h(t_1, t_2)$ is the naive logarithmic height on $\mathbb{A}^2(\overline{\mathbb{Q}})$.*

We find an upper bound that depends on the number of common zeroes of $\hat{h}_{t_1}$ and $\hat{h}_{t_2}$ as well as the heights of the parameters $t_1$ and $t_2$:

**Theorem 1.6.** *For all $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that*

$$\hat{h}_{t_1} \cdot \hat{h}_{t_2} \leq \left( \varepsilon + \frac{C(\varepsilon)}{N(t_1, t_2)} \right) (h(t_1, t_2) + 1)$$

*for all $t_1$ and $t_2$ in $\overline{\mathbb{Q}} \setminus \{0, 1\}$, where $N(t_1, t_2) := |\pi(E_{t_1}^{tors}) \cap \pi(E_{t_2}^{tors})|$.*

The three theorems combine to give a uniform bound on the number $N(t_1, t_2)$ of common zeroes of $\hat{h}_{t_1}$ and $\hat{h}_{t_2}$ for all $t_1 \neq t_2$ in $\overline{\mathbb{Q}} \setminus \{0, 1\}$.

Theorems 1.4 and 1.5 follow from estimates on the local height functions and the local equilibrium measures on the $v$-adic Berkovich projective line at each place $v$ of a number field containing $t_1$ and $t_2$, computing using the dynamical Lattès map $f_t : \mathbb{P}^1 \to \mathbb{P}^1$ induced by multiplication by 2 on a Legendre curve $E_t$. The non-archimedean contributions to $\hat{h}_{t_1} \cdot \hat{h}_{t_2}$ turn out to be straightforward to compute for these heights. Significant technical issues arise when $v$ is archimedean and both parameters $t_i$ are tending to the singularity set $\{0, 1, \infty\}$ for this family; we resolve these issues by appealing to the theory of degenerations of complex dynamical systems on $\mathbb{P}^1(\mathbb{C})$, in which a family of complex rational maps degenerates to a non-archimedean dynamical system acting on a Berkovich space, as in the work of DeMarco-Faber [DF1] and Favre [Fa], using the formalism of hybrid space as discussed by Boucksom-Jonsson in [BJ].

For Theorem 1.6, we expand upon the quantitative equidistribution results of Favre-Rivera-Letelier [FRL1] and Fili [Fi] to analyze the rates of convergence of measures supported on finite sets of zeroes of a height $h$ to the associated equilibrium measures at each place $v$. To do so requires control on the modulus of continuity of the local heights, and again we rely on estimates from the hybrid space to treat the cases where a parameter $t$ is tending to one of the singularities for the family $E_t$.

Although Theorem 1.4 alone was not enough to prove Theorem 1.3, it implies a uniform bound of a different sort, when combined with Zhang's inequality on the essential minimum of a height function [Zh2]:

**Proposition 1.7.** *Choose any $b$ satisfying $0 < b < \delta/2$ for the $\delta$ of Theorem 1.4. Then the set*

$$S(b, t_1, t_2) := \{x \in \mathbb{P}^1(\overline{\mathbb{Q}}) : \hat{h}_{t_1}(x) + \hat{h}_{t_2}(x) \leq b\}$$

*is finite for each pair $t_1 \neq t_2 \in \overline{\mathbb{Q}} \setminus \{0, 1\}$.*

Our proof of Theorem 1.1, however, gives more; it provides a uniform version of the Bogomolov Conjecture for this family of genus 2 curves. The Bogomolov Conjecture was proved for each individual curve $X$ over $\overline{\mathbb{Q}}$ in [Ul, Zh1]. To state our result precisely, we fix ample and symmetric line bundles on the family of Jacobians $J(X)$ for the genus 2 curves $X$ defined over $\overline{\mathbb{Q}}$ that we consider in Theorem 1.1. Specifically, we take $L_X = \Phi^* L_D$ for the isogeny $\Phi : J(X) \to E_1 \times E_2$ of Proposition 9.1, with $L_D$ the line bundle associated to the divisor $D = \{O_1\} \times E_2 + E_1 \times \{O_2\}$, where $O_i$ is the identity element of $E_i$.

**Theorem 1.8.** *There exist constants $B$ and $b > 0$ such that*

$$\left|\{x \in j_P(X)(\overline{\mathbb{Q}}) : \hat{h}_{L_X}(x) \leq b\}\right| \ \leq \ B$$

*for all smooth curves $X$ over $\overline{\mathbb{Q}}$ of genus 2 admitting a degree-two map to an elliptic curve and all Weierstrass points $P$ on $X$, where $\hat{h}_{L_X}$ is the Néron-Tate canonical height on $J(X)$.*

Finally, we mention that we implement this general strategy towards uniform boundedness in a follow-up article [DKY] in another setting, providing a uniform bound on the number of common preperiodic points for distinct polynomials of the form $f_c(z) = z^2 + c$ with $c \in \mathbb{C}$.

**Outline of the paper.** We fix our notation and provide background in Section 2. Sections 3, 4, and 5 provide the estimates on local height functions and local measures needed to prove all of our theorems. Theorem 1.5 is proved in Section 6, and from it we deduce Theorem 1.4 and Proposition 1.7. A generalization of Theorem 1.6 is proved in Section 7 which treats points of small height, not only of height 0. We prove Theorem 1.3 in Section 8 and finally Theorems 1.1 and 1.8 in Section 9.

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## 2. Heights, measures, and energies

This section develops the background and notation needed for the proofs that follow. Throughout, $K$ is a number field with places $M_K$.

2.1. **The canonical height.** Fix $t \in \overline{\mathbb{Q}} \setminus \{0,1\}$. Let $E_t$ be the Legendre elliptic curve and $\pi : E_t \to \mathbb{P}^1$ the projection defined by $\pi(x,y) = x$. The multiplication-by-two endomorphism on $E_t$ descends via $\pi$ to a morphism of degree 4 on $\mathbb{P}^1$ given by

$$f_t(x) = \frac{(x^2-t)^2}{4x(x-1)(x-t)}. \tag{2.1}$$

The canonical height on the elliptic curve

$$\hat{h}_{E_t} : E_t(\overline{\mathbb{Q}}) \to \mathbb{R}$$

can be defined via the projection $\pi$ and the iteration of $f_t$ as $h_{E_t}(P) := \frac{1}{2}\hat{h}_t(\pi(P))$ where

$$\hat{h}_t : \mathbb{P}^1(\overline{\mathbb{Q}}) \to \mathbb{R}$$

is the dynamical canonical height defined by

$$\hat{h}_t(x) = \lim_{n\to\infty} \frac{1}{4^n} h(f_t^n(x)). \tag{2.2}$$

Here, $h$ is the (logarithmic) Weil height on $\mathbb{P}^1(\overline{\mathbb{Q}})$. Note that $\hat{h}_t(x) \geq 0$ for all $x \in \mathbb{P}^1(\overline{\mathbb{Q}})$, and

$$\hat{h}_t(x) = 0 \iff x \in \pi(E_t^{tors})$$

[Si], [CS].

The height $\hat{h}_t$ has a local decomposition as follows: for any number field $K$ containing $t$, and for each place $v \in M_K$, there exists a local height function $\lambda_{t,v}$ such that

$$\hat{h}_t(x) = \sum_{v\in M_K} \frac{r_v}{|\operatorname{Gal}(\overline{K}/K)\cdot x|} \sum_{y\in \operatorname{Gal}(\overline{K}/K)\cdot x} \lambda_{t,v}(y)$$

for all $x \in \overline{K}$, where

$$r_v := \frac{[K_v : \mathbb{Q}_v]}{[K:\mathbb{Q}]}.$$

The local heights $\lambda_{t,v}$ can be chosen to extend continuously to $\mathbb{P}^1(\mathbb{C}_v) \setminus \{\infty\}$, where $\mathbb{C}_v$ is the completion (w.r.t. $v$) of an algebraic closure of the completion $K_v$, and to satisfy

$$\lambda_{t,v}(x) = \log|x|_v + \mathcal{O}(1)$$

as $|x|_v \to \infty$.

2.2. **Local heights and escape rates.** To compute the local heights, we will often lift the maps $f_t : \mathbb{P}^1 \to \mathbb{P}^1$ of (2.1) to homogeneous coordinates. Define

$$F_t(z,w) := \big((z^2-tw^2)^2, 4zw(z-w)(z-tw)\big)$$

for $z$ and $w$ in $\mathbb{C}_v$. As observed in [BR, Chapter 10], its escape-rate function

$$G_{F_t,v}(z,w) := \lim_{n\to\infty} \frac{1}{4^n} \log\|F_t^n(z,w)\|_v, \tag{2.3}$$

where $\|(z,w)\|_v = \max\{|x|_v, |y|_v\}$, satisfies

$$\hat{h}_t(x) = \sum_{v \in M_K} \frac{r_v}{|\operatorname{Gal}(\overline{K}/K) \cdot \tilde{x}|} \sum_{\tilde{y} \in \operatorname{Gal}(\overline{K}/K)\cdot\tilde{x}} G_{F_t,v}(\tilde{y})$$

for $x \in \mathbb{P}^1(\overline{K})$ and $\tilde{x}$ any choice of lift of $x$ to $\overline{K}^2$. In particular, we may take

$$\lambda_{t,v}(x) = G_{F_t,v}(x,1) \tag{2.4}$$

as a local height for $\hat{h}_t$.

The elliptic curves $E_t$ and $E_{1-t}$ and $E_{1/t}$ are isomorphic, with the following transformation formulas for the local heights:

**Proposition 2.1.** *Fix any number field $K$ and $v \in M_K$. Then, for all $t \in K \setminus \{0,1\}$, we have*

$$G_{F_{1-t},v}(1-z,1) = G_{F_t,v}(z,1) = G_{F_{1/t},v}(z,t) = G_{F_{1/t},v}(z/t,1) + \log|t|_v.$$

*Proof.* Let $A$ be the automorphism $A(z,w) = (w-z,w)$. Then

$$A \circ F_t^n = -F_{1-t}^n \circ A$$

for all iterates, proving the first equality. Similarly, let $B(z,w) = (z,tw)$. Then

$$B \circ F_t^n = F_{1/t}^n \circ B$$

for all iterates, proving the second equality. The final equality follows from the logarithmic homogeneity of $G$. □

2.3. **The Berkovich projective line.** Let $K$ be a number field. For each $v \in M_K$, let $\mathbb{A}_v^{1,an}$ denote the Berkovich affine line over $\mathbb{C}_v$. The points of $\mathbb{A}_v^{1,an}$ come in four types. For $v \in M_K^0$, the Type I points in $\mathbb{A}_v^{1,an}$ are, by definition, the elements of the field $\mathbb{C}_v$; the Type II points are in one-to-one correspondence with disks $D(a,r) = \{x \in \mathbb{C}_v : |x-a|_v \le r\}$ with $r > 0$ rational. (We will not need the Type III or Type IV points in this article.) A Type II point corresponding to $D(a,r)$ will be denoted by $\zeta_{a,r}$. The *Gauss point* $\zeta_{0,1}$ is the Type II point identified with the unit disk. The Berkovich projective line $\mathbb{P}_v^{1,an} = \mathbb{A}_v^{1,an} \cup \{\infty\}$ is the one point compactification of $\mathbb{A}_v^{1,an}$, which is a canonically-defined path-connected compact Hausdorff space containing $\mathbb{P}^1(\mathbb{C}_v)$ as a dense subspace. If $v$ is archimedean, then $\mathbb{C}_v \simeq \mathbb{C}$ and $\mathbb{P}_v^{1,an} = \mathbb{P}^1(\mathbb{C})$.

For each $v \in M_K$ there is a distribution-valued Laplacian operator $\Delta$ on $\mathbb{P}_v^{1,an}$. The function $\log^+|z|_v$ on $\mathbb{P}^1(\mathbb{C}_v)$ extends naturally to a continuous real valued function $\mathbb{P}_v^{1,an} \to \mathbb{R} \cup \{\infty\}$, and the Laplacian is normalized such that

$$\Delta \log^+|z|_v = \omega_v - \delta_\infty$$

on $\mathbb{P}_v^{1,an}$, where $\omega_v = m_{S^1}$ is the Lebesgue probability measure on the unit circle when $v$ is archimedean, and $\omega_v = \delta_G$ is a point mass at the Gauss point of $\mathbb{P}_v^{1,an}$ when $v$ is non-archimedean. A probability measure $\mu_v$ on $\mathbb{P}_v^{1,an}$ is said to have *continuous*

*potentials* if $\mu_v - \omega_v = \Delta g$ with $g : \mathbb{P}^{1,an}_v \to \mathbb{R}$ continuous. The continuous potential for $\mu_v$ is unique up to the addition of a constant. See [BR] for more details. Note that the Laplacian used here is the negative of the one appearing in [PST] and [BR], but agrees with the usual Laplacian (up to a factor of $2\pi$) at the archimedean places.

Suppose $v \in M_K^0$ with $M_K^0$ being the set of non-archimedean places. We set

$$\mathbb{H} := \mathbb{A}^{1,an}_v \setminus \mathbb{C}_v.$$

The hyperbolic distance $d_{\text{hyp}}$ on $\mathbb{H}$ gives it the structure of a metrized $\mathbb{R}$-tree and satisfies

$$d_{\text{hyp}}(\zeta_{a,r_1}, \zeta_{a,r_2}) = \log(r_1/r_2)$$

for any $a \in \mathbb{C}_v$ and any $r_1 \geq r_2$. We will say that a probability measure $\mu_v$ on $\mathbb{H}$ is an *interval measure* if it is the uniform distribution on an interval $[\zeta_1, \zeta_2] \subset \mathbb{H}$ with respect to the linear structure induced from the hyperbolic metric $d_{\text{hyp}}$.

2.4. **Canonical measures and good reduction.** For each Legendre elliptic curve $E_t$ with $t$ in a number field $K$ and each $v \in M_K$, the local height $\lambda_{t,v}$ of (2.4) extends to define a continuous and subharmonic function on $\mathbb{A}^{1,an}_v$ with logarithmic singularity at $\infty$. We have

$$\Delta\lambda_{t,v} = \mu_{t,v} - \delta_\infty$$

on $\mathbb{P}^{1,an}_v$, where $\mu_{t,v}$ is the canonical probability measure for the dynamical system $f_t$ at $v$ [FRL1], [BR].

For archimedean $v \in M_K$, the measure $\mu_{t,v}$ is the unique $f_t$-invariant measure on $\mathbb{P}^1(\mathbb{C})$ achieving the maximal entropy $\log 4$. It is the push-forward of the Haar measure on $E_t(\mathbb{C})$ via the projection $\pi$ introduced in §2.1. See, for example, [Mi] for a dynamical discussion of the maps $f_t$ on the Riemann sphere.

For non-archimedean $v \in M_K$, if the curve $E_t$ and the map $f_t$ have good reduction, the measure $\mu_{t,v}$ is the point mass $\delta_G$ supported on the Gauss point $\zeta_{0,1}$. The map $f_t$ has potential good reduction, meaning that it has good reduction under a suitable change of coordinates on $\mathbb{P}^1$, if and only if the measure $\mu_{t,v}$ is supported at a single Type II point in $\mathbb{H}$. In general, the support of $\mu_{t,v}$ is equal to the Julia set of $f_t$ in $\mathbb{P}^{1,an}_v$.

Recall that the $j$-invariant of the elliptic curve $E_t$ over $\mathbb{C}$ is given by

$$j(t) = \frac{256(1-t+t^2)^3}{(1-t)^2t^2}. \tag{2.5}$$

For $t \in K$ and non-archimedean $v \in M_K$, the map $f_t$ has potential good reduction at $v$ if and only if the curve $E_t$ has potential good reduction at $v$. This equivalence can be proved via equidistribution of torsion points on $E_t$ at all places [BPe, Theorem 1] (thus implying that the measure $\mu_{t,v}$ will also be supported at a single point of $\mathbb{P}^{1,an}_v$) or via a direct calculation showing that the Julia set of $f_t$ is a singleton if and only if $|j(t)|_v \leq 1$.

2.5. **The height as an adelic metric.** Suppose $t \in K \setminus \{0,1\}$. The height $\hat{h}_t$ on $\mathbb{P}^1(\overline{\mathbb{Q}})$, introduced in §2.1, is induced from an adelic metric on $\mathcal{O}_{\mathbb{P}^1}(1)$, in the sense of Zhang [Zh2]. Fixing coordinates on $\mathbb{P}^1$ and a section $s$ of $\mathcal{O}_{\mathbb{P}^1}(1)$ with $(s) = (\infty)$, then a metric $\|\cdot\|_{t,v}$ can be defined at each place $v$ of $K$ by setting

$$-\log\|s(z)\|_{t,v} = \lambda_{t,v}(z) = G_{F_t,v}(z,1),$$

for the function $G_{F_t,v}$ of (2.3). The height $\hat{h}_t$ satisfies

$$\hat{h}_t(x) = \sum_{v\in M_K} \frac{r_v}{|\operatorname{Gal}(\overline{K}/K)\cdot x|} \sum_{y\in \operatorname{Gal}(\overline{K}/K)\cdot x} (-\log\|s(y)\|_{t,v})$$

for all $x \neq \infty$ in $\mathbb{P}^1(\overline{\mathbb{Q}})$. Writing $\lambda_{t,v}(z) = \log|z|_v + c_v + o(1)$ as $|z|_v \to \infty$ with a constant $c_v$ at each place $v$ of $K$, we may compute that

$$0 = \hat{h}_t(\infty) = \sum_{v\in M_K} r_v\, c_v, \tag{2.6}$$

because $\infty$ is the projection of the origin of $E_t$.

2.6. **The intersection pairing.** For these heights $\hat{h}_t$ coming from the Legendre family of elliptic curves, with $t \in \overline{\mathbb{Q}} \setminus \{0,1\}$, we have

$$\hat{h}_{t_1} = \hat{h}_{t_2} \iff t_1 = t_2. \tag{2.7}$$

Indeed, any height coming from an adelic metric on $\mathcal{O}_{\mathbb{P}^1}(1)$ is uniquely determined, up to an additive constant, by the associated curvature distributions; see, for example, [FRL1]. For heights of the form $\hat{h}_t$, at each archimedean place $v$ of a number field containing $t$, the curvature distribution $\mu_{t,v}$ on $\mathbb{P}^1(\mathbb{C})$ is the push-forward of the Haar measure on $E_t(\mathbb{C})$ by $\pi$; it therefore has a greater density at the four branch points $\{0,1,t,\infty\}$ of $\pi$, and thus determines $t$.

There is a well-defined intersection number between any pair of such heights, as in [Zh2] (see also [CL2]). By the non-degeneracy of this height pairing and (2.7),

$$\hat{h}_{t_1}\cdot\hat{h}_{t_2} \geq 0 \text{ with equality if and only if } t_1 = t_2, \tag{2.8}$$

as computed in [PST].

To define the pairing $\hat{h}_{t_1}\cdot\hat{h}_{t_2}$, we fix sections $s$ and $u$ of $\mathcal{O}_{\mathbb{P}^1}(1)$ such that their divisors do not intersect. Given $t_1$ and $t_2$ in a number field $K$, and a place $v$ of $K$, we set

$$\langle \hat{h}_{t_1}, \hat{h}_{t_2}\rangle_v^{s,u} := \int \log\|s\|_{t_1,v}^{-1}\, \Delta(\log\|u\|_{t_2,v}^{-1}) = \langle \hat{h}_{t_2}, \hat{h}_{t_1}\rangle_v^{u,s}.$$

The integral is over the Berkovich analytification $\mathbb{P}_v^{1,an}$ of $\mathbb{P}^1$, over the field $\mathbb{C}_v$. The metrics satisfy

$$\Delta(\log\|s\|_{t,v}^{-1}) = \mu_{t,v} - \delta_{(s)},$$

and $\mu_{t,v}$ is the associated curvature distribution.

The height pairing is then defined as

$$\hat{h}_{t_1} \cdot \hat{h}_{t_2} := \hat{h}_{t_1}((u)) + \hat{h}_{t_2}((s)) + \sum_{v \in M_K} r_v \, \langle \hat{h}_{t_1}, \hat{h}_{t_2} \rangle_v^{s,u} \tag{2.9}$$

which is independent of the choices of $s$ and $u$. This pairing is easily seen to be symmetric, and since $\hat{h}_t(\infty) = 0$ for all $t$, it can be expressed as

$$\begin{aligned} \hat{h}_{t_1} \cdot \hat{h}_{t_2} &= \hat{h}_{t_2}(\infty) + \sum_{v \in M_K} r_v \int (\log \|s\|_{t_1,v}^{-1}) \, d\mu_{t_2,v} = \sum_{v \in M_K} r_v \int \lambda_{t_1,v} \, d\mu_{t_2,v} \\ &= \hat{h}_{t_1}(\infty) + \sum_{v \in M_K} r_v \int (\log \|s\|_{t_2,v}^{-1}) \, d\mu_{t_1,v} = \sum_{v \in M_K} r_v \int \lambda_{t_2,v} \, d\mu_{t_1,v} \end{aligned} \tag{2.10}$$

when $(s) = (\infty)$.

As $\hat{h}_t \cdot \hat{h}_t = 0$ for all $t \in \overline{\mathbb{Q}} \setminus \{0,1\}$ from (2.8), note that

$$\sum_{v \in M_K} r_v \int \lambda_{t,v} \, d\mu_{t,v} = 0 = \sum_{v \in M_K} r_v \, c_v, \tag{2.11}$$

by combining (2.10) and (2.6). The pairing can be rewritten as:

$$\begin{aligned} \hat{h}_{t_1} \cdot \hat{h}_{t_2} &= \frac{1}{2} \left( \hat{h}_{t_2}(\infty) + \hat{h}_{t_1}(\infty) + \sum_{v \in M_K} r_v \left( \int \lambda_{t_1,v} \, d\mu_{t_2,v} + \int \lambda_{t_2,v} \, d\mu_{t_1,v} \right) \right) \\ &= \frac{1}{2} \sum_{v \in M_K} r_v \left( \int (\lambda_{t_1,v} - \lambda_{t_2,v}) \, d\mu_{t_2,v} + \int (\lambda_{t_2,v} - \lambda_{t_1,v}) \, d\mu_{t_1,v} \right). \end{aligned} \tag{2.12}$$

The advantage of working with (2.12) is the following local version of the non-degeneracy property (2.8):

**Proposition 2.2.** [FRL1, Propositions 2.6 and 4.5] *Let $K$ be a number field and $v \in M_K$ For any $t_1, t_2 \in K \setminus \{0,1\}$, the local energy*

$$E_v(t_1, t_2) := \frac{1}{2} \left( \int (\lambda_{t_1,v} - \lambda_{t_2,v}) \, d\mu_{t_2,v} + \int (\lambda_{t_2,v} - \lambda_{t_1,v}) \, d\mu_{t_1,v} \right)$$

*is non-negative; it is equal to 0 if and only if $\mu_{t_1,v} = \mu_{t_2,v}$.*

**Proposition 2.3.** *Let $v \in M_K$, and fix $t_1, t_2 \in K \setminus \{0,1\}$. We have*

$$E_v(t_2, t_1) = E_v(t_1, t_2) = E_v(1 - t_1, 1 - t_2) = E_v(1/t_1, 1/t_2).$$

*Proof.* Given measures $\mu_{t_1,v}$ and $\mu_{t_2,v}$, the local energy $E_v(t_1, t_2)$ can be expressed as

$$-\frac{1}{2} \int g \, d(\mu_{t_1,v} - \mu_{t_2,v})$$

for any continuous potential $g$ of the signed measure $\mu_{t_1,v} - \mu_{t_2,v}$, because $g = \lambda_{t_1,v} - \lambda_{t_2,v} + c$ for some constant $c$. We have

$$f_{1-t} = \alpha \circ f_t \circ \alpha^{-1}$$

for $\alpha(z) = 1 - z = \alpha^{-1}(z)$, such that $\mu_{1-t,v} = \alpha_* \mu_{t,v}$ and $g = (\lambda_{t_1,v} - \lambda_{t_2,v}) \circ \alpha^{-1}$ is a potential for the measure $\mu_{1-t_1,v} - \mu_{1-t_2,v}$. Therefore, $E_v(1-t_1, 1-t_2) = E_v(t_1, t_2)$. Similarly, we have $f_{1/t}(z) = \alpha \circ f_t \circ \alpha^{-1}(z)$ for $\alpha(z) = z/t$, so $E_v(1/t_1, 1/t_2) = E_v(t_1, t_2)$. □

2.7. **Measures and mutual energy.** Suppose that $\nu_1$ and $\nu_2$ are signed measures on $\mathbb{P}^1(\mathbb{C})$ with trace measures $|\nu_i|$ for which the function $\log|z-w| \in L^1(|\nu_1| \otimes |\nu_2|)$ on $\mathbb{C}^2 \setminus \mathrm{Diag}$. The mutual energy of $\nu_1$ and $\nu_2$ is defined in [FRL1] by

$$(\nu_1, \nu_2) := -\int_{\mathbb{C}^2 \setminus \mathrm{Diag}} \log|z-w| \, d\nu_1 \otimes d\nu_2. \tag{2.13}$$

This definition extends to the non-archimedean setting by replacing $|z-w|$ with the Hsia kernel $\delta_v(z,w)$ based at the point at $\infty$. In this way, for $v \in M_K^0$, a pairing is defined similarly as

$$(\nu_1, \nu_2)_v := -\int_{\mathbb{A}_v^{1,an} \times \mathbb{A}_v^{1,an} \setminus \mathrm{Diag}} \log \delta_v(z,w) \, d\nu_1 \otimes d\nu_2. \tag{2.14}$$

See [FRL1, §4.4] and [BR, Chapter 4].

For measures $\nu_i$ of total mass 0 with continuous potentials on $\mathbb{P}_v^{1,an}$ we have

$$(\nu_1, \nu_2)_v = -\int g_1 \, d\nu_2$$

for any choice of continuous potential $g_1$ for $\nu_1$. Further, $(\nu_1, \nu_2)_v \geq 0$ with equality if and only if $\nu_1 = \nu_2$ [FRL1, Propositions 2.6 and 4.5]. Note that Proposition 2.2 is a special case of this fact. Indeed, in this notation, the local energy $E_v(t_1, t_2)$ defined in Proposition 2.2 is given by

$$E_v(t_1, t_2) = \frac{1}{2} (\mu_{t_1,v} - \mu_{t_2,v}, \ \mu_{t_1,v} - \mu_{t_2,v})_v \tag{2.15}$$

at each place $v$ of a number field containing $t_1$ and $t_2$, for the canonical measures introduced in §2.4.

The mutual energy $(\cdot,\cdot)_v$ of (2.13) and (2.14) can also be defined for discrete measures. If $F = \{x_1, \ldots, x_n\}$ is any finite set in a number field $K$, and $v \in M_K$, then denote by $[F]_v$ the probability measure supported equally on the elements of $F \subset \mathbb{C}_v$. Then

$$\sum_v r_v \, ([F]_v, [F]_v)_v = \sum_v r_v \, \frac{1}{|F|^2} \sum_{i \neq j} \log|x_i - x_j|_v = 0 \tag{2.16}$$

by the product formula.

2.8. **A metric on the space of adelic heights.** The height pairing gives rise to a metric on the space of continuous, semipositive, adelic metrics on $\mathcal{O}_{\mathbb{P}^1}(1)$ [Fi, Theorem 1]. Given a number field $K$ and any collection of probability measures $\{\mu_v\}_{v\in M_K}$ on $\mathbb{P}^{1,an}_v$ with continuous potentials for which $\mu_v = \omega_v$ at all but finitely many places (where $\omega_v$ is a point mass supported on the Gauss point), then there is a unique metric on $\mathcal{O}_{\mathbb{P}^1}(1)$ with curvature distributions given by $\{\mu_v\}_{v\in M_K}$, normalized such that its associated height function $h : \mathbb{P}^1(\overline{\mathbb{Q}}) \to \mathbb{R}$ satisfies $h \cdot h = 0$ [FRL1]. The height pairing between any two such heights is computed as

$$h_1 \cdot h_2 = \frac{1}{2} \sum_{v\in M_K} r_v \, (\mu_{1,v} - \mu_{2,v},\ \mu_{1,v} - \mu_{2,v})_v.$$

Fili observed that a distance between $h_1$ and $h_2$ can be defined by

$$\operatorname{dist}(h_1, h_2) := (h_1 \cdot h_2)^{1/2}.$$

Indeed, we have already seen that $h_1 \cdot h_2 = 0$ if and only if $h_1 = h_2$ because of the non-degeneracy of the mutual energy $(\cdot,\cdot)_v$ at each place. Furthermore, $\operatorname{dist}(\cdot,\cdot)$ satisfies a triangle inequality: at each place, the mutual energy induces a non-degenerate, symmetric, bilinear form on the vector space of measures of mass 0 with continuous potentials on $\mathbb{P}^{1,an}_v$, and so the triangle inequality for $\operatorname{dist}(\cdot,\cdot)$ follows from an $\ell^2$ triangle inequality.

## 3. Non-archimedean energy

Throughout this section, we fix a number field $K$ and a non-archimedean place $v \in M_K$, and provide a lower bound on the non-archimedean local energy defined in Proposition 2.2:

**Theorem 3.1.** *For $t_1, t_2 \in K \setminus \{0,1\}$, we have*

$$E_v(t_1,t_2) - \frac{4}{3}\log|2|_v \geq \begin{cases} \frac{\log^2|t_1/t_2|_v}{6\log\max\{|t_2|_v,|t_1|_v\}}, & \min\{|t_2|_v,|t_1|_v\} > 1 \\ \frac{\log^2|t_1/t_2|_v}{-6\log\min\{|t_2|_v,|t_1|_v\}}, & \max\{|t_2|_v,|t_1|_v\} < 1 \\ \frac{|\log|t_1/t_2|_v|}{6}, & \text{otherwise.} \end{cases}$$

*Equality holds for $v \nmid 2$ with $\min\{|t_1-1|_v, |t_2-1|_v\} \geq 1$.*

3.1. **Measure and escape rate for $v \nmid 2$.**

**Proposition 3.2.** *Suppose $t \in K \setminus \{0,1\}$ and $v \nmid 2$. Then $f_t$ has good reduction at $v$ if and only if $|t(t-1)|_v = 1$. If $|t(t-1)|_v \neq 1$, then $f_t$ fails to have potential good reduction at $v$, and the canonical measure $\mu_{t,v}$ on $\mathbb{P}^{1,an}_v$ of $f_t$ is the interval measure supported on*

$$I = \begin{cases} [\zeta_{0,1}, \zeta_{0,|t|_v}] & \text{for } |t|_v > 1 \text{ or } |t|_v < 1, \\ [\zeta_{0,1}, \zeta_{1,|1-t|_v}] & \text{for } |1-t|_v < 1. \end{cases}$$

*Proof.* By Proposition 2.1, it suffices to treat the cases with $|t|_v > 1$. By [FRL2, §5.1], $f_t^{-1}(I) = I$ and the action of $f_t$ on $I$ is by a tent map of degree 2. That is,

$$f_t(\zeta_{0,|t|_v^r}) = \begin{cases} \zeta_{0,|t|_v^{2r-1}} & \text{if } 1/2 \le r \le 1, \\ \zeta_{0,|t|_v^{1-2r}} & \text{if } 0 \le r \le 1/2. \end{cases}$$

The proposition follows. □

We may now compute the local height $\lambda_{t,v}(z) = G_{F_t,v}(z,1)$ on $\mathbb{A}_v^{1,an}$, which is locally constant away from the interval $[0,\infty] \subset \mathbb{A}_v^{1,an}$.

**Proposition 3.3.** *Suppose $v \nmid 2$ is non-archimedean and $|t(t-1)|_v \ge 1$. The escape-rate function $G_{F_t,v}$ satisfies*

$$G_{F_t,v}(z,1) = \begin{cases} \log|z|_v & \text{for } |z|_v \ge |t|_v \\ \frac{1}{2}\left(\frac{\log^2|z|_v}{\log|t|_v} + \log|t|_v\right) & \text{for } 1 < |z|_v < |t|_v \\ \frac{1}{2}\log|t|_v & \text{for } |z|_v \le 1 \end{cases} \tag{3.1}$$

*for all $z \in \mathbb{C}_v$.*

*Proof.* Let $\lambda$ be the continuous extension of the expression on the right hand side of the formula (3.1) to $\mathbb{A}_v^{1,an}$. By Proposition 3.2, $\mu_{t,v}$ is the interval measure corresponding to $[\zeta_{0,1}, \zeta_{0,|t|_v}]$, and a direct computation shows that

$$\Delta\lambda = \mu_{t,v} - \delta_\infty.$$

Thus it suffices to show that $G_{F_t,v}(\cdot,1)$ and $\lambda$ agree at a single point. For any $z_0 \in \mathbb{C}_v$ with $|z_0|_v > |t|_v$, define $(z_n, w_n) := F_t^n(z_0,1)$, so that

$$(z_{n+1}, w_{n+1}) = F_t(z_n, w_n) = ((z_n^2 - tw_n^2)^2, 4z_nw_n(z_n - w_n)(z_n - tw_n)). \tag{3.2}$$

Inductively,

$$|z_n|_v = |z_0|_v^{4^n} > |t|_v|w_n|_v > |w_n|_v.$$

Consequently,

$$G_{F_t,v}(z_0,1) = \lim_{n\to\infty} \frac{1}{4^n}\log\|F_t^n(z_0,1)\|_v = \log|z_0|_v = \lambda(z_0).$$

□

A similar application of Proposition 3.2 yields

**Proposition 3.4.** *Suppose $v \nmid 2$ is non-archimedean and $|t|_v < 1$. The escape-rate function $G_{F_t,v}$ satisfies*

$$G_{F_t,v}(z,1) = \begin{cases} \log|z|_v & \text{for } |z|_v \ge 1 \\ -\frac{\log^2|z|_v}{2\log|t|_v} + \log|z|_v & \text{for } |t|_v < |z|_v < 1 \\ \frac{1}{2}\log|t|_v & \text{for } |z|_v \le |t|_v \end{cases} \tag{3.3}$$

*for all $z \in \mathbb{C}_v$.*

3.2. **Proof of Theorem 3.1 for $v \nmid 2$.** We compute the local energy $E_v(t_1, t_2)$ by cases.

**Case (1):** $|t_1|_v > 1$ and $|t_2|_v < 1$. Recall the local energy can be expressed as

$$2\, E_v(t_1, t_2) = \int_{\mathbb{P}^{1,an}_v} (\lambda_{t_1,v} - \lambda_{t_2,v})\; d\mu_{t_2,v} + \int_{\mathbb{P}^{1,an}_v} (\lambda_{t_2,v} - \lambda_{t_1,v})\; d\mu_{t_1,v}.$$

Therefore by Proposition 3.2, 3.3 and 3.4,

$$\begin{aligned} 2E_v(t_1, t_2) &= \int_{\mathbb{P}^{1,an}_v} (\lambda_{t_1,v} - \lambda_{t_2,v})\; d\mu_{t_2,v} + \int_{\mathbb{P}^{1,an}_v} (\lambda_{t_2,v} - \lambda_{t_1,v})\; d\mu_{t_1,v}. \\ &= \int_{\log|t_2|_v}^{0} \left( \frac{\log|t_1|_v}{2} - \left( -\frac{x^2}{2\log|t_2|_v} + x \right) \right) \frac{dx}{-\log|t_2|_v} \\ &\quad + \int_{0}^{\log|t_1|_v} \left( x - \frac{1}{2}\left( \frac{x^2}{\log|t_1|_v} + \log|t_1|_v \right) \right) \frac{dx}{\log|t_1|_v} \\ &= \frac{\log|t_1/t_2|_v}{3}. \end{aligned}$$

**Case (2):** $|t_1|_v > 1$ and $|t_2|_v > 1$. Without loss of generality, we assume that $|t_1|_v = \max\{|t_1|_v, |t_2|_v\}$. By Proposition 3.2 and 3.3,

$$\begin{aligned} 2E_v(t_1, t_2) &= \int_{\mathbb{P}^{1,an}_v} (\lambda_{t_1,v} - \lambda_{t_2,v})\; d\mu_{t_2,v} + \int_{\mathbb{P}^{1,an}_v} (\lambda_{t_2,v} - \lambda_{t_1,v})\; d\mu_{t_1,v}. \\ &= \int_{0}^{\log|t_2|_v} \left( \frac{1}{2}\left( \frac{x^2}{\log|t_1|_v} + \log|t_1|_v \right) - \frac{1}{2}\left( \frac{x^2}{\log|t_2|_v} + \log|t_2|_v \right) \right) \frac{dx}{\log|t_2|_v} \\ &\quad + \int_{0}^{\log|t_2|_v} \left( \frac{1}{2}\left( \frac{x^2}{\log|t_2|_v} + \log|t_2|_v \right) - \frac{1}{2}\left( \frac{x^2}{\log|t_1|_v} + \log|t_1|_v \right) \right) \frac{dx}{\log|t_1|_v} \\ &\quad + \int_{\log|t_2|_v}^{\log|t_1|_v} \left( x - \frac{1}{2}\left( \frac{x^2}{\log|t_1|_v} + \log|t_1|_v \right) \right) \frac{dx}{\log|t_1|_v} \\ &= \frac{\log^2|t_1/t_2|_v}{3\log\max\{|t_1|_v, |t_2|_v\}}. \end{aligned}$$

**Case (3):** $|t_2(t_2-1)|_v = 1$ and $|t_1 - 1|_v \geq 1$. In this case, $f_{t_2}$ has good reduction, so $\mu_{t_2,v}$ is a point mass supported on the Gauss point $\zeta_{0,1}$. Hence

$$2E_v(t_1, t_2) = \frac{|\log|t_1|_v|}{3} = \frac{|\log|t_1/t_2|_v|}{3}.$$

**Case (4):** The remaining cases reduce to the above three by the symmetry relations of Proposition 2.3. This completes the proof of Theorem 3.1 under the assumption that $v \nmid 2$.

3.3. **Measure and escape rate for $v \mid 2$.**

**Proposition 3.5.** *Suppose $v \mid 2$ is non-archimedean. The canonical measure $\mu_{t,v}$ on $\mathbb{P}^{1,an}_v$ of $f_t$ is the interval measure corresponding to the interval $I$ with*

$$I = \begin{cases} [\zeta_{0,|t/4|_v}, \zeta_{0,|4|_v}] & \text{for } |t|_v < |16|_v, \\ [\zeta_{0,|1/4|_v}, \zeta_{0,|4t|_v}] & \text{for } |t|_v > 1/|16|_v, \\ [\zeta_{1,|1-t|_v/|4|_v}, \zeta_{1,|4|_v}] & \text{for } |1-t|_v < |16|_v. \end{cases}$$

*For $|16|_v \le |t|_v \le 1/|16|_v$ with $|1-t|_v \ge |16|_v$, $f_t(z)$ has potential good reduction, and $\mu_{t,v}$ is supported on a single point in $\mathbb{H}$.*

*Proof.* We proceed as in the computations of [FRL2, §5.1], though the authors had assumed for simplicity that the residue characteristic of their field is not 2. If $|t|_v > |1/16|_v$, the interval $[\zeta_{0,|1/4|_v}, \zeta_{0,|4t|_v}]$ is totally invariant by $f_t$, and

$$f_t(\zeta_{0,|4t|_v|16t|_v^{-r}}) = \zeta_{0,|4t|_v|16t|_v^{-2r}} \quad \text{and} \quad f_t(\zeta_{0,|4t|_v|16t|_v^{r-1}}) = \zeta_{0,|4t|_v|16t|_v^{-2r}}$$

for $r \in [0, 1/2]$. Thus $\mu_{t,v}$ is the interval measure on $[\zeta_{0,|1/4|_v}, \zeta_{0,|4t|_v}]$. The cases $|t|_v < |16|_v$ or $|1-t|_v < |16|_v$ can then be deduced from Proposition 2.1.

For all $|16|_v \le |t|_v \le 1/|16|_v$ with $|1-t|_v \ge |16|_v$, we have $|j(t)|_v \le 1$, so $f_t$ has potential good reduction. □

Following the proofs of Propositions 3.3 and 3.4, from Proposition 3.5 we obtain

**Proposition 3.6.**

$$G_{F_t,v}(z,1) = \begin{cases} \log|z|_v & \text{for } |z|_v \ge |4t|_v \\ \frac{1}{2}\left(\frac{\log^2|4z|_v}{\log|16t|_v} + \log|t|_v\right) & \text{for } 1/|4|_v < |z|_v < |4t|_v \\ \frac{1}{2}\log|t|_v & \text{for } |z|_v \le 1/|4|_v \end{cases} \tag{3.4}$$

*for $t$ with $|t|_v \ge 1/|16|_v$, and*

$$G_{F_t,v}(z,1) = \begin{cases} \log|z|_v & \text{for } |z|_v \ge |4|_v \\ \frac{1}{2}\left(\frac{\log^2|4z/t|_v}{\log|16/t|_v} + \log|t|_v\right) & \text{for } |4|_v < |z|_v < |t/4|_v \\ \frac{1}{2}\log|t|_v & \text{for } |z|_v \le |t/4|_v \end{cases} \tag{3.5}$$

*for $t$ with $|t|_v \le |16|_v$.*

3.4. **Proof of Theorem 3.1 for $v \mid 2$.** We compute as in the case where $v \nmid 2$.

**Case (1):** $\{t_1, t_2\}$ with $\min\{|t_1|_v, |t_2|_v\} \ge 1/|16|_v$ and $\max\{|t_1|_v, |t_2|_v\} > 1/|16|_v$. By Proposition 3.6 yield

$$2\,E_v(t_1,t_2) = \frac{\log^2|t_1/t_2|_v}{3\log\max\{|16t_1|_v, |16t_2|_v\}} \ge \frac{\log^2|t_1/t_2|_v}{3\log\max\{|t_1|_v, |t_2|_v\}}.$$

**Case (2):** $|t_1|_v \ge 1/|16|_v$ and $|t_2|_v \le |16|_v$. Again by Proposition 3.6,

$$2\,E_v(t_1,t_2) = \frac{\log|16t_1|_v - \log|t_2/16|_v}{3} - \log|16|_v \ge \frac{\log|t_1/t_2|_v}{3}.$$

**Case (3):** $|t_1|_v > 1/|16|_v$, $|16|_v \geq |t_2|_v \geq 1/|16|_v$ and $|1-t_2|_v \geq |16|_v$. Let $\zeta_{t_2} \in \mathbb{H}$ be in the support of $\mu_{t_2,v}$. For any $z \in \mathbb{C}_v$ with $|z|_v > 1/|4|_v$,

$$|f_{t_2}(z)|_v = \frac{|(z^2-t_2)^2|_v}{|4z(z-1)(z-t_2)|_v} > |z|_v.$$

Hence $\zeta_{0,1/|4|_v} \in [\zeta_{t_2}, \infty)$. Let $z_0 \in \mathbb{C}_v$ with $|z_0|_v > 1/|4|_v$, and let $(z_n, w_n) := F_{t_2}^n(z_0, 1)$. From the recursive formula (3.2), inductively we have $|z_n| = |z_0|_v^{4^n} > |w_n|_v/|4|_v$. Consequently

$$\lambda_{t_2,v}(z) = G_{F_{t_2},v}(z,1) = \lim_{n\to\infty} \frac{\log \|F_{t_2}^n\|_v}{4^n} = \log|z|_v$$

for $z$ with $|z|_v > 1/|4|_v$, and then $\lambda_{t_2,v}(\zeta_{0,r}) = \log r$ for $r \geq 1/|4|_v$. Moreover, as $\Delta\lambda_{t_2,v} = \delta_{\zeta_{t_2}} - \delta_\infty$, the function $\lambda_{t_2,v}$ is increasing at a constant rate along the ray $[\zeta_{t_2}, \infty)$, with respect to the hyperbolic metric. Therefore $\lambda_{t_2,v}(\zeta_{t_2}) \leq \lambda_{t_2,v}(\zeta_{0,1/|4|_v}) = -\log|4|_v$. Hence by Propositions 3.5 and 3.6,

$$\begin{aligned} 2\,E_v(t_1,t_2) &= \int_{\mathbb{P}_v^{1,an}} (\lambda_{t_1,v} - \lambda_{t_2,v})\, d\mu_{t_2,v} + \int_{\mathbb{P}_v^{1,an}} (\lambda_{t_2,v} - \lambda_{t_1,v})\, d\mu_{t_1,v}. \\ &= (\lambda_{t_1,v}(\zeta_{t_2}) - \lambda_{t_2,v}(\zeta_{t_2})) \\ &\quad + \int_{\log|1/4|_v}^{\log|4t_1|_v} \left( x - \frac{1}{2}\left( \frac{(x+\log|4|_v)^2}{\log|16t_1|_v} + \log|t_1|_v \right) \right) \frac{dx}{\log|16t_1|_v} \\ &\geq \frac{\log|16t_1|_v}{3}. \end{aligned}$$

Here we have used $\lambda_{t_2,v}(\zeta_{t_2}) \leq -\log|4|_v$ and $\lambda_{t_1,v}(\zeta_{t_2}) = \frac{1}{2}\log|t_1|_v$ for the last inequality.

Of course, for $|16|_v \leq |t_i|_v \leq 1/|16|_v$ and $|1-t_i|_v \geq |16|_v$ for $i = 1, 2$, we have

$$2\,E_v(t_1,t_2) \;\geq\; 0 \;\geq\; \frac{|\log|t_1/t_2|_v|}{3} + \frac{8}{3}\log|2|_v.$$

**Case (4):** The remaining cases reduce to the above three by the symmetry relations of Proposition 2.3. This completes the proof of Theorem 3.1.

## 4. Archimedean places and the hybrid space

In this section, we provide some of the estimates we need to control the archimedean contributions to the height pairings. Throughout this section, we assume our parameter $t \in \mathbb{C} \setminus \{0,1\}$ is complex. We let $\mu_t$ denote the probability measure on $\mathbb{P}^1(\mathbb{C})$ which is the push forward of the Haar measure on the Legendre elliptic curve

$E_t(\mathbb{C})$ via $\pi(x,y)=x$. This measure is also the unique measure of maximal entropy for the dynamical system defined by the Lattès map

$$f_t(z)=\frac{(z^2-t)^2}{4z(z-1)(z-t)},$$

as noted in [Mi, §7]. We study degenerations of the probability measures $\mu_t$ and their potentials as $t\to 0$. (The cases of $t\to 1$ and $t\to\infty$ are similar.) To this end, we consider the action of $f_t$ sending $(t,z)$ to $(t,f_t(z))$ on the complex surface $X=\mathbb{D}^*\times\mathbb{P}^1(\mathbb{C})$, where $\mathbb{D}^*$ is the punctured unit disk. We make use of the hybrid space $X^{hyb}$, in which the Berkovich projective line over the field of formal Laurent series $\mathbb{C}((t))$ creates a central fiber of $X$ over $t=0$ in the unit disk $\mathbb{D}$. We appeal to the topological description of the hybrid space from [BJ] and the associated dynamical degenerations described in [Fa].

4.1. **The family of Lattès maps and their escape rates.** In homogeneous coordinates on $\mathbb{C}^2$, recall that the maps $f_t$ may be presented as

$$F_t(z,w):=\left((z^2-tw^2)^2,4zw(z-w)(z-tw)\right).$$

They have escape rate functions

$$G_{F_t}(z,w):=\lim_{t\to\infty}\frac{1}{4^n}\log\|F_t^n(z,w)\|, \tag{4.1}$$

as in (2.3).

View the families $f_t$ and $F_t$ as maps $f$ and $F$ defined over the field $k=\mathbb{C}(t)$, and consider the non-archimedean absolute value $|\cdot|_0$ on $k$ satisfying $|g(t)|_0=e^{-\operatorname{ord}_0 g}$. Let $k_0=\mathbb{C}((t))$ denote the completion of $\mathbb{C}(t)$ with respect to this absolute value. Let $\mathbb{L}$ denote a (minimal) complete and algebraically closed field containing $k_0$. The non-archimedean escape rate $\hat{G}_F$ on $\mathbb{L}^2$ is defined as in (2.3). Since $|t|_0<1$, it is given for $x\in\mathbb{L}$ by the following formula, exactly as in Proposition 3.4:

$$\begin{aligned}\hat{g}_f(x):=\hat{G}_F(x,1) &= \begin{cases}\log|x|_0 & \text{for } |x|_0\geq 1\\ \log|x|_0-\frac{(\log|x|_0)^2}{2\log|t|_0} & \text{for } |t|_0<|x|_0<1\\ \frac{1}{2}\log|t|_0 & \text{for } |x|_0\leq|t|_0\end{cases}\\ &= \begin{cases}-a & \text{for } |x|_0=|t|^a \text{ with } a\leq 0\\ -a+\frac{1}{2}a^2 & \text{for } |x|_0=|t|^a \text{ with } 0\leq a\leq 1\\ -\frac{1}{2} & \text{for } |x|_0=|t|^a \text{ with } a\geq 1\end{cases}\end{aligned} \tag{4.2}$$

The function $\hat{g}_f$ extends naturally to the Berkovich space $\mathbb{P}^{1,an}_{\mathbb{L}}$; away from the point at $\infty$, it is a continuous potential for the equilibrium measure $\hat{\mu}_f$ of $f$.

The potential $\hat{g}_f$ and the measure $\hat{\mu}_f$ are invariant under the action of $\mathrm{Gal}(\mathbb{L}/k_0)$ on $\mathbb{P}^{1,an}_{\mathbb{L}}$. They descend to define a function and probability measure – that we will also denote by $\hat{g}_f$ and $\hat{\mu}_f$ – on the quotient Berkovich line $\mathbb{P}^{1,an}_{k_0}$ (see [Be, §4.2] for details on this quotient map). As computed in Proposition 3.2, the measure $\hat{\mu}_f$ is supported

on the interval $[\zeta_{0,|t|_0}, \zeta_{0,1}]$, and it is uniform with respect the linear structure from the hyperbolic metric.

4.2. **Convergence of measures.** The family $f_t$ acts on the product space $\mathbb{D}^* \times \mathbb{P}^1$ sending $(t,z)$ to $(t, f_t(z))$. It extends meromorphically to $X_0 := \mathbb{D} \times \mathbb{P}^1$, or indeed to any model complex surface $X \to \mathbb{D}$ which is isomorphic to $\mathbb{D}^* \times \mathbb{P}^1$ over $\mathbb{D}^*$ and has a simple normal crossings divisor as its central fiber.

Fixing a surface $X \to \mathbb{D}$ and letting $t \to 0$, the degeneration of the measures $\mu_t$ of maximal entropy for $f_t$ – or indeed for any meromorphic family of rational maps on $\mathbb{P}^1$ – to the central fiber of $X$ is now well understood. [DF1, DF2] compute the limit of the measures $\mu_t$ for any choice of model $X$, and a relation is shown between these limits and the non-archimedean measure $\hat{\mu}_f$. In particular, if we set

$$A_{a,b,C} := \{(t,z) \in \mathbb{D}^* \times \mathbb{C} : C^{-1}|t|^a \leq |z| \leq C|t|^b\}$$

for any $C > 1$ and any real numbers $a$ and $b$ with $a \geq b$, then

$$\mu_t(A_{a,b,C}) \to \hat{\mu}_f([\zeta_{0,|t|_0^a}, \zeta_{0,|t|_0^b}]) = \text{length}_{\mathbb{R}}([0,1] \cap [b,a]). \tag{4.3}$$

This follows from [DF1, Theorem B] (allowing for changes of coordinates on $\mathbb{P}^1$ and base changes, passing to covers of the punctured disk $\mathbb{D}^*$) or from the computations described in [DF2, Theorem D] (taking $\Gamma$ to be a vertex set in the interval $[\zeta_{0,1}, \zeta_{0,|t|_0}]$). Another proof is described below in §4.3. In particular, this convergence implies:

**Lemma 4.1.** *Given any $\varepsilon > 0$ and integer $n \geq 1$, there exists $\delta > 0$ such that*

$$\frac{1}{n} - \varepsilon < \mu_t(\{|t|^{(i+1)/n} \leq |z| \leq |t|^{i/n}\}) < \frac{1}{n} + \varepsilon$$

*for all $0 < |t| < \delta$ and $i = 0, \ldots, n-1$.*

Taking $\varepsilon = 1/n^2$ in Lemma 4.1, we observe that for any given $n$, there is a $\delta > 0$ such that we also have

$$\mu_t\left(\{|z| \geq 1\} \cup \{|z| \leq |t|\}\right) < \frac{1}{n} \tag{4.4}$$

for all $0 < |t| < \delta$.

4.3. **Convergence in the hybrid space.** In [Fa], Favre gives an alternate proof of (4.3) by showing that

$$\mu_t \to \hat{\mu}_f \tag{4.5}$$

weakly in the hybrid space $X^{hyb}$ [Fa, Theorem B]. The hybrid space consists of replacing the central fiber in the models $X$ above with the Berkovich line $\mathbb{P}^{1,an}_{k_0}$, carrying an appropriate topology. The convergence of measures follows from the convergence of their potentials to the potential of the measure $\hat{\mu}_f$ in the Berkovich line. We describe this convergence here, as we will use it for proving our main result.

Let $m_1$ denote the Lebesgue measure on the unit circle in $\mathbb{C}$, normalized to have total length 1. Let $\Phi_t(z)$ denote a continuous potential on $\mathbb{P}^1(\mathbb{C})$ for the measure $\mu_t - m_1$. Explicitly, in local coordinates $z \in \mathbb{C} \subset \mathbb{P}^1$, we can take

$$\Phi_t(z) = G_{F_t}(z,1) - \log^+ |z| \tag{4.6}$$

with $G_{F_t}$ as in (4.1). In [Fa], Favre proves that the function

$$\varphi(t,z) := \frac{\Phi_t(z)}{\log |t|^{-1}} \tag{4.7}$$

extends to define a continuous function on $X^{hyb}$, taking the values of a potential of the limiting measure $\hat{\mu}_f - \omega_0$ on the central fiber. Here $\omega_0$ is the delta mass on the Gauss point $\zeta_{0,1}$ of the Berkovich line $\mathbb{P}^{1,an}_{k_0}$. More precisely, we consider the function

$$\begin{aligned} \hat{\varphi}_f(x) &:= \begin{cases} 0 & \text{for } |x|_0 \geq 1 \\ \log |x|_0 - \frac{(\log |x|_0)^2}{2 \log |t|_0} & \text{for } |t|_0 < |x|_0 < 1 \\ \frac{1}{2} \log |t|_0 & \text{for } |x|_0 \leq |t|_0 \end{cases} \\ &= \begin{cases} 0 & \text{for } |x|_0 = |t|^a \text{ with } a \leq 0 \\ -a + \frac{1}{2}a^2 & \text{for } |x|_0 = |t|^a \text{ with } 0 \leq a \leq 1 \\ -\frac{1}{2} & \text{for } |x|_0 = |t|^a \text{ with } a \geq 1 \end{cases} \end{aligned} \tag{4.8}$$

for $x \in \mathbb{L}$, similar to the formula for $\hat{g}_f$ in (4.2). This function $\hat{\varphi}_f$ extends continuously to all of $\mathbb{P}^{1,an}_{\mathbb{L}}$; it is Galois invariant over $k_0$; and it descends to the quotient $\mathbb{P}^{1,an}_{k_0}$. Favre's theorem implies that the function $\varphi$ of (4.7) extends continuously to $X^{hyb}$, coinciding with $\hat{\varphi}_f$ over $t = 0$:

**Proposition 4.2.** *Given any $\varepsilon > 0$, there exists $\delta > 0$, such that*

$$\left| \varphi(t,z) - \hat{\varphi}_f(\zeta_{0,|t|_0^a}) \right| < \varepsilon$$

*for all $0 < |t| < \delta$, for all $a \in \mathbb{R}$, and all $z$ for which*

$$\left| \frac{\log |z|}{\log |t|} - a \right| < \delta.$$

*Proof.* Recall that the absolute value $|\cdot|_0$ on $\mathbb{L}$ induces a continuous function on the Berkovich space that we will also denote by $|\cdot|_0 : \mathbb{P}^{1,an}_{k_0} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$. We use the standard absolute value $|\cdot|$ on $\mathbb{C}$, extended to a continuous function $\mathbb{P}^1(\mathbb{C}) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$.

The topology on $\mathbb{P}^{1,an}_{k_0}$ is such that annuli of the form

$$A(r_1, r_2) := \{x \in \mathbb{P}^{1,an}_{k_0} : r_1 < |x|_0 < r_2\}$$

are open for any choice of $0 \leq r_1 < r_2 \leq \infty$, as are the Berkovich disks of the form

$$D_0(r) := \{x \in \mathbb{P}^{1,an}_{k_0} : |x|_0 < r\} \text{ and } D_\infty(r) := \{x \in \mathbb{P}^{1,an}_{k_0} : |x|_0 > r\}$$

for any $0 < r < \infty$. The topology on $X^{hyb}$ is such that an annular set of the form

$$\{(t,z) \in \mathbb{D}^* \times \mathbb{P}^1(\mathbb{C}) : |t|^{a+\delta} < |z| < |t|^{a-\delta} \text{ and } 0 < |t| < \delta\} \cup A(|t|_0^{a+\delta}, |t|_0^{a-\delta})$$

is an open neighborhood of $\zeta_{0,|t|_0^a}$ on the central fiber for any $a$ and any $\delta > 0$. Similarly, the disk-like sets

$$\{(t,z) \in \mathbb{D}^* \times \mathbb{P}^1(\mathbb{C}) : |z| < |t|^{a} \text{ and } 0 < |t| < \delta\} \cup D_0(|t|_0^a)$$

and

$$\{(t,z) \in \mathbb{D}^* \times \mathbb{P}^1(\mathbb{C}) : |z| > |t|^{a} \text{ and } 0 < |t| < \delta\} \cup D_\infty(|t|_0^a)$$

are open for any $a \in \mathbb{R}$, and allowing $a$ and $\delta$ to vary provides open neighborhoods at 0 and $\infty$ respectively in the central fiber. See [BJ, §2.2 and Definition 4.9] for details on the hybrid topology. Note in particular that the hybrid topology restricted to the central fiber is the usual Berkovich topology.

By the continuity statement of [Fa, Theorem 2.10] and exhibiting $\varphi$ as a uniform limit of model functions ([Fa, Section 4.3] provides the details in the dynamical case) the function $\varphi$ extends to define a continuous function on $X^{hyb}$, taking the values of $\hat{\varphi}_f$ on the central fiber. Let $L$ denote the closed segment in $\mathbb{P}^{1,an}_{k_0}$ between 0 and $\infty$. We may by compactness cover $L$ by finitely many neighborhoods on which $|\varphi(x) - \varphi(y)| \leq \epsilon$. As the values of $\hat{\varphi}_f$ depend only on the values of $\varphi$ on $L$, each open neighborhood of a point in the interior of $L$ contains an open interval in $L$, and $\hat{\varphi}$ is constant near 0 and $\infty$, we may assume these neighborhoods are annular or disk-like as defined above. Thus we obtain a uniform $\delta$ as claimed. □

As $\Phi_t(z) = G_{F_t}(z,1) - \log^+|z|$, we also have a uniform continuity statement for $G$ when $|z|$ is bounded from above:

**Proposition 4.3.** *Given any $\varepsilon > 0$ and $M > 1$, there exists $\delta > 0$ such that*

$$\left| \frac{G_{F_t}(z,1)}{\log|t|^{-1}} - \hat{g}_f(\zeta_{0,|t|_0^a}) \right| < \varepsilon$$

*for all $0 < |t| < \delta$, for all $a \in \mathbb{R}$, and all $|z| \leq M$ for which*

$$\left| \frac{\log|z|}{\log|t|} - a \right| < \delta.$$

4.4. **Discrete measures and regularizations.** Let $F$ be any finite set in $\mathbb{C}$. Denote by $[F]$ the probability measure supported equally on the elements of $F$, and for $r > 0$, denote by $[F]_r$ the probability measure supported equally and uniformly on circles of radius $r$ about each element of $F$.

**Proposition 4.4.** *For every $\varepsilon > 0$, there exists $c = c(\varepsilon) > 0$ such that*

$$|(\mu_t, [F]) - (\mu_t, [F]_r)| \ < \ \varepsilon \ \max\{\log|t|^{-1}, \log|t-1|^{-1}, \log|t|, 1\}$$

*for all* $t \in \mathbb{C} \setminus \{0,1\}$ *and any finite set* $F$ *in* $\mathbb{C}$ *and any*

$$r \leq c \, \min\{|t|^2, |t-1|^2, |t|^{-2}\}.$$

*Proof.* For any $x \in \mathbb{C}$ and any $r > 0$, let $m_{x,r}$ be the probability measure supported on the circle of radius $r$ around $x$. Recall that

$$(\rho, \sigma) := -\iint_{\mathbb{C}\times\mathbb{C}\setminus\Delta} \log|z-w| \, d\rho(z) \, d\sigma(w).$$

For each fixed $t$, the function $G_{F_t}(\cdot, 1)$ is a potential for $\mu_t$ in $\mathbb{C}$, and therefore, there exists a constant $C_t$ such that

$$\int_{\mathbb{C}} \log|z-w| \, d\mu_t(z) = G_{F_t}(w,1) + C_t.$$

Now let $F$ be any finite set in $\mathbb{C}$. Then, assuming $r < 1$, we have

$$\begin{aligned}(\mu_t, [F]_r) - (\mu_t, [F]) &= \frac{1}{|F|} \sum_{x\in F} \left( G_{F_t}(x,1) - \int G_{F_t}(\zeta,1) \, dm_{x,r}(\zeta) \right) \\ &= \frac{1}{|F|} \sum_{\{|x|>2\}} \left( \Phi_t(x) - \int \Phi_t(\zeta) \, dm_{x,r}(\zeta) \right) \\ &\quad + \frac{1}{|F|} \sum_{\{|x|\leq 2\}} \left( G_{F_t}(x,1) - \int G_{F_t}(\zeta,1) \, dm_{x,r}(\zeta) \right)\end{aligned}$$

because the function $\log^+|z|$ is harmonic away from the unit circle on $\mathbb{C}$.

By Proposition 4.3, there exists $\delta > 0$ such that

$$\left| \frac{G_{F_t}(z,1)}{\log|t|^{-1}} - \hat{g}_f(\zeta_{0,|t|_0^a}) \right| < \varepsilon/2 \tag{4.9}$$

for all $|z| \leq 2$ satisfying

$$\left| \frac{\log|z|}{\log|t|} - a \right| < \delta$$

and all $|t| < \delta$ and any $a \in \mathbb{R}$. Shrinking $\delta$ if needed, we have

$$|\varphi(t,z)| < \varepsilon/2 \tag{4.10}$$

for $|z| \geq 1$ and all $|t| < \delta$, by Proposition 4.2.

Let $C_\delta$ be the compact subset of $\mathbb{C} \setminus \{0,1\}$ consisting of all $t$ with $|t| \geq \delta$ and $|t-1| \geq \delta$ and $|1/t| \geq \delta$. Over $C_\delta \times \mathbb{P}^1$, the family of potentials $\{\Phi_t\}$ is uniformly continuous. So there exists $c_1 = c_1(\delta)$ such that

$$|\Phi_t(z) - \Phi_t(z')| < \varepsilon$$

whenever $\mathrm{dist}(z,z') < c_1$ and for all $t \in C_\delta$. Here, dist represents the chordal distance on $\mathbb{P}^1$. Furthermore, we may take $c_1$ such that we also have

$$|G_{F_t}(z,1) - G_{F_t}(z',1)| < \varepsilon$$

for all $|z - z'| < c_1$ with $|z| \leq 2$, and all $t \in C_\delta$. Thus

$$|(\mu_t, [F]_r) - (\mu_t, [F])| < \varepsilon$$

for any choice of finite set $F$, $t \in C_\delta$, and $r < c_1$.

Now assume that $|t| < \delta$. We will consider three cases. First, suppose $|t|^{1+\delta} \leq |z| \leq 2$. Choose any $c_2 = c_2(\delta)$ such that

$$\left|\log(1 \pm c_2\delta^{1-\delta})\right| < \delta(\log \delta^{-1}). \tag{4.11}$$

Then

$$\begin{aligned}
|\log|z'/z|| &= \left|\log\left|\frac{z'-z}{z} + 1\right|\right| \\
&\leq \max\left\{\log\left(1 + \left|\frac{z'-z}{z}\right|\right), \left|\log\left(1 - \left|\frac{z'-z}{z}\right|\right)\right|\right\} \\
&\leq \max\left|\log\left(1 \pm \frac{c_2|t|^2}{|z|}\right)\right| \\
&\leq \max\left|\log\left(1 \pm c_2\delta^{1-\delta}\right)\right| \\
&< \delta(\log \delta^{-1}) \ \leq \ \delta \log|t|^{-1}
\end{aligned}$$

for all $|t| < \delta$. This is equivalent to

$$\left|\frac{\log|z|}{\log|t|} - \frac{\log|z'|}{\log|t|}\right| < \delta \tag{4.12}$$

for all $|z - z'| < c_2|t|^2$ with $|t|^{1+\delta} \leq |z| \leq 2$. Combined with (4.9) and setting $a = (\log|z|)(\log|t|)$, this implies that

$$|G_{F_t}(z, 1) - G_{F_t}(z', 1)| < \varepsilon \log|t|^{-1}$$

for such pairs $z$ and $z'$.

Second, suppose that $|z| \leq |t|^{1+\delta}$. By shrinking $c_2$ further if necessary, we have $c_2 < (1 - \delta^\delta)/\delta$, and therefore if $|z| \leq |t|^{1+\delta}$ and $|z - z'| < c_2|t|^2$, with $|t| < \delta$, we also have $|z'| \leq |t|$. Applying the convergence (4.9) to the limit value of $-1/2$ for all $|z| \leq |t|^{1+\delta}$, we have

$$|G_{F_t}(z, 1) - G_{F_t}(z', 1)| < \varepsilon \log|t|^{-1}$$

for such pairs $z$ and $z'$.

Third, for $|z| \geq 2$, by the convergence (4.10),

$$|\Phi_t(z) - \Phi_t(z')| < \varepsilon \log|t|^{-1}$$

for all $|z| \geq 2$ and $|z - z'| < c_2|t|^2$ and $|t| < \delta$.

Together these three cases yield

$$|(\mu_t, [F]_r) - (\mu_t, [F])| < \varepsilon \log|t|^{-1}$$

for any choice of finite set $F$ and all $|t| < \delta$, with $r < c_2|t|^2$.

If $|t-1| < \delta$, the arguments above go through by replacing $z$ with $1-z$, as

$$G_{F_{1-t}}(1-z,1) = G_{F_t}(z,1)$$

by Proposition 2.1. It follows that

$$|(\mu_t,[F]_r) - (\mu_t,[F])| < \varepsilon \log|t-1|^{-1}$$

for any choice of finite set $F$, $|t-1| < \delta$, and $r < c_2\,|t-1|^2$, with $\delta$ and $c_2$ as above.

For $t$ near $\infty$, more care is needed, as

$$G_{F_t}(z,1) = G_{F_{1/t}}(z,t) = G_{F_{1/t}}(z/t,1) + \log|t|$$

by Proposition 2.1. Setting $s = 1/t$,

$$\frac{G_{F_t}(z,1)}{\log|t|} = \frac{G_{F_s}(s\,z,1) - \log|s|}{\log|s|^{-1}} = \frac{G_{F_s}(s\,z,1)}{\log|s|^{-1}} + 1$$

and (4.9) implies that

$$\left|\frac{G_{F_t}(z,1)}{\log|t|} - \left(\hat{g}_f(\zeta_{0,|s|_0^{1-a}}) + 1\right)\right| < \varepsilon/2 \tag{4.13}$$

for all $|z| \le 2|t|$ satisfying

$$\left|\frac{\log|z|}{\log|t|} - a\right| < \delta$$

with $|t| > \delta^{-1}$ and any $a \in \mathbb{R}$. As in (4.10), we also have

$$|\varphi(t,z)| < \varepsilon/2$$

for $|z| \ge |t|$ and $|t| > \delta^{-1}$, because $\hat{g}_f(\zeta_{0,|s|_0^{1-a}}) + 1 = a$ for all $a \ge 1$. The choice of $c_2$ in (4.11) is similar. It follows that

$$|(\mu_t,[F]_r) - (\mu_t,[F])| < \varepsilon \log|t|$$

for any choice of finite set $F$ and all $|t| > 1/\delta$, with $r < c_2\,|t|^{-2}$.

Let $c := \min\{c_1, c_2\}$ to complete the proof. □

## 5. Archimedean energy

As in Section 4, assume $t \in \mathbb{C} \setminus \{0,1\}$ is a complex parameter, with $\mu_t$ on $\mathbb{P}^1(\mathbb{C})$ the push-forward of the Haar measure on $E_t(\mathbb{C})$, and $\lambda_t(z) = G_{F_t}(z,1)$ a potential for $\mu_t - \delta_\infty$ on $\mathbb{P}^1(\mathbb{C})$. In this section we provide estimates on the archimedean local energy (introduced in Proposition 2.2)

$$E_\infty(s,t) := \frac{1}{2}\left(\int(\lambda_s - \lambda_t)\,d\mu_t + \int(\lambda_t - \lambda_s)\,d\mu_s\right),$$

for $s,t \in \mathbb{C} \setminus \{0,1\}$ as one or both of the parameters tends to 0, 1, or $\infty$. We treat three cases separately: where only one parameter escapes into a cusp, where both

parameters escape into a cusp, and where the two parameters head to two different cusps. By the symmetry established in Proposition 2.3, we focus on the case where $s$ tends to 0.

Throughout, we work in hybrid space and make use of the convergence of potentials to $\hat{g}_f$ and measures to $\hat{\mu}_f$ as $t \to 0$, as proved in [Fa, Theorem B] and explained in §4.3.

### 5.1. **A single escaping parameter.**

**Theorem 5.1.** *Given $\varepsilon > 0$ and any compact set $C \subset \mathbb{C} \setminus \{0,1\}$, there exists $\delta > 0$ such that*

$$\left(\frac{1}{6} - \varepsilon\right) \log|s|^{-1} \leq E_\infty(s,t) \leq \left(\frac{1}{6} + \varepsilon\right) \log|s|^{-1}$$

*for all $s$ satisfying $0 < |s| < \delta$ and all $t \in C$.*

*Proof.* Suppose that $C \subset \mathbb{C} \setminus \{0,1\}$ is compact, and $t \in C$. Recall that for any $s \in \mathbb{C} \setminus \{0,1\}$, $\Phi_s(z) = G_{F_s}(z,1) - \log^+|z|$ as in (4.6) and $\varphi(s,z) = \Phi_s(z)/(\log|s|^{-1})$. The local energy $E_\infty(s,t)$ satisfies

$$\begin{aligned} \frac{E_\infty(s,t)}{\log|s|^{-1}} &= \frac{1}{2\log|s|^{-1}} \left( \int (\Phi_s - \Phi_t)\, d\mu_t + \int (\Phi_t - \Phi_s)\, d\mu_s \right) \\ &= \frac{1}{2} \left( \int \left( \varphi(s,z) - \frac{\Phi_t}{\log|s|^{-1}} \right) d\mu_t + \int \frac{\Phi_t}{\log|s|^{-1}}\, d\mu_s - \int \varphi(s,z)\, d\mu_s \right). \end{aligned}$$

The $\Phi_t$ functions are uniformly bounded for all $t \in C$ and all $z \in \mathbb{P}^1(\mathbb{C})$, so there is a $\delta$ such that

$$\left| \int \frac{\Phi_t}{\log|s|^{-1}}\, d\mu_s \right| < \varepsilon$$

for all $|s| < \delta$ and all $t \in C$. We can also find a small $r = r(C)$ such that

$$\mu_t(\{|z| \leq r\}) < \varepsilon$$

for all $t \in C$. By Proposition 4.2, (shrinking $\delta$ if needed)

$$|\varphi(s,z)| < \varepsilon$$

for all $|z| > r$ and $|s| < \delta$, and

$$|\varphi(s,z)| < 1$$

for all $z$ and all $|s| < \delta$. Consequently,

$$\begin{aligned} \left| \int \left( \varphi(s,z) - \frac{\Phi_t}{\log|s|^{-1}} \right) d\mu_t \right| &\leq \int_{\{|z|\leq r\}} |\varphi(s,z)|\, d\mu_t + \int_{\{|z|>r\}} |\varphi(s,z)|\, d\mu_t + \int \left| \frac{\Phi_t}{\log|s|^{-1}} \right| d\mu_t \\ &< 3\varepsilon. \end{aligned}$$

Finally, by the weak convergence of $\mu_s \to \hat{\mu}_f$ and convergence of $\varphi(s,z)$ to $\hat{\varphi}_f$, we can shrink $\delta$ again such that

$$\left| \int \varphi(s,z)\, d\mu_s - \int \hat{\varphi}_f\, d\hat{\mu}_f \right| < \varepsilon$$

for all $|s| < \delta$. Since

$$\int \hat{\varphi}_f\, d\hat{\mu}_f = \int_0^1 \left(-a + \frac{a^2}{2}\right) da = -\frac{1}{3},$$

we have

$$\left(\frac{1}{6} - 4\varepsilon\right) \log|s|^{-1} \le E_\infty(s,t) \le \left(\frac{1}{6} + 4\varepsilon\right) \log|s|^{-1}$$

for all $|s| < \delta$ and all $t \in C$. □

5.2. **Both parameters escaping to the same cusp.**

**Theorem 5.2.** *Given $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\left(\frac{1}{6}\left(1 - \frac{1}{b}\right)^2 - \varepsilon\right) \log|s|^{-1} \le E_\infty(s,t) \le \left(\frac{1}{6}\left(1 - \frac{1}{b}\right)^2 + \varepsilon\right) \log|s|^{-1}$$

*for all $s,t$ satisfying $0 < |s| = |t|^b \le |t| < \delta$ and for all $b \ge 1$.*

*Proof.* For each real number $b \ge 1$, consider the function

$$\begin{aligned} \hat{\varphi}_b(x) &:= \begin{cases} 0 & \text{for } |x|_0 \ge 1 \\ \log|x|_0 - \frac{(\log|x|_0)^2}{2b \log|t|_0} & \text{for } |t|_0^b \le |x|_0 \le 1 \\ \frac{b}{2}\log|t|_0 & \text{for } |x|_0 \le |t|_0^b \end{cases} \\ &= \begin{cases} 0 & \text{for } |x|_0 = |t|^a \text{ with } a \le 0 \\ -a + a^2/(2b) & \text{for } |x|_0 = |t|^a \text{ with } 0 \le a \le b \\ -b/2 & \text{for } |x|_0 = |t|^a \text{ with } a \ge b \end{cases} \end{aligned}$$

for all $x \in \mathbb{L}$. Note that $\hat{\varphi}_1 = \hat{\varphi}_f$ from (4.8). As with $\hat{\varphi}_1$, each $\hat{\varphi}_b$ extends naturally to a function on the Berkovich projective line $\mathbb{P}^{1,an}_{k_0}$ and is a potential of the measure $\hat{\mu}_b - \delta_G$, where $\hat{\mu}_b$ is interval measure on $[\zeta_{0,|t|_0^b}, \zeta_{0,1}]$ and $\delta_G$ is the delta-mass at the Gauss point $\zeta_{0,1}$.

For each $b \ge 1$, the non-archimedean local energy $E_0(\hat{\mu}_1, \hat{\mu}_b)$ is given by

$$\begin{aligned} E_0(\hat{\mu}_1, \hat{\mu}_b) &:= \frac{1}{2}\left(\int (\hat{\varphi}_1 - \hat{\varphi}_b)\, d\hat{\mu}_b + \int (\hat{\varphi}_b - \hat{\varphi}_1)\, d\hat{\mu}_1\right) \\ &= \frac{\left(\log(|t|_0/|t|_0^b)\right)^2}{-6 \log \min\{|t|_0, |t|_0^b\}} \\ &= \frac{(b-1)^2}{6b}, \end{aligned}$$

as computed in Theorem 3.1 (in the case $v \nmid 2$).

Fix $s$ and $t$ in $\mathbb{C}\setminus\{0,1\}$. When both $s$ and $t$ are close to one of the three cusps, we can estimate the archimedean local energy $E_\infty(s,t)$ in terms of the non-archimedean pairing using the degeneration description in hybrid space. We first prove a special case of Theorem 5.2:

**Proposition 5.3.** *Given $\varepsilon>0$ and $B>2$, there exists $\delta>0$ such that*

$$(E_0(\hat\mu_1,\hat\mu_b)-\varepsilon)\log|t|^{-1}\le E_\infty(s,t)\le(E_0(\hat\mu_1,\hat\mu_b)+\varepsilon)\log|t|^{-1}$$

*for all $s,t$ satisfying $0<|s|=|t|^b\le|t|<\delta$ with $1\le b\le B$.*

This proposition is an immediate consequence of the weak convergence of measures $\mu_t\to\hat\mu_1$ in the hybrid space, and the convergence of potentials as described in §4.3. We give the details to clarify how the bound $b\le B$ is used.

Fix $\varepsilon>0$ and $B>2$.

For $s$ and $t$ in the punctured unit disk $\mathbb{D}^*$, consider the functions

$$\varphi(t,z):=\frac{\Phi_{F_t}(z,1)}{\log|t|^{-1}}\quad\text{and}\quad b\varphi(s,z):=\frac{\Phi_{F_s}(z,1)}{\log|t|^{-1}} \tag{5.1}$$

defined on the fiber $\{t\}\times\mathbb{C}$ in the hybrid space. By Proposition 4.2, there exists $\delta_1>0$ such that

$$\left|\varphi(t,z)-\hat\varphi_1(\zeta_{0,|t|_0^a})\right|<\varepsilon/(4B) \tag{5.2}$$

for all $|t|<\delta_1$, $a\ge0$, and all $|t|^{a+\delta}<|z|<|t|^{a-\delta}$. In particular,

$$|\varphi(t,z)|<\varepsilon/(4B)$$

for all $|z|\ge1$ and all $|t|<\delta_1$.

Now fix any $1\le b\le B$. It follows that $b\varphi(s,z)\to\hat\varphi_b$ in the hybrid space as $s$ and $t$ tend to 0 with $|s|=|t|^b$, and uniformly in $b$ for $b\le B$. This is because the annulus $A_t(a,\delta)$ can be written in terms of $s$ as

$$A_t(a,\delta)=\left\{z\in\mathbb{C}:\quad |s|^{a/b+\delta/b}<|z|<|s|^{a/b-\delta/b}\right\},$$

such that

$$\left|b\varphi(s,z)-\hat\varphi_b(\zeta_{0,|t|_0^a})\right|=\left|\frac{b\,\Phi_{F_s}(z,1)}{\log|s|^{-1}}-b\,\hat\varphi_1(\zeta_{0,|t|_0^{a/b}})\right|<b\,\varepsilon/(4B)<\varepsilon/4 \tag{5.3}$$

for all $z\in A_t(a,\delta_1)$. In particular,

$$|b\,\varphi(s,z)|<\varepsilon/4$$

for all $|z|\ge1$ and all $|t|<\delta_1$. Thus the rate of convergence of $\varphi_t^s$ to the functions $\hat\varphi_b$ on the central fiber in $X^{hyb}$, as $t\to0$, is uniform over all $1\le b\le B$.

Recall that the measures $\mu_t$ on the fiber over $t$ converge weakly in $X^{hyb}$ to the measure $\hat\mu_1$ on the central fiber. For each $s$ with $|s|=|t|^b$, let $\mu_t^s$ denote the measure associated to $f_s$ but viewed in the fiber $\{t\}\times\mathbb{P}^1$. The measures $\mu_t^s$ converge to the

measure $\hat{\mu}_b$ as $t \to 0$ with $|s| = |t|^b$, and this convergence can also be made uniform in $b$ with $b \leq B$. That is, by Lemma 4.1, for any $n$ there exists $\delta_2 > 0$ such that

$$\frac{1}{n} - \frac{1}{n^2} < \mu_t(\{|t|^{(i+1)/n} \leq |z| \leq |t|^{i/n}\}) < \frac{1}{n} + \frac{1}{n^2} \tag{5.4}$$

for all $|t| < \delta_2$ and each $i = 0, \ldots, n$. Note that this implies that

$$\mu_t(\{|z| \leq |t|\} \cup \{|z| \geq 1\}) < \frac{1}{n}.$$

Therefore, we also have

$$\frac{1}{n} - \frac{1}{n^2} < \mu_t^s(\{|t|^{b(i+1)/n} \leq |z| \leq |t|^{bi/n}\}) < \frac{1}{n} + \frac{1}{n^2} \tag{5.5}$$

and

$$\mu_t^s(\{|z| \leq |t|^b\} \cup \{|z| \geq 1\}) < \frac{1}{n}.$$

for all $|t| < \delta_2$. Thus, the measure $\hat{\mu}_b$ on small sub-annuli of the annulus $\{|t|^b \leq |z| \leq 1\}$ is controlled uniformly for all $1 \leq b \leq B$.

Putting all the pieces together,

$$\frac{E_\infty(s,t)}{\log|t|^{-1}} = \frac{1}{2}\left(\int (b\varphi(s,z) - \varphi(t,z))\, d\mu_t + \int (\varphi(t,z) - b\varphi(s,z))\, d\mu_t^s\right)$$

is within $\varepsilon$ of

$$E_0(\hat{\mu}_1, \hat{\mu}_b) = \frac{1}{2}\left(\int (\hat{\varphi}_b - \hat{\varphi}_1)\, d\hat{\mu}_1 + \int (\hat{\varphi}_1 - \hat{\varphi}_b)\, d\hat{\mu}_b\right) = \frac{(b-1)^2}{6b}$$

for all $t$ sufficiently small and all $s$ with $|s| = |t|^b$, for any $1 \leq b \leq B$.

□

Here is a reformulation of Theorem 5.2 in terms of growth of $|t|$:

**Theorem 5.4.** *Given $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\left(\frac{(b-1)^2}{6b} - b\,\varepsilon\right)\log|t|^{-1} \leq E_\infty(s,t) \leq \left(\frac{(b-1)^2}{6b} + b\,\varepsilon\right)\log|t|^{-1}$$

*for all $s, t$ satisfying $0 < |s| \leq |t| < \delta$, where $b = (\log|s|)/(\log|t|) \geq 1$.*

Comparing Theorem 5.4 to the statement of Proposition 5.3, we see that we have only adjusted the $\varepsilon$ term to allow $b$ to be arbitrarily large.

*Proof of Theorem 5.4.* Fix $\varepsilon > 0$.

As in the proof of Proposition 5.3, we make use of the weak convergence of measures $\mu_t \to \hat{\mu}_1$ and convergence of the potentials $\varphi(t,z) \to \hat{\varphi}_1$ in the hybrid space as $t \to 0$.

Choose $r$ satisfying $0 < r < \varepsilon/100$. There is a $\delta_2$ such that

$$\mu_t(\{|z| \leq |t|\}) < \varepsilon/50$$

and

$$|\varphi(t,z)| < \varepsilon/50 \text{ for } |z| \geq |t|^r$$

and

$$|\varphi(t,z)| < \frac{1}{2} + \varepsilon/50 \text{ for all } z$$

for all $|t| < \delta_2$. Thus, for $s \in \mathbb{C}^*$ with $|s| = |t|^b$ and any choice of $b > 1/r$, we have

$$|b\varphi(s,z)| < b\,\varepsilon/50 \text{ for } |z| \geq |t|$$

and

$$|b\varphi(s,z)| < \frac{b}{2} + b\,\varepsilon/50 \text{ for all } z$$

for all $|t| < \delta_2$. By shrinking $\delta_2$ further if necessary, we appeal to the weak convergence of measures $\mu_t \to \hat{\mu}_1$ to deduce that

$$\begin{aligned} \left| \int \varphi(t,z)\, d\mu_t - \int \hat{\varphi}_1\, d\hat{\mu}_1 \right| &= \left| \int \varphi(t,z)\, d\mu_t + \int_0^1 \left( x - \frac{x^2}{2} \right)\, dx \right| \\ &= \left| \int \varphi(t,z)\, d\mu_t + \frac{1}{3} \right| < \varepsilon/10 \end{aligned} \tag{5.6}$$

for all $|t| < \delta_2$.

Now fix $B > 1/r$ such that

$$\left| \frac{(b-1)^2}{6\,b} - \frac{b}{6} \right| < b\,\varepsilon/50 \tag{5.7}$$

for all $b \geq B$. For this $B$, we can find a $\delta_B > 0$ such that Proposition 5.3 is satisfied for all $0 < |s| = |t|^b \leq |t| < \delta_B$ with $1 \leq b \leq B$. Choose any $\delta \leq \min\{\delta_B, \delta_2\}$, and we obtain the theorem for $b \leq B$.

Now suppose $b \geq B$. We will estimate

$$\frac{E_\infty(s,t)}{\log |t|^{-1}} = \frac{1}{2} \left( \int (b\varphi(s,z) - \varphi(t,z))\, d\mu_t + \int (\varphi(t,z) - b\varphi(s,z))\, d\mu_t^s \right)$$

for all $|t| < \delta$ and any $s$ with $|s| = |t|^b$ by estimating the two integrals separately.

As shown above,

$$|b\varphi(s,z) - \varphi(t,z)| \leq 1/2 + \varepsilon/50 + b\,\varepsilon/50 < b\,\varepsilon/10$$

for all $|z| \geq |t|$ and $0 < |s| = |t|^b \leq |t| < \delta$ with $b \geq B$, and

$$|b\varphi(s,z) - \varphi(t,z)| \leq 1/2 + \varepsilon/50 + b/2 + b\,\varepsilon/50$$

for all $z$ and $0 < |s| = |t|^b \leq |t| < \delta$. Writing the first integral as

$$\int (b\varphi(s,z) - \varphi(t,z))\, d\mu_t = \int_{|z|\geq|t|} (b\varphi(s,z) - \varphi(t,z))\, d\mu_t + \int_{|z|\leq|t|} (b\varphi(s,z) - \varphi(t,z))\, d\mu_t,$$

it follows that

$$\left|\int (b\varphi(s,z)-\varphi(t,z))\,d\mu_t\right| \le b\,\varepsilon/10 + (1/2+\varepsilon/50+b/2+b\,\varepsilon/50)(\varepsilon/50) < b\,\varepsilon/5$$

for all $b \ge B$ and $0 < |s| = |t|^b \le |t| < \delta$.

Write the second integral as

$$\int (\varphi(t,z)-b\varphi(s,z))\ d\mu_t^s = \int \varphi(t,z)\,d\mu_t^s - \int b\varphi(s,z)\,d\mu_t^s.$$

As $|\varphi_t|$ is bounded by $1/2+\varepsilon/50$ for all $z$, we have

$$\left|\int \varphi(t,z)\,d\mu_t^s\right| \le \frac{1}{2} + \varepsilon/50 < b\,\varepsilon/25$$

for all $b \ge B$. On the other hand, we have

$$\int b\varphi(s,z)\,d\mu_t^s = b\int \varphi(s,z)\,d\mu_s$$

such that

$$\left|\int b\varphi(s,z)\,d\mu_t^s + \frac{b}{3}\right| < b\,\varepsilon/10$$

for all $0 < |s| = |t|^b \le |t| < \delta$ from (5.6).

We conclude that

$$\left|\frac{E_\infty(s,t)}{\log|t|^{-1}} - \frac{b}{6}\right| < b\,\varepsilon/2$$

for all $b$ sufficiently large and all $|t| < \delta$. On the other hand, we also have

$$\left|\frac{b}{6} - \frac{(b-1)^2}{6b}\right| < b\,\varepsilon/50$$

for all $b \ge B$ by our choice of $B$, so the theorem is proved. □

### 5.3. **Parameters escaping to different cusps.**

**Theorem 5.5.** *Given $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\left(\frac{1}{6}\left(1+\frac{1}{b}\right)-\varepsilon\right)\log|s|^{-1} \le E_\infty(s,t) \le \left(\frac{1}{6}\left(1+\frac{1}{b}\right)+\varepsilon\right)\log|s|^{-1}$$

*for all $s,t$ satisfying $0 < |t| < \delta$ and $|s| = 1/|t|^b$ and for all $b \ge 1$.*

*Proof.* Suppose that $t \to \infty$ and $s \to 0$. We aim to estimate $E_\infty(s,t)$. The proof is nearly identical to that of Theorem 5.2, working in the same hybrid space over a unit disk that we will parameterize by $u \in \mathbb{D}$. For $t = 1/u$ near infinity and $|s| = |u|^b$ near 0, consider the functions

$$g_u(z) = \frac{G_{F_{1/u}}(z,1)}{\log|u|^{-1}} = \frac{G_{F_t}(z,1)}{\log|t|}$$

and

$$bg_s(z) = b\frac{G_{F_s}(z,1)}{\log|s|^{-1}}$$

in the fiber $\{u\} \times \mathbb{P}^1$.

As in Proposition 3.3, the limit of $g_u(z)$ as $u \to 0$ with $|z| = |u|^a$ is

$$\hat{g}_\infty(a) := \begin{cases} -a & \text{for } a \le -1 \\ \frac{1}{2}(a^2+1) & \text{for } -1 \le a \le 0 \\ \frac{1}{2} & \text{for } a \ge 0 \end{cases}$$

The measures $\mu_t$ on $\{u\} \times \mathbb{P}^1$ with $u = 1/t$ will converge the uniform measure $\hat{\mu}_\infty$ on the interval $[\zeta_{0,1}, \zeta_{0,|u|_0^{-1}}]$ as $u \to 0$.

As $s \to -0$ with $|s| = 1/|t|^b$, $b \ge 1$, $bg_s \to \hat{g}_b$ exactly as in the proof of Theorem 5.2. The non-archimedean local energy is computed in Theorem 3.1 as

$$E(\hat{\mu}_\infty, \hat{\mu}_b) = \frac{b+1}{6}.$$

We conclude as in the proof of Theorem 5.2 that, for all given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left(\frac{b+1}{6} - b\,\varepsilon\right)\log|t| \le E_\infty(s,t) \le \left(\frac{b+1}{6} + b\,\varepsilon\right)\log|t|$$

for all $|t| > 1/\delta$ as claimed. □

## 6. Proof of Theorems 1.4 and 1.5

In this section, we first prove Theorem 1.5, which states there exist constants $\alpha, \beta > 0$ such that

$$\hat{h}_{t_1} \cdot \hat{h}_{t_2} \;\ge\; \alpha\, h(t_1,t_2) - \beta$$

for all $t_1 \ne t_2$ in $\overline{\mathbb{Q}} \setminus \{0,1\}$. We then use this lower bound to prove Theorem 1.4 and Proposition 1.7.

### 6.1. Balancing local contributions.

Fix any $r$ such that

$$0 < r \le 1/16.$$

Fix $t_1, t_2 \in \overline{\mathbb{Q}} \setminus \{0,1\}$, and let $K$ be any number field containing $t_1$ and $t_2$. We split the places $M_K$ into "good" and "bad" subsets, depending on the pair $t_1, t_2$ and the choice of $r$. Let $M_{\text{good}}(t_1,t_2)$ be the set the places $v \in M_K$ with

$$|\log|t_2/t_1|_v| \ge r \cdot \max\{|\log|t_2|_v|, |\log|t_1|_v|\};$$

and set $M_{\text{bad}}(t_1,t_2) = M_K \setminus M_{\text{good}}(t_1,t_2)$. We further decompose $M_{\text{good}}(t_1,t_2)$ into its archimedean ($M^\infty_{\text{good}}$) and non-archimedean ($M^0_{\text{good}}$) places.

**Lemma 6.1.** *There exists an integer $N_0$ such that*

$$6\,E_v(t_1,t_2) \geq \frac{3r}{4}|\log|t_1/t_2|_v| - \log N_0$$

*for any choice of $t_1$ and $t_2$ in $\overline{\mathbb{Q}}\setminus\{0,1\}$ and for all archimedean $v \in M^\infty_{\text{good}}(t_1,t_2)$.*

*Proof.* Let

$$\epsilon = \frac{r^2}{24}$$

and let $\delta_1$ be the minimum of the $\delta$'s from Theorems 5.2 and 5.5 for this choice of $\epsilon$. Let $\delta_2$ be the $\delta$ of Theorem 5.1 for the compact set

$$\{t \in \mathbb{C} : \delta_1 \leq |t| \leq 1/\delta_1 \text{ and } |t-1| \geq \delta_1\}$$

in $\mathbb{C}\setminus\{0,1\}$. Let $\delta_0$ be the minimum of $\delta_1$ and $\delta_2$, and let $N_0$ be an integer bigger than $1/\delta_0$.

Now fix $t_1, t_2$ and any number field $K$ containing $t_1$ and $t_2$, and fix a place $v \in M^\infty_{\text{good}}(t_1,t_2) \subset M_K$. If $\delta_0 \leq |t_i|_v \leq 1/\delta_0$ for $i = 1,2$, we have

$$6\,E_v(t_1,t_2) \;\geq\; 0 \;\geq\; \frac{3r}{4}|\log|t_1/t_2|_v| - \log N_0.$$

As $v \in M^\infty_{\text{good}}$, if $|t_2|_v \leq |t_1|_v < 1$, then

$$|t_2|_v = |t_1|_v^b < |t_1|_v \text{ for } \frac{b-1}{b} \geq r$$

and therefore, by Theorem 5.2, if additionally $|t_1|_v < \delta_1$, then we have

$$\begin{aligned} 6\,E_v(t_1,t_2) &\geq \left(\frac{(b-1)^2}{b^2} - 6\epsilon\right)|\log|t_1|_v^b| \\ &\geq \left(\frac{(b-1)r}{b} - \frac{r^2}{4}\right)|\log|t_1|_v^b| \\ &\geq \frac{3r}{4}\frac{b-1}{b}|\log|t_1|_v^b| = \frac{3r}{4}|\log|t_1/t_2|_v|. \end{aligned}$$

If $|t_1|_v < \delta_1$ and $|t_2|_v = 1/|t_1|_v^b$ for some $b \geq 1$, then by Theorem 5.5

$$6\,E_v(t_1,t_2) \geq \left(\frac{(b+1)}{b} - 6\epsilon\right)|\log|t_1|_v^b| \geq \frac{3(b+1)r}{4}|\log|t_1|_v| = \frac{3r}{4}|\log|t_1/t_2|_v|;$$

if $\delta_1 \leq |t_1|_v \leq 1/\delta_1$, $|t_1-1|_v \geq \delta_1$ and $|t_2|_v < \delta_0$, we have by Theorem 5.1 that

$$6\,E_v(t_1,t_2) \geq (1-6\epsilon)|\log|t_2|_v| \geq \frac{3r}{4}|\log|t_1/t_2|_v| - \log N_0.$$

Combining the above inequalities with the symmetry relations of Proposition 2.3, we obtain

$$6\,E_v(t_1,t_2) \geq \frac{3r}{4}|\log|t_1/t_2|_v| - \log N_0.$$

□

**Lemma 6.2.** *There is a constant $C > 0$ such that*

$$\sum_{v \in M_{\mathrm{good}}(t_1,t_2)} 3r_v E_v(t_1,t_2) \geq \frac{3r}{4} h(t_2/t_1) - \frac{3r^2}{2} h(t_1,t_2) - C$$

*for any $t_1 \neq t_2 \in \overline{\mathbb{Q}} \backslash \{0,1\}$.*

*Proof.* Fix $t_1$ and $t_2$ and any number field $K$ containing them. For the non-archimedean places $v \in M^0_{\mathrm{good}}(t_1,t_2)$, by Theorem 3.1, we have

$$6E_v(t_1,t_2) \geq r \cdot |\log |t_2/t_1|_v| - 8 \log^+ |1/2|_v,$$

and thus

$$\sum_{v \in M^0_{\mathrm{good}}(t_1,t_2)} 6r_v E_v(t_1,t_2) \geq \sum_{v \in M^0_{\mathrm{good}}(t_1,t_2)} r_v \left( r \cdot |\log |t_2/t_1|_v| - 8 \log^+ |1/2|_v \right). \tag{6.1}$$

From Lemma 6.1, for each archimedean $v \in M^\infty_{\mathrm{good}}(t_1,t_2)$,

$$6E_v(t_1,t_2) \geq \frac{3r}{4} |\log |t_2/t_1|_v| - \log N_0,$$

where $N_0$ is an integer which is independent of the choice of $v$. With $h$ the naive logarithmic height on $\overline{\mathbb{Q}}$, we set

$$C = 4\, h(2) + \frac{1}{2}\, h(N_0) = \frac{1}{2} \log(2^8 N_0).$$

Then, we have

$$\begin{aligned}
\sum_{v \in M_{\mathrm{good}}(t_1,t_2)} 6r_v E_v(t_1,t_2) &= \sum_{v \in M^\infty_{\mathrm{good}}(t_1,t_2)} 6r_v E_v(t_1,t_2) + \sum_{v \in M^0_{\mathrm{good}}(t_1,t_2)} 6r_v E_v(t_1,t_2) \\
&\geq \sum_{v \in M_{\mathrm{good}}(t_1,t_2)} r_v \frac{3r}{4} |\log |t_2/t_1|_v| - \sum_{v \in M_K} r_v \left( 8 \log^+ |2|_v + \log^+ |N_0|_v \right) \\
&= \sum_{v \in M_{\mathrm{good}}(t_1,t_2)} r_v \frac{3r}{4} |\log |t_2/t_1|_v| - 2C \\
&= \sum_{v \in M_K} r_v \frac{3r}{4} |\log |t_2/t_1|_v| - \sum_{v \in M_{\mathrm{bad}}(t_1,t_2)} r_v \frac{3r}{4} |\log |t_2/t_1|_v| - 2C \\
&\geq \sum_{v \in M_K} \frac{3r_v r}{4} |\log |t_2/t_1|_v| - \sum_{v \in M_{\mathrm{bad}}(t_1,t_2)} \frac{3r^2 r_v}{4} \max\{|\log |t_2|_v|, |\log |t_1|_v|\} - 2C \\
&\geq \frac{3r}{4}\, 2\, h(t_2/t_1) - \frac{3r^2}{4} \cdot 4\, h(t_2,t_1) - 2C
\end{aligned}$$

For the last inequality, we use the facts $2\, h(x) = \sum_{v \in M_K} r_v |\log |x|_v|$, and

$$\sum_{v \in M_K} r_v \max\{|\log |t_2|_v|, |\log |t_1|_v|\} \leq 2\,(h(t_2) + h(t_1)) \leq 4\, h(t_2,t_1).$$

□

6.2. **Proof of Theorem 1.5.** We begin with a standard lemma.

**Lemma 6.3.** *There is a constant $C > 0$, such that*

$$h\left(\frac{t_2}{t_1}, \frac{1-t_2}{1-t_1}\right) \geq \frac{1}{2}\, h(t_1, t_2) - C$$

*for $t_1 \neq t_2 \in \overline{\mathbb{Q}} \setminus \{0, 1\}$. Here the $h$ is the naive logarithmic height on $\mathbb{A}^2(\overline{\mathbb{Q}})$.*

*Proof.* Let $g(t_1, t_2) = (t_2/t_1, (1-t_2)/(1-t_1))$; it is birational on $\mathbb{P}^2$ with inverse

$$g^{-1}(x, y) = \left(\frac{1-y}{x-y}, \frac{x(1-y)}{x-y}\right)$$

of degree $d = 2$. For each rational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, there exists a constant $C$ depending only on $f$ such that

$$h(f(x : y : z)) \leq (\deg f)\, h(x : y : z) + C$$

outside of the indeterminacy set for $f$ [HS, Theorem B.2.5]. Applying this to $f = g^{-1}$ we have a constant $C > 0$ such that

$$h(g^{-1}(x : y : z)) \leq 2\, h(x : y : z) + C$$

for $(x : y : z) \notin \{(0 : 1 : 0), (1 : 0 : 0), (1 : 1 : 1)\}$. Therefore, letting $(t_1 : t_2 : 1) = g^{-1}(x : y : 1)$, we obtain

$$h(t_1, t_2) \leq 2\, h(g(t_1, t_2)) + C$$

for all $t_1 \neq t_2$ in $\overline{\mathbb{Q}} \setminus \{0, 1\}$. In other words,

$$h\left(\frac{t_2}{t_1}, \frac{t_2-1}{t_1-1}\right) \geq \frac{1}{2}\, h(t_1, t_2) - \frac{1}{2}\, C.$$

□

Now fix $t_1 \neq t_2$ in $\overline{\mathbb{Q}} \setminus \{0, 1\}$. From Lemma 6.2, we know there is a constant $C$ (independent of $t_1$ and $t_2$) such that

$$\begin{aligned} 3\, \hat{h}_{t_1} \cdot \hat{h}_{t_2} &= 3 \sum_{v \in M_K} r_v\, E_v(t_1, t_2) \\ &\geq 3 \sum_{v \in M_{\mathrm{good}}(t_1, t_2)} r_v\, E_v(t_1, t_2) \\ &\geq \frac{3r}{4} h(t_2/t_1) - \frac{3r^2}{2} h(t_2, t_1) - C \end{aligned} \tag{6.2}$$

for any $t_1 \neq t_2 \in \overline{\mathbb{Q}} \setminus \{0, 1\}$. Substituting $t_i$ in the (6.2) by $1 - t_i$ for $i = 1, 2$, we also have

$$3\, \hat{h}_{1-t_1} \cdot \hat{h}_{1-t_2} \geq \frac{3r}{4} h\left(\frac{1-t_2}{1-t_1}\right) - \frac{3r^2}{2} h(1-t_2, 1-t_1) - C.$$

Combining this with Proposition 2.3, we find that

$$3\,\hat{h}_{t_1}\cdot\hat{h}_{t_2} \geq \frac{3r}{4}h\left(\frac{1-t_2}{1-t_1}\right) - \frac{3r^2}{2}h(1-t_2,1-t_1) - C. \tag{6.3}$$

Consequently, by adding the inequalities (6.2) and (6.3), we have

$$6\,\hat{h}_{t_1}\cdot\hat{h}_{t_2} \geq \frac{3r}{4}\left(h(t_2/t_1) + h\left(\frac{1-t_2}{1-t_1}\right)\right) - \frac{3r^2}{2}\left(h(t_2,t_1)+h(1-t_2,1-t_1)\right) - 2\,C.$$

Observe that there is a constant $C'>0$ from Lemma 6.3 such that

$$h(t_2/t_1) + h\left(\frac{1-t_2}{1-t_1}\right) \geq h\left(\frac{t_2}{t_1},\frac{1-t_2}{1-t_1}\right) \geq \frac{1}{2}\cdot h(t_1,t_2) - C'.$$

Since $|h(1-t_2,1-t_1) - h(t_1,t_2)|$ is uniformly bounded over all pairs $t_1,t_2\in\overline{\mathbb{Q}}$, we may combine the above inequality with the previous to conclude that

$$6\,\hat{h}_{t_1}\cdot\hat{h}_{t_2} \geq \frac{3r}{8}h(t_1,t_2) - 3r^2\cdot h(t_1,t_2) - 6\,C'',$$

In other words,

$$\hat{h}_{t_1}\cdot\hat{h}_{t_2} \geq (r/16 - r^2/2)h(t_2,t_1) - C'',$$

and the proof of Theorem 1.5 is complete by taking $\alpha = r/16 - r^2/2$ and $\beta = C''$. □

**Remark.** If we set $r = 1/16$, the constant $\alpha>0$ in Theorem 1.5 can be taken as $\alpha = 1/512$.

6.3. **Proof of Theorem 1.4.** From Theorem 1.5, we deduce a uniform lower bound on the height pairing $\hat{h}_s\cdot\hat{h}_t$ for all $s\neq t$ in $\overline{\mathbb{Q}}\setminus\{0,1\}$.

Suppose there exist parameters $s_n\neq t_n\in\overline{\mathbb{Q}}$ such that

$$\hat{h}_{s_n}\cdot\hat{h}_{t_n}\to 0 \qquad \text{as } n\to\infty.$$

Fix $\epsilon>0$. For each $n$, choose a number field $K_n$ containing $s_n$ and $t_n$. By assumption and non-negativity of the local energies $E_v$ (Proposition 2.2), there is $N\in\mathbb{N}$ such that for all $n>N$, the archimedean contribution to the pairing is less than $\epsilon$; that is, for $n>N$,

$$\sum_{v\in M_{K_n}^\infty} r_v\,E_v(s_n,t_n) < \epsilon,$$

recalling that $r_v = \frac{[K_{nv}:\mathbb{Q}_v]}{[K_n:\mathbb{Q}]}$ now depends on $n$. Let $M_n$ be the set of archimedean places $v$ in $M_{K_n}^\infty$ such that $E_v(s_n,t_n)<2\epsilon$, noting that for $n>N$, we have

$$\sum_{v\in M_n} r_v \geq \frac{1}{2}.$$

Recall that the local energy $E_v(s,t)$ is continuous in $s$ and $t$, and it vanishes if and only if $s=t$. So there exists a $\delta$, depending only on $\epsilon$, such that one of the following holds

(1) $|t_n - s_n|_v < \delta$
(2) $\min\{|s_n|_v, |t_n|_v\} < \delta$
(3) $\min\{|s_n - 1|_v, |t_n - 1|_v\} < \delta$
(4) $\max\{|s_n|_v, |t_n|_v\} > 1/\delta$

for every $v \in M_n$ and for all $n > N$. Note that we can take $\delta \to 0$ as $\epsilon \to 0$. We may then choose a subset $M_n'$ of $M_n$ for which all places $v \in M_n'$ satisfy one of the four conditions, and such that

$$\sum_{v \in M_n'} r_v \geq \frac{1}{8},$$

and we conclude by the product formula that

$$\max\{h(s_n - t_n), h(s_n, t_n), h(s_n - 1, t_n - 1)\} > \frac{1}{8\delta}.$$

We conclude by the triangle inequality that as $\epsilon \to 0$, $h(s_n, t_n) \to \infty$, and therefore by the inequality of Theorem 1.5, we must have $\hat{h}_{s_n} \cdot \hat{h}_{t_n} \to \infty$ as well, a contradiction.

6.4. **Proof of Proposition 1.7.** Fix a number field $K$, and fix $t_1 \neq t_2$ in $K \setminus \{0, 1\}$. Let $\|\cdot\|_i$ denote the adelic metric on the line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$ associated to the height $\hat{h}_{t_i}$. Let $\overline{L}$ denote the line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$ equipped with the metric $(\|\cdot\|_1 \|\cdot\|_2)^{1/2}$; its associated height function is

$$h_{\overline{L}}(x) = \frac{1}{2}\left(\hat{h}_{t_1}(x) + \hat{h}_{t_2}(x)\right).$$

Zhang's inequality on the essential minimum of a height function implies that

$$\liminf_{n \to \infty} h_{\overline{L}}(x_n) \;\geq\; (h_{\overline{L}} \cdot h_{\overline{L}})/(2 \deg L) = \frac{1}{2}(h_{\overline{L}} \cdot h_{\overline{L}})$$

along any infinite sequence of distinct points $x_n \in \mathbb{P}^1(\overline{\mathbb{Q}})$ [Zh2, Theorem 1.10]. In particular, the set

$$\{x \in \mathbb{P}^1(\overline{\mathbb{Q}}) : \hat{h}_{t_1}(x) + \hat{h}_{t_2}(x) \leq b\}$$

is finite for any choice of $b < h_{\overline{L}} \cdot h_{\overline{L}}$.

By the linearity of the intersection pairing, we see that

$$h_{\overline{L}} \cdot h_{\overline{L}} = \frac{1}{4}\hat{h}_{t_1} \cdot \hat{h}_{t_1} + \frac{1}{2}\hat{h}_{t_1} \cdot \hat{h}_{t_2} + \frac{1}{4}\hat{h}_{t_2} \cdot \hat{h}_{t_2} = \frac{1}{2}\hat{h}_{t_1} \cdot \hat{h}_{t_2}.$$

Therefore, we may choose any $b < \delta/2$ for the $\delta$ of Theorem 1.4, and the proposition is proved.

## 7. Proof of Theorem 1.6

Fix any $b \geq 0$ such that $b < \delta/2$ for the the $\delta$ of Theorem 1.4. Recall from Proposition 1.7 that the set

$$S(b, t_1, t_2) := \{x \in \mathbb{P}^1(\overline{\mathbb{Q}}) : \hat{h}_{t_1}(x) + \hat{h}_{t_2}(x) \leq b\}$$

is finite for every pair $t_1 \neq t_2 \in \overline{\mathbb{Q}} \setminus \{0, 1\}$. In this section, we prove the following generalization of Theorem 1.6.

**Theorem 7.1.** *Let $b \geq 0$ be chosen so that $b < \delta/2$ for the $\delta$ of Theorem 1.4. For all $\varepsilon > 0$, there exists a constant $C(\varepsilon)$ so that*

$$\hat{h}_{t_1} \cdot \hat{h}_{t_2} \leq 4b + \left(\varepsilon + \frac{C(\varepsilon)}{|S(b, t_1, t_2)|}\right)(h(t_1, t_2) + 1),$$

*for all $t_1 \neq t_2$ in $\overline{\mathbb{Q}} \setminus \{0, 1\}$.*

Note that Theorem 1.6 follows from Theorem 7.1 by setting $b = 0$.

7.1. **Adelic measures and heights associated to a finite set.** Fix a number field $K$, and suppose that $F$ is a finite set in $\overline{K}$ which is $\mathrm{Gal}(\overline{K}/K)$-invariant. Let $\eta$ be a collection of positive real numbers

$$\eta := \{\eta_v\}_{v \in M_K}$$

with $\eta_v = 1$ for all but finitely many $v \in M_K$. For archimedean $v \in M_K$ and $x \in F$, we let $m_{x,v}$ denote the Lebesgue probability measure on the circle of radius $\eta_v$ centered at the point $x \in F$. We then set

$$m_{F,\eta,v} = \frac{1}{|F|} \sum_{x \in F} m_{x,v}.$$

Similarly, for each non-archimedean $v \in M_K$, we let $m_{F,\eta,v}$ denote the probability measure distributed uniformly on the points $\zeta_{x,\eta_v}$ in $\mathbb{A}^{1,an}_v$ over all $x \in F$. Then $m_{F,\eta} = \{m_{F,\eta,v}\}_{v \in M_K}$ is an adelic measure in the sense of [FRL1]. It gives rise to a unique height $h_{F,\eta}$ on $\mathbb{P}^1(\overline{\mathbb{Q}})$ associated to a continuous and semipositive adelic metric on $\mathcal{O}_{\mathbb{P}^1}(1)$ with curvature distributions given by $m_{F,\eta,v}$ and satisfying

$$h_{F,\eta} \cdot h_{F,\eta} = 0.$$

The local heights

$$\lambda_{F,\eta,v}(z) = \alpha_v + \frac{1}{|F|} \sum_{x \in F} \log \max\{|z - x|_v, \eta_v\},$$

for $z \in \mathbb{C}_v$ satisfy

$$\alpha_v = -\frac{1}{2\,|F|} \sum_{x \in F} \int \log \max\{|z - x|_v, \eta_v\}\, dm_{F,\eta,v}.$$

**Remark.** The height $h_{F,\eta}$ will generally not admit sequences of "small" points. In fact, for any choices of $F$ and $\eta$ such that $\sum_v r_v\,\alpha_v \neq 0$, the essential minimum of $h_{F,\eta}$ is positive.

7.2. **An upper bound on the height pairing.** Now suppose that $t_1$ and $t_2$ lie in $K\backslash\{0,1\}$. Recall that $\mu_t$ and $\hat{h}_t$ respectively denote the measure and height associated to the curve $E_t$. By the triangle inequality for the distance function of §2.8, we have

$$\left(\hat{h}_{t_1}\cdot\hat{h}_{t_2}\right)^{1/2} \leq \left(\hat{h}_{t_1}\cdot h_{F,\eta}\right)^{1/2} + \left(\hat{h}_{t_2}\cdot h_{F,\eta}\right)^{1/2} \tag{7.1}$$

for any choice of $F$ and $\eta$. By symmetry and bilinearity of the mutual energy,

$$\begin{aligned}\hat{h}_{t_i}\cdot h_{F,\eta} &= \frac{1}{2}\sum_{v\in M_K} r_v\,(\mu_{t_i,v}-m_{F,\eta,v},\mu_{t_i,v}-m_{F,\eta,v})_v\\ &= \frac{1}{2}\sum_{v\in M_K} r_v\,\left((\mu_{t_i,v},\mu_{t_i,v})_v - 2\,(m_{F,\eta,v},\mu_{t_i,v})_v + (m_{F,\eta,v},m_{F,\eta,v})_v\right)\end{aligned}$$

for $i=1,2$. For fixed $i$, writing the local height for $\hat{h}_{t_i}$ as $\lambda_{t_i,v}=\log|z|_v+c_v+o(1)$ for $|z|_v\to\infty$ yields

$$\sum_v r_v\,(\mu_{t_i,v},\mu_{t_i,v})_v = -\sum_v r_v\int(\lambda_{t_i,v}-c_v)\,d\mu_{t_i,v} = 0$$

from (2.11). Therefore,

$$\hat{h}_{t_i}\cdot h_{F,\eta} = \frac{1}{2}\sum_{v\in M_K} r_v\,\left(-2(\mu_{t_i,v},m_{F,\eta,v})_v + (m_{F,\eta,v},m_{F,\eta,v})_v\right).$$

Recall from §2.7 that $[F]_v$ is the probability measure on $\mathbb{P}^{1,an}_v$ distributed equally on the elements of $F$ for each $v\in M_K$. By [FRL1, Lemma 4.11] and [Fi, Lemma 12], we have

$$(m_{F,\eta,v},m_{F,\eta,v})_v \leq ([F]_v,[F]_v)_v + \frac{-\log\eta_v}{|F|}.$$

It follows that

$$\begin{aligned}\hat{h}_{t_i}\cdot h_{F,\eta} &\leq \frac{1}{2}\sum_{v\in M_K} r_v\cdot\left(-2\,(\mu_{t_i,v},m_{F,\eta,v})_v + ([F]_v,[F]_v)_v + \frac{-\log\eta_v}{|F|}\right)\\ &= \frac{1}{2}\sum_{v\in M_K} r_v\cdot\left(-2\,(\mu_{t_i,v},m_{F,\eta,v})_v + \frac{-\log\eta_v}{|F|}\right) \qquad (7.2)\end{aligned}$$

with the final equality following from (2.16).

**Proposition 7.2.** *Suppose $t\neq 0,1$ lies in a number field $K$, and fix any $b\geq 0$. Assume that $F$ is a finite, $\mathrm{Gal}(\overline{K}/K)$-invariant set of points for which $\hat{h}_t(F)\leq b$.*

*Then*

$$\hat{h}_t \cdot h_{F,\eta} \;\le\; b + \sum_{v\in M_K} r_v \left( -(\mu_{t,v}, m_{F,\eta,v})_v + (\mu_{t,v}, [F]_v)_v + \frac{-\log \eta_v}{2\,|F|} \right).$$

*for any choice of* $\eta = \{\eta_v\}_v$ *with* $\eta_v = 1$ *for all but finitely many* $v \in M_K$.

*Proof.* The height of $F$ can be computed as

$$b \ge \hat{h}_t(F) = \frac{1}{|F|}\sum_{x\in F} \hat{h}_t(x) = \hat{h}_t(\infty) - \sum_{v\in M_K} r_v\,(\mu_{t,v}, [F]_v)_v = -\sum_{v\in M_K} r_v\,(\mu_{t,v}, [F]_v)_v,$$

and therefore we may add $b + \sum_v r_v\,(\mu_{t,v}, [F]_v)_v$ to the right hand side of (7.2). □

7.3. **Proof of Theorem 7.1.** Fix $0 \le b < \delta/2$ so that Proposition 1.7 is satisfied for all $t_1 \ne t_2$ in $\overline{\mathbb{Q}} \setminus \{0,1\}$. Now fix $t_1 \ne t_2$ in $\overline{\mathbb{Q}} \setminus \{0,1\}$ and a number field $K$ containing $t_1$ and $t_2$. Set

$$F = \{x \in \mathbb{P}^1(\overline{\mathbb{Q}}) : \hat{h}_{t_1}(x) + \hat{h}_{t_2}(x) \le b\} \setminus \{\infty\} = S(b, t_1, t_2) \setminus \{\infty\},$$

so $F$ is a finite, $\mathrm{Gal}(\overline{K}/K)$-invariant set. At each non-archimedean place $v$ of $K$, we set

$$\eta_v := \min\{1, |t_1(t_1-1)|_v, |t_2(t_2-1)|_v\}.$$

Now fix $\varepsilon' > 0$. For each archimedean $v$, we set

$$\eta_v := c(\varepsilon') \min_{i=1,2} \min\{|t_i|_v^2, |t_i - 1|_v^2, |t_i|_v^{-2}\}$$

where the constant $c(\varepsilon')$ is from Proposition 4.4. Let $\eta = \{\eta_v\}_v$; observe that $\eta_v = 1$ for all but finitely many $v$, and

$$\sum_{v\in M_K} -r_v \log \eta_v \le 2\,(h(t_1) + h(1-t_1) + h(t_2) + h(1-t_2)) - \frac{1}{2}\log c(\varepsilon'). \tag{7.3}$$

For non-archimedean $v$, the explicit form of the measure $\mu_{t_i,v}$ (described in Section 3) implies that

$$(\mu_{t_i,v}, m_{F,\eta,v})_v = (\mu_{t_i,v}, [F]_v)_v$$

for this choice of $\eta$, because the potentials for $\mu_{t_i,v}$ are constant on disks of radius $\eta_v$.

We thus obtain from Proposition 7.2 that

$$\begin{aligned} \hat{h}_{t_i} \cdot h_{F,\eta} \;\le\; & b + \sum_{v\in M_K^0} r_v \frac{-\log \eta_v}{2|F|} \\ & + \sum_{v\in M_K^\infty} r_v \left( -(\mu_{t_1,v}, m_{F,\eta,v})_v + (\mu_{t_1,v}, [F]_v)_v + \frac{-\log \eta_v}{2|F|} \right) \end{aligned}$$

for $i = 1, 2$, where $M_K^0$ denotes the non-archimedean places and $M_K^\infty$ the archimedean places.

We have for $v \in M_K^\infty$ that

$$-(\mu_{t_i,v}, m_{F,\eta,v})_v + (\mu_{t_i}, [F]_v)_v \le \varepsilon' \log \max\{|t_i|_v, |t_i|_v^{-1}, |t_i - 1|_v^{-1}\}$$

for $i = 1, 2$ by Proposition 4.4.

Since the logarithmic Weil height satisfies $2\,h(x) = \sum_v |\log |x|_v|$, we thus obtain

$$\begin{aligned} \hat{h}_{t_i} \cdot h_{F,\eta} &\le b + 2\varepsilon' \left(h(t_i) + h(t_i - 1)\right) \\ &\quad + \frac{2\,\left(h(t_1) + h(1-t_1) + h(t_2) + h(1-t_2)\right) - \frac{1}{2}\log c(\varepsilon')}{|F|} \end{aligned}$$

for $i = 1, 2$. Since $h(1-t_i) \le h(t_i) + \log 2 \le h(t_1, t_2) + \log 2$ for $i = 1, 2$, this inequality becomes

$$\hat{h}_{t_i} \cdot h_{F,\eta} \le b + 2\varepsilon' \left(2h(t_1, t_2) + \log 2\right) + \frac{1}{|F|}(8h(t_1, t_2) + 4\log 2 - \frac{1}{2}\log c(\varepsilon'))$$

for $i = 1, 2$.

By the triangle inequality (7.1), we have

$$\begin{aligned} \left(\hat{h}_{t_1} \cdot \hat{h}_{t_2}\right)^{1/2} &\le \left(\hat{h}_{t_1} \cdot h_{F,\eta}\right)^{1/2} + \left(\hat{h}_{t_2} \cdot h_{F,\eta}\right)^{1/2} \\ &\le 2\left(b + 2\varepsilon' \left(2h(t_1, t_2) + \log 2\right) + \frac{1}{|F|}(8h(t_1, t_2) + 4\log 2 - \frac{1}{2}\log c(\varepsilon'))\right)^{1/2}, \end{aligned}$$

so

$$\begin{aligned} \hat{h}_{t_1} \cdot \hat{h}_{t_2} &\le 4\left(b + 2\varepsilon' \left(2h(t_1, t_2) + \log 2\right) + \frac{1}{|F|}(8h(t_1, t_2) + 4\log 2 - \frac{1}{2}\log c(\varepsilon'))\right) \\ &= 4b + \left(\frac{32}{|F|} + 16\varepsilon'\right) h(t_1, t_2) + \frac{16\log 2 - 2\log c(\varepsilon')}{|F|} + 8\varepsilon' \log 2. \end{aligned} \tag{7.4}$$

Fix any $\varepsilon > 0$, and choose $\varepsilon' < \varepsilon/16$. Since $|F| = |S(b, t_1, t_2)| - 1 \ge 2$, we can find a large constant $C(\varepsilon)$ satisfying

$$\frac{32}{|F|} + 16\varepsilon' \le \varepsilon + \frac{C(\varepsilon)}{|S(b, t_1, t_2)|}$$

and

$$\frac{16\log 2 - 2\log c(\varepsilon')}{|F|} + 8\varepsilon' \log 2 \le \varepsilon + \frac{C(\varepsilon)}{|S(b, t_1, t_2)|}.$$

The inequality (7.4) then yields

$$\hat{h}_{t_1} \cdot \hat{h}_{t_2} \le 4b + \left(\varepsilon + \frac{C(\varepsilon)}{|S(b, t_1, t_2)|}\right)(h(t_1, t_2) + 1),$$

concluding the proof of Theorem 7.1.

## 8. Proof of Theorem 1.3

In this section, we deduce Theorem 1.3 from Theorems 1.4, 1.5, 1.6 for algebraic values of $t_1$ and $t_2$; we then extend the result to hold for parameters $t_i$ in $\mathbb{C}$, via a specialization argument. In fact, we prove the following stronger result over $\overline{\mathbb{Q}}$:

**Theorem 8.1.** *There exist constants $B$ and $b > 0$ so that*

$$\left|\{x \in \mathbb{P}^1(\overline{\mathbb{Q}}) : \hat{h}_{t_1}(x) + \hat{h}_{t_2}(x) \leq b\}\right| \ \leq \ B$$

*for all $t_1 \neq t_2$ in $\overline{\mathbb{Q}} \setminus \{0, 1\}$.*

8.1. **Proof of Theorem 8.1.** Let $\delta > 0$ be as in Theorem 1.4 so that

$$\hat{h}_{t_1} \cdot \hat{h}_{t_2} \geq \delta$$

for all $t_1 \neq t_2$ in $\overline{\mathbb{Q}} \setminus \{0, 1\}$. Fix

$$0 < b < \delta/8$$

so that, from Proposition 1.7, the set

$$S(b, t_1, t_2) = \{x \in \mathbb{P}^1(\overline{\mathbb{Q}}) : \hat{h}_{t_1}(x) + \hat{h}_{t_2}(x) \leq b\}$$

is finite for all $t_1 \neq t_2$ in $\overline{\mathbb{Q}} \setminus \{0, 1\}$. Let $h(t_1, t_2)$ be the naive logarithmic height on $\mathbb{A}^2(\overline{\mathbb{Q}})$. Fix $H > \frac{2\beta}{\alpha}$ for the $\alpha, \beta$ of Theorem 1.5 and such that

$$\frac{H - 8b/\alpha}{H + 1} > 3/4. \tag{8.1}$$

Suppose that $t_1 \neq t_2 \in \overline{\mathbb{Q}}$ satisfy $h(t_1, t_2) \geq H$. Then for $\varepsilon = \frac{\alpha}{4}$, there exists by Theorem 7.1 a constant $C$ such that

$$\hat{h}_{t_1} \cdot \hat{h}_{t_2} \leq 4b + \left(\frac{\alpha}{4} + \frac{C}{|S(b, t_1, t_2)|}\right)(h(t_1, t_2) + 1).$$

On the other hand, by Theorem 1.5 and the choice of $H$, we have

$$\frac{\alpha}{2}\, h(t_1, t_2) \leq \alpha h(t_1, t_2) - \beta \leq \hat{h}_{t_1} \cdot \hat{h}_{t_2}.$$

Therefore

$$\frac{\alpha}{2}\, h(t_1, t_2) \leq 4b + \left(\frac{\alpha}{4} + \frac{C}{|S(b, t_1, t_2)|}\right)(h(t_1, t_2) + 1),$$

and so

$$|S(b, t_1, t_2)| \leq \frac{C}{\left(\frac{\alpha h/2 - 4b}{h+1}\right) - \frac{\alpha}{4}} = \frac{C}{\frac{\alpha}{2}\left(\frac{h - 8b/\alpha}{h+1}\right) - \frac{\alpha}{4}} \leq \frac{8C}{\alpha}$$

for $h := h(t_1, t_2) \geq H$, from (8.1).

Suppose now that $t_1 \neq t_2 \in \overline{\mathbb{Q}}$ satisfy $h(t_1, t_2) < H$. Set $\varepsilon' = \frac{\delta}{4(H+1)}$, and find a constant $C'$ as in Theorem 7.1 so that

$$\hat{h}_{t_1} \cdot \hat{h}_{t_2} \leq 4b + \left(\varepsilon' + \frac{C'}{|S(b, t_1, t_2)|}\right)(h(t_1, t_2) + 1),$$

and thus, since $b < \delta/8$, we have

$$\delta/2 < \delta - 4b \ \leq \ \left(\varepsilon' + \frac{C'}{|S(b,t_1,t_2)|}\right)(h(t_1,t_2)+1).$$

We conclude that

$$|S(b,t_1,t_2)| \leq \frac{4(H+1)C'}{\delta},$$

providing a uniform bound also for $t_1$ and $t_2$ satisfying $h(t_1,t_2) < H$. This completes the proof of Theorem 8.1.

8.2. **Specialization: proof of Theorem 1.3.** We implement a standard specialization argument to deduce Theorem 1.3 from Theorem 8.1. Note that the division polynomials for the Legendre curve $E_t$ have coefficients in $\mathbb{Q}[t]$. Let $B$ be the uniform bound obtained in Theorem 8.1, so that

$$|\pi(E_{t_1}^{tors}) \cap \pi(E_{t_2}^{tors})| = \left|\{x \in \mathbb{P}^1(\overline{\mathbb{Q}}) : \hat{h}_{t_1}(x) = \hat{h}_{t_2}(x) = 0\}\right| \leq B \tag{8.2}$$

for all $t_1 \neq t_2 \in \overline{\mathbb{Q}} \setminus \{0,1\}$. Assume that there exist $t_1 \neq t_2 \in \mathbb{C} \setminus \{0,1\}$ with

$$N(t_1,t_2) := |\pi(E_{t_1}^{tors}) \cap \pi(E_{t_2}^{tors})| > B$$

and $t_1$ transcendental. If $x \in \pi(E_{t_1}^{tors}) \cap \mathbb{P}^1(\overline{\mathbb{Q}})$, then $x \in \pi(E_t^{tors})$ for all $t \in \mathbb{C} \setminus \{0,1\}$ as it is a root of a division polynomial. It follows that there is at least one non-algebraic point $x \in \pi(E_{t_1}^{tors}) \cap \pi(E_{t_2}^{tors})$, as only $x = 0, 1, \infty$ are torsion images for all $t \in \mathbb{C} \setminus \{0,1\}$ [DWY, Proposition 1.4].

Now let

$$S := \{x_1, x_2, \cdots, x_N\} = \pi(E_{t_1}^{tors}) \cap \pi(E_{t_2}^{tors}),$$

where $N = N(t_1,t_2)$, and assume that $x_1$ is transcendental. Because it is a torsion image for both parameters, $\mathbb{Q}(x_1,t_1,t_2)$ and therefore also the field

$$L := \mathbb{Q}(t_1,t_2,x_1,\cdots,x_N)$$

are of transcendence degree one. Consequently $L$ is isomorphic to a function field $k = K(X)$ for a number field $K$ and an algebraic curve $X$ defined over $\overline{\mathbb{Q}}$. Via the identification of $L$ with $k$, there exists an algebraic point $\gamma \in X(\overline{K})$ with distinct specializations $x_i(\gamma) \in \mathbb{P}^1(\overline{\mathbb{Q}})$ for $i = 1, \ldots, N$ and

$$t_1(\gamma) \neq t_2(\gamma) \in \overline{\mathbb{Q}} \setminus \{0,1\}.$$

The division relations in $L$ imply that the specializations $E_{t_1(\gamma)}$ and $E_{t_2(\gamma)}$ have at least $N$ common torsion images, contradicting (8.2). Therefore, we must have

$$|\pi(E_{t_1}^{tors}) \cap \pi(E_{t_2}^{tors})| \leq B$$

for all $t_1 \neq t_2 \in \mathbb{C} \setminus \{0,1\}$, and the proof of Theorem 1.3 is complete.

8.3. **Common torsion images.** We obtain the following immediate corollary of Theorem 1.3, which is a special case of Conjecture 1.2.

**Corollary 8.2.** *There exists a uniform bound $B$ such that*

$$|\pi_1(E_1^{tors}) \cap \pi_2(E_2^{tors})| \leq B$$

*for any pair of elliptic curves $E_i$ over $\mathbb{C}$ and any pair of standard projections $\pi_i$ for which*

$$|\pi_1(E_1[2]) \cap \pi_2(E_2[2])| = 3.$$

*Proof.* By fixing coordinates on $\mathbb{P}^1$, we may assume that $\pi_1(E_1[2]) \cap \pi_2(E_2[2]) = \{0, 1, \infty\}$. For each $e \in E_i[2]$ the composition $\pi_i^e(P) = \pi_i(P + e)$ is again a standard projection and satisfies $\pi_i^e(E_i^{tors}) = \pi_i^e(E_i^{tors})$. Therefore, we may assume that $\pi_i(O_i) = \infty$ for the origin $O_i$ of $E_i$, $i = 1, 2$. Putting each $E_i$ into Legendre form now shows that the corollary follows from Theorem 1.3. □

## 9. Proof of Theorems 1.1 and 1.8

Throughout this section, we let $\mathcal{L}_2$ denote the hypersurface in the moduli space $\mathcal{M}_2$ consisting of all genus 2 curves $X$ over $\mathbb{C}$ that admit a degree-two map to an elliptic curve; see, e.g., [SV] for details on $\mathcal{L}_2$. The surface $\mathcal{L}_2$ consists of all $X$ whose Jacobians admit real multiplication by the real quadratic order of discriminant 4, as explained in the proof of [Mc, Theorem 4.10].

For any smooth, compact, genus 2 curve over $\mathbb{C}$, and for any Weierstrass point $P$ on $X$,

$$|j_P(X) \cap J(X)^{tors}| \geq 6$$

as the difference of two Weierstrass points is torsion. On the other hand, any curve $X$ of genus $g \geq 2$ has $|j_P(X) \cap J(X)^{tors}| \leq 2$ for all but finitely many $P$, by Baker and Poonen [BP], so an Abel-Jacobi map based at a Weierstrass point has in this sense a large number of torsion images.

In this section we deduce Theorem 1.1 from Corollary 8.2, providing a uniform upper bound on $|j_P(X) \cap J(X)^{tors}|$ for all $X$ in $\mathcal{L}_2$. We also deduce Theorem 1.8 from Theorem 8.1.

9.1. **Genus 2 curve from a pair of elliptic curves.** Suppose that $\pi_1 : E_1 \to \mathbb{P}^1$ and $\pi_2 : E_2 \to \mathbb{P}^1$ are standard projections on elliptic curves $E_i$ such that

$$|\pi_1(E_1[2]) \cap \pi_2(E_2[2])| = 3,$$

as in Corollary 8.2. Consider the diagonal $D \subset \mathbb{P}^1 \times \mathbb{P}^1$, and lift $D$ to a curve $C \subset E_1 \times E_2$ via $\Pi = \pi_1 \times \pi_2$. Let $\nu : X \to C$ normalize $C$, noting that the degree four map $\Pi \circ \nu : X \to D$ has branch locus $\pi_1(E_1[2]) \cup \pi_2(E_2[2])$, with each branch

point the image of two points in $X$, each of multiplicity two. By Riemann-Hurwitz, the genus of $X$ is 2, and by construction, the curve $X$ is in $\mathcal{L}_2$ in $\mathcal{M}_2$.

9.2. **From a genus 2 curve to a pair of elliptic curves.** Here we observe that every $X \in \mathcal{L}_2$ arises from the construction described in §9.1:

**Proposition 9.1.** *Every $X \in \mathcal{L}_2$ is the lift of the diagonal under a product of standard projections $\pi_i$ on elliptic curves $E_i$ for which*

$$|\pi_1(E_1[2]) \cap \pi_2(E_2[2])| = 3.$$

*Moreover, there is a Weierstrass point $Q \in X(\mathbb{C})$ and a degree-four isogeny $\Phi : J(X) \to E_1 \times E_2$ such that*

$$\Phi \circ j_Q(X) = (\pi_1 \times \pi_2)^{-1} D \quad \text{in} \quad E_1 \times E_2$$

*where $D$ is the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$, $J(X)$ is the Jacobian of $X$, and $j_Q$ is the Abel-Jacobi embedding associated to $Q$.*

*Proof.* As noted by [SV], each curve $X \in \mathcal{L}_2$ has an affine model

$$C : y^2 = x^6 - s_1 x^4 + s_2 x^2 - 1,$$

where the polynomial on the right has non-zero discriminant. $C$ admits degree two maps $(x, y) \mapsto (x^2, y)$ and $(x, y) \mapsto (1/x^2, iy/x^3)$ to elliptic curves with affine presentation

$$E_1 : y^2 = x^3 - s_1 x^2 + s_2 x - 1$$

and

$$E_2 : y^2 = x^3 - s_2 x^2 + s_1 x - 1,$$

respectively, defining a map $\nu : X \to E_1 \times E_2$. For each these curves, the $x$-coordinate projection $\pi_x$ is standard, so $\pi_1 := \pi_x$ and $\pi_2 := 1/\pi_x$ are standard projections for $E_1$ and $E_2$ respectively. The projection $\pi_1$ ramifies over $\{\infty, r_1, r_2, r_3\}$ and $\pi_2$ ramifies over $\{0, r_1, r_2, r_3\}$, where $\{r_1, r_2, r_3\}$ are the distinct, nonzero roots of $x^3 - s_1 x^2 + s_2 x - 1$. Thus

$$|\pi_1(E_1[2]) \cap \pi_2(E_2[2])| = 3.$$

Define $\Pi := \pi_1 \times \pi_2$, noting that for $(x, y) \in C$, $\Pi \circ \nu(x, y) = \Pi(x^2, 1/x^2) = (x^2, x^2)$. Thus $\Pi \circ \nu(X) = D$, where $D \subset \mathbb{P}^1 \times \mathbb{P}^1$ is the diagonal.

Fix $r \in \pi_1(E_1[2]) \cap \pi_2(E_2[2])$, and equip each $E_i$ with a group structure such that the identity lies above $r$. Observe that the $[-1]$-involution on $E_1 \times E_2$ induces the hyperelliptic involution on $X$. In particular, the Weierstrass points on $X$ are the six preimages of $\pi_1(E_1[2]) \cap \pi_2(E_2[2])$ under $\Pi \circ \nu$. Choose $Q \in X$ such that $\Pi(\nu(Q)) = (r, r)$, so that $Q$ is Weierstrass and $\nu$ factors as $\Phi \circ j_Q$ for some isogeny $\Phi : J(X) \to E_1 \times E_2$. The nontrivial elements of the kernel of $\Phi$ are precisely the three 2-torsion points in $J(X)$ which are differences of Weierstrass points mapping

to the same point in the diagonal $D \subset \mathbb{P}^1 \times \mathbb{P}^1$. Thus $\Phi$ is degree four as claimed, completing the proof. □

9.3. **Proof of Theorem 1.1.** Fix $X \in \mathcal{L}_2$. From Proposition 9.1, we have elliptic curves $E_1$ and $E_2$ and a Weierstrass point $Q \in X$ such that

$$|j_Q(X) \cap J(X)^{tors}| \leq 16\,|\pi_1(E_1^{tors}) \cap \pi_2(E_2^{tors})|,$$

for a pair of standard projections $\pi_i : E_i \to \mathbb{P}^1$ satisfying $|\pi_1(E_1[2]) \cap \pi_2(E_2[2])| = 3$. Given any other Weierstrass point $P \in X$, we have $[P-Q] \in J(X)^{tors}$, so we conclude that

$$|j_P(X) \cap J(X)^{tors}| = |j_Q(X) \cap J(X)^{tors}| \leq 16B,$$

where $B$ is the constant of Corollary 8.2.

9.4. **Proof of Theorem 1.8.** Fix $X \in \mathcal{L}_2 \subset \mathcal{M}_2$, defined over $\overline{\mathbb{Q}}$. From Proposition 9.1 there is a pair $t_1 \neq t_2$ in $\overline{\mathbb{Q}} \setminus \{0,1\}$ and an isogeny $\Phi : J(X) \to E_{t_1} \times E_{t_2}$ of degree 4 so that $\Pi \circ \Phi \circ j_Q(X)$ is the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$, where $\Pi = \pi \times \pi$ and $Q$ is a Weierstrass point on $X$. Recall from §2.1 that the Néron-Tate canonical height on $\hat{h}_{E_t}$ on $E_t$ satisfies

$$\hat{h}_{E_t}(P) = \frac{1}{2}\,\hat{h}_t(\pi(P))$$

for all $P \in E_t(\overline{\mathbb{Q}})$ and each $t \in \overline{\mathbb{Q}} \setminus \{0,1\}$.

Let

$$D = \{O_1\} \times E_{t_2} + E_{t_1} \times \{O_2\}$$

be a divisor on $E_{t_1} \times E_{t_2}$ where $O_i$ denotes the identity element of $E_{t_i}$, and let $L_D$ be the associated line bundle. Let $L_X = \Phi^* L_D$ on $J(X)$, and let $\hat{h}_{L_X}$ be the associated Néron-Tate canonical height on $J(X)(\overline{\mathbb{Q}})$. By the functoriality of canonical heights [HS, Theorem B.5.6], we have

$$\begin{aligned} \hat{h}_{L_X}(x) &= \hat{h}_{L_D}(\Phi(x)) \\ &= \hat{h}_{E_{t_1}}(\Phi(x)_1) + \hat{h}_{E_{t_2}}(\Phi(x)_2) \\ &= \frac{1}{2}\left(\hat{h}_{t_1}(\pi(\Phi(x)_1)) + \hat{h}_{t_2}(\pi(\Phi(x)_2))\right), \end{aligned}$$

where $\Phi(x) = (\Phi(x)_1, \Phi(x)_2)$ in $E_{t_1} \times E_{t_2}$. Restricting to the points $x \in j_P(X)(\overline{\mathbb{Q}})$, so that $\pi(\Phi(x)_1) = \pi(\Phi(x)_2)$ in $\mathbb{P}^1$, the theorem now follows from Theorem 8.1.

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