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Some consequences of Schanuel's Conjecture

by

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During the Arizona Winter School 2008 (held in Tucson, AZ) we worked on the following problems:

a) (Expanding a remark by S. Lang [1]). Define $E_0 = \overline{\mathbb{Q}}$. Inductively, for $n \geq 1$, define E_n as the algebraic closure of the field generated over E_{n-1} by the numbers $\exp(x) = e^x$, where x ranges over E_{n-1}. Let E be the union of E_n, $n \geq 0$. Show that Schanuel's Conjecture implies that the numbers $\pi, \log\pi, \log\log\pi, \log\log\log\pi, \ldots$ are algebraically independent over E.

b) Try to get a (conjectural) generalization involving the field L defined as follows. Define $L_0 = \overline{\mathbb{Q}}$. Inductively, for $n \geq 1$, define L_n as the algebraic closure of the field generated over L_{n-1} by the numbers y, where y ranges over the set of complex numbers such that $e^y \in L_{n-1}$. Let L be the union of L_n, $n \geq 0$.

We were able to prove the more general result:

Theorem 1. *Schanuel's Conjecture implies E and L are linearly disjoint over $\overline{\mathbb{Q}}$.*

And deduced from it the following ones:

1. $\pi \notin E$ and $e \notin L$.
2. $\pi, \log\pi, \log\log\pi, \ldots$ are algebraically independent over E.
3. $e, e^e, e^{e^e}, \ldots$ are algebraically independent over L.
4. $E \cap L = \overline{\mathbb{Q}}$.

Remember:

Conjecture 1 (Schanuel)**.** *Let $x_1, \dots, x_n$ be $\mathbb{Q}$-linearly independent complex numbers. Then the transcendence degree over $\mathbb{Q}$ of the field*

$$\mathbb{Q}(x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n})$$

is at least n.

Definition 1. *Let F/K be a field extension and $F_1, F_2 \subseteq F$ two subextensions. We say they are* linearly disjoint *over K when the following holds:*

$\{x_1, \dots, x_n\} \subseteq F_1$ linearly independent over $K \Rightarrow \{x_1, \dots, x_n\}$ linearly independent over F_2.

We say they are free *(or* algebraically disjoint*) over K when:*

$\{x_1, \dots, x_n\} \subseteq F_1$ algebraically independent over $K \Rightarrow \{x_1, \dots, x_n\}$ algebraically independent over F_2.

Remark 1. *Linear disjointness is equivalent to the multiplication map*

$$F_1 \otimes_K F_2 \longrightarrow F$$

being injective. Therefore this is a symmetric condition in F_1 and F_2.

Remark 2. *Algebraic disjointness is equivalent to the existence of transcendence basis B_1, B_2 of the extensions F_1/K and F_2/K (respectively) such that $B_1 \cup B_2$ is algebraically independent over K. Therefore this one is also a symmetric condition in F_1 and F_2.*

Remark 3. *For a set $S \subseteq F_1$ to be algebraically independent over K means all its monomials being linearly independent over K. Thus linearly disjointness* implies *freeness, and in general the converse is not true (although we are going to use a partial converse to this fact, proved in [3]).*

Remark 4. *If F_1, F_2 are linearly disjoint over K then we must have $F_1 \cap F_2 = K$, since $k \in F_1 \cap F_2$ will be F_2-linearly dependent together with 1 whence, they should also be K-linearly dependent.*

Before going to the proof of the Theorem, we need a couple of technical lemmas involving a key construction.

Lemma 1. *We have $E_n = \overline{\mathbb{Q}(\exp(E_{n-1}))}$*[1]*.*

[1] $\overline{F}$ meaning the algebraic closure of the field F.

Proof. With induction in n, the base case follows by definition since

$$E_1 = \overline{E_0(\exp(E_0))} = \overline{\overline{\mathbb{Q}}(\exp(E_0))} = \overline{\mathbb{Q}(\exp(E_0))},$$

and $E_0 = \overline{\mathbb{Q}}$.

In general

$$\begin{aligned} E_n &= \overline{E_{n-1}(\exp(E_{n-1}))} \\ E_n &= \overline{\overline{\mathbb{Q}(\exp(E_{n-2}))}(\exp(E_{n-1}))} \\ E_n &= \overline{\mathbb{Q}(\exp(E_{n-2}))(\exp(E_{n-1}))} \\ E_n &= \overline{\mathbb{Q}(\exp(E_{n-1}))}, \end{aligned} \tag{1}$$

since $E_{n-2} \subseteq E_{n-1}$. ☐

Lemma 2. *For every $x \in E_n$ there is a finite set $A_{n-1} \subseteq E_{n-1}$ such that x is algebraic over $\mathbb{Q}(\exp(A_{n-1}))$ (or equivalently, $x \in \overline{\mathbb{Q}(\exp(A_{n-1}))}$).*

Proof. We have $x \in E_n = \overline{\mathbb{Q}(\exp(E_{n-1}))}$ which means it is a root of a nontrivial polynomial with coefficients in $\mathbb{Q}(\exp(E_{n-1}))$. Each coefficient involves only finitely many exponentials of elements in E_{n-1}. Therefore, taking A_{n-1} the union of those exponents will work. ☐

Lemma 3 (the Key Lemma)**.** *For every $x \in E_n$ there is a finite set $A \subseteq E_{n-1}$ such that $x \in \overline{\mathbb{Q}(\exp(A))}$ and A is also algebraic over $\mathbb{Q}(\exp(A))$.*

Proof. Start with A_{n-1} as in the previous lemma and iterate the reasoning finding a sequence of subsets $A_{n-1}, A_{n-2}, A_{n-3}, \ldots, A_0$ as follows:

- Since $A_{n-1} \subseteq E_{n-1}$ is finite, it follows that A_{n-1} is algebraic over $\mathbb{Q}(\exp(A_{n-2}))$ for some finite $A_{n-2} \subseteq E_{n-2}$.

- Next A_{n-2} is algebraic over $\mathbb{Q}(\exp(A_{n-3}))$ for some finite $A_{n-3} \subseteq E_{n-3}$.
  ...

- Finally A_1 is algebraic over $\mathbb{Q}(\exp(A_0))$ for some finite $A_0 \subseteq E_0 = \overline{\mathbb{Q}}$.

Then just take $A = \bigcup_{m \leq n-1} A_m \subseteq E_{n-1}$. Since $A_{n-1} \subseteq A$ we get $x \in \overline{\mathbb{Q}(\exp(A))}$ and since each A_m is algebraic over $\mathbb{Q}(\exp(A_{m-1}))$ then it is so over $\mathbb{Q}(\exp(A))$ and therefore, the whole set A is algebraic over $\mathbb{Q}(\exp(A))$. ☐

In a similar way we get analogues of these lemmas in the case of the logarithmic extensions L_m. Let us state them for the sake of preciseness.

Lemma 4. *We have* $L_n = \overline{\mathbb{Q}(\exp^{-1}(L_{n-1}))}$.

Lemma 5. *For every* $x \in L_n$ *there is a finite set* $C_n \subseteq \mathbb{C}$ *such that* $\exp(C_n) \subseteq L_{n-1}$ *and that* x *is algebraic over* $\mathbb{Q}(C_n)$.

Lemma 6 (the Key Lemma)**.** *For every* $x \in L_n$ *there is a finite set* $C \subseteq \mathbb{C}$ *with* $\exp(C) \subseteq L_{n-1}$ *such that* $\exp(C) \cup \{x\}$ *is algebraic over* $\mathbb{Q}(C)$.

The proofs follow the same outline as in the exponential case.

Now we are ready to go the proof of the theorem:

Assuming the Schanuel's Conjecture to be true, let us prove E_m and L_n are linearly disjoint for arbitrary m and n (this will be enough since E is the union of the E_m and L is the union of the L_n).

Proceeding by induction, let us assume it is true that E_{m-1} and L_n are linearly disjoint over $\overline{\mathbb{Q}}$.

Suppose E_m and L_n are not linearly disjoint. Let us take $\{l_i\} \subseteq L_n$ linearly independent over $\overline{\mathbb{Q}}$ and $\{e_i\} \subseteq E_m$ such that $\sum l_i e_i = 0$.

By the Key Lemmas:

- $\exists$ finite $A \subseteq E_{m-1}$ such that $A \cup \{e_i\}$ algebraic over $\mathbb{Q}(\exp(A))$.
- $\exists$ finite $C \subseteq L_n$ finite such that $\exp(C) \cup \{l_i\}$ algebraic over $\mathbb{Q}(C)$.

Now take $B \subseteq A$ such that $\exp(B)$ is a transcendence basis of $\mathbb{Q}(\exp(A))$, and take $D \subseteq C$ such that D is a transcendence basis of $\mathbb{Q}(C)$.

We claim $B \cup D$ is linearly independent over $\mathbb{Q}$.

Consider any linear relation over $\mathbb{Q}$ and by cleaning denominators if necessary take

$$\sum_{b \in B} p_b b = \sum_{d \in D} q_d d$$

with all the p_b, q_d integers.

Since the expression on the left is an element in E_{m-1} and that of the right is an element of L_n, and by hypothesis these two fields were linearly disjoint over $\overline{\mathbb{Q}}$, we should have $E_{m-1} \cap L_n = \overline{\mathbb{Q}}$ and both expressions would represent an element $r \in \overline{\mathbb{Q}}$.

But $\sum_{d \in D} q_d d = r$ is an algebraic relation of D with coefficients in $\overline{\mathbb{Q}}$, hence it must be the trivial relation (keep in mind that D was taken to be algebraically independent over $\overline{\mathbb{Q}}$).

We get at once $r = 0 = q_d$ for all $d \in D$.

Now from $\sum_{b\in B} p_b b = 0$ taking exponentials on both sides we get

$$\prod_{b\in B} (\exp(b))^{p_b} = 1$$

which is an algebraic relation with coefficients in $\mathbb{Q}$ (and hence in $\overline{\mathbb{Q}}$) among the elements of the set $\exp(B)$, taken algebraically independent. Therefore, the (Laurent) monomial $\prod_{b\in B}(X_b)^{p_b}$ should be the trivial one, i.e. the integers p_b must be all equal to zero.

Summarizing, we have proven $B \cup D$ is $\mathbb{Q}$-linearly independent.

By Schanuel's Conjecture $\text{trdeg}_{\mathbb{Q}}\mathbb{Q}(B, D, \exp(B), \exp(D)) \geq |B| + |D|$.

However

$$\text{trdeg}_{\mathbb{Q}}\mathbb{Q}(B, D, \exp(B), \exp(D)) = \text{trdeg}_{\mathbb{Q}}\mathbb{Q}(B, C, \exp(A), \exp(D))$$

since $\exp(A)$ is algebraic over $\mathbb{Q}(\exp(B))$ and C is algebraic over $\mathbb{Q}(D)$.

We also have

$$\text{trdeg}_{\mathbb{Q}}\mathbb{Q}(B, C, \exp(A), \exp(D)) = \text{trdeg}_{\mathbb{Q}}\mathbb{Q}(C, \exp(A))$$

because $B \subseteq A$ and the latter was algebraic over $\mathbb{Q}(\exp(A))$, and similarly $\exp(D) \subseteq \exp(C) \subseteq \overline{\mathbb{Q}(C)}$.

Finally

$$\text{trdeg}_{\mathbb{Q}}\mathbb{Q}(C, \exp(A)) = \text{trdeg}_{\mathbb{Q}}\mathbb{Q}(D, \exp(B)) \leq |B| + |D|$$

since C was algebraic over $\mathbb{Q}(D)$ and $\exp(A)$ was so over $\mathbb{Q}(\exp(B))$.

From

$$|B| + |D| \geq \text{trdeg}_{\mathbb{Q}}\mathbb{Q}(D, \exp(B)) \geq |B| + |D|$$

we conclude $\text{trdeg}_{\mathbb{Q}}\mathbb{Q}(D, \exp(B)) = |B| + |D|$ and the set $\exp(B) \cup D$ turns out to be algebraically independent over $\mathbb{Q}$, whence over $\overline{\mathbb{Q}}$.

Therefore $\mathbb{Q}(\exp(B))$ and $\mathbb{Q}(D)$ are free over $\overline{\mathbb{Q}}$, and the same is true for $\overline{\mathbb{Q}(\exp(B))}$ and $\overline{\mathbb{Q}(D)}$.

Since $\overline{\mathbb{Q}}$ is algebraically closed, $\overline{\mathbb{Q}(\exp(B))}$ and $\overline{\mathbb{Q}(D)}$ are linearly disjoint over $\overline{\mathbb{Q}}$ (see [3] Theorem 4.12, page 367).

But the $\{l_i\}$ are algebraic over $\mathbb{Q}(C)$ and the $\{e_i\}$ are algebraic over $\mathbb{Q}(\exp(A))$, which means $\{l_i\} \subseteq \overline{\mathbb{Q}(D)}$ and $\{e_i\} \subseteq \overline{\mathbb{Q}(\exp(B))}$ giving to us the nontrivial linear relation $\sum l_i e_i = 0$. Contradiction.

Corollary 1. *We have* $L \cap E = \overline{\mathbb{Q}}$.

Proof. It follows directly from the linear disjointness. ☐

Corollary 2. *We have* $\pi \notin E$ *and* $e \notin L$.

Proof. We have $e = \exp(1) \in E_1 \subseteq E$. Since $e \notin \overline{\mathbb{Q}}$ it cannot be also in L.

If π were in E, $i\pi$ should also be there. But $i\pi \in L_1 \subseteq L$ since it is a logarithm of -1. We conclude $i\pi \notin E$ because it is not in $\overline{\mathbb{Q}}$. ☐

Corollary 3. *The numbers* $\pi, \log \pi, \log\log \pi, \ldots$ *are algebraically independent over* E.

Proof. We are actually going to prove that $i\pi, \log \pi, \log\log \pi, \ldots$ are algebraically independent over E (which is an equivalent statement).

Let us write $\log_{[k]} \pi$ for the $k^{th}-$ iterated logarithm of π.

Observe that the whole sequence $i\pi, \log \pi, \log\log \pi, \ldots$ lies in L.

Since we are assuming E and L linearly independent over $\overline{\mathbb{Q}}$, they are going be free, and it will be enough to prove $i\pi, \log \pi, \log\log \pi, \ldots$ they are algebraically independent over $\overline{\mathbb{Q}}$, or, which is the same, they are algebraically independent over $\mathbb{Q}$.

To prove $i\pi, \log \pi, \log\log \pi, \ldots, \log_{[n]} \pi$ are $\mathbb{Q}$-algebraically independent, we use Schanuel's Conjecture again.

Without loss of generality, we may assume the statement true for

$$i\pi, \log \pi, \log\log \pi, \ldots, \log_{[n-1]} \pi$$

(by induction).

As before, any nontrivial $\mathbb{Q}$-linear relation among the $i\pi, \log \pi, \ldots, \log_{[n]} \pi$ can be thought as a nontrivial $\mathbb{Z}$-linear combination (by clearing denominators) and then as an algebraic relation among their exponentials (by applying exp(.) at both sides).

More precisely:

$$i\pi q + \sum_{k=1}^{n} q_k \log_{[k]} \pi = 0$$

with $q, q_k \in \mathbb{Z}$ leads us to

$$(-1)^q \prod_{k=1}^{n} (\log_{[k-1]} \pi)^{q_k} = 1$$

which equals

$$\prod_{k=0}^{n-1} (\log_{[k]} \pi)^{q_{k+1}} = (-1)^q$$

Since we are assuming $i\pi, \log \pi, \log\log \pi, \ldots, \log_{[n-1]} \pi$ are $\mathbb{Q}$-algebraically independent (also $\pi, \log \pi, \log\log \pi, \ldots, \log_{[n-1]} \pi$) this last algebraic relation must be the trivial one, i.e. $q_k = 0$ for all $1 \leq k \leq n$ and q even (but this is no so important). Returning to our linear relation we get $i\pi q = 0$ meaning $q = 0$.

Therefore $A = \{i\pi, \log \pi, \log\log \pi, \ldots, \log_{[n]} \pi\}$ are linearly independent over $\mathbb{Q}$ and by Schanuel's Conjecture, the transcendence degree of $\mathbb{Q}(A, \exp(A))$ should be at least $n + 1$.

Since $\exp(A)$ is algebraic over $\mathbb{Q}(A)$, this means

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(i\pi, \log \pi, \log\log \pi, \ldots, \log_{[n]} \pi) \geq n + 1,$$

i.e. $i\pi, \log \pi, \log\log \pi, \ldots, \log_{[n]} \pi$ are algebraically independent over $\mathbb{Q}$ (then over $\overline{\mathbb{Q}}$ and hence over E). □

Corollary 4. *The numbers* $e, e^e, e^{e^e}, \ldots$ *are* $L-$*algebraically independent.*

Proof. As before, we only have to prove they are so over $\mathbb{Q}$. Again, this follows by induction.

Name $\exp^{[n]}(1) = \exp(\exp^{[n-1]}(1))$ and $\exp^{[0]}(1) = 1$.

Let us assume the $\{\exp^{[k]}(1)\}_{k=1}^{n}$ are algebraically independent over $\mathbb{Q}$. Then the set

$$A = \{1, e, e^e, \ldots, \exp^{[n]}(1)\} = \{\exp^{[k]}(1)\}_{k=0}^{n}$$

is $\mathbb{Q}$-linearly independent and by Schanuel's Conjecture we should have

$$n + 1 \leq \operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(A, \exp(A)) = \operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(\exp(A))$$

because A is algebraic over $\mathbb{Q}(\exp(A))$.

But $\exp(A) = \{\exp^{[k]}(1)\}_{k=1}^{n+1}$ would be algebraically independent over $\mathbb{Q}$. This finishes the inductive step. □

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