Group actions on trees

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November 10, 2004

1 Trees

Let X be a metric space. Recall that a *geodesic* in X is an isometrically embedded arc $\alpha : I \to X$, and X is called *geodesic* if every pair of points of X is joined by a geodesic.

Definition 1.1 *A geodesic metric space T is a* (metric) tree *if every geodesic triangle is a tripod.*

Simplicial trees with any combinatorial metric (in which the infimum of the distances between vertices is strictly greater than 0) are metric trees. Here is an example of a metric tree that isn't simplicial.

Example 1.2 *(The SNCF metric) Endow $\mathbb{R}^2$ with the metric given by*

$$d((x, y_1), (x, y_2)) = |y_2 - y_1|$$

and

$$d((x_1, y_1), (x_2, y_2)) = |y_1| + |x_2 - x_1| + |y_2|$$

for $x_1 \neq x_2$. This is clearly a metric tree, but not simplicial. For an example with second countable topology, take the subspace consisting of the union of the x-axis and those vertical lines with rational x-coordinate.

2 Isometries of trees

For any metric space X and any isometry γ, define

$$l(\gamma) = \inf_{x \in X} d(x, \gamma x).$$

In general, isometries of X fall into three classes.

Definition 2.1 *Let γ be an isometry of X.*

1. *If γ fixes a point of X then γ is called* elliptic.

2. *If there exists* $x \in X$ *such that*

$$d(x, \gamma(x)) = l(\gamma) > 0$$

then γ *is* hyperbolic*.*

3. *Otherwise,* γ *is called* parabolic*.*

Intuitively, parabolic isometries fix a point at infinity, and are the hardest to describe. But in the case of trees, these don't arise.

Lemma 2.2 *Let* T *be a tree, and* γ *an isometry that doesn't fix a point. Then there exists a unique embedded line*

$$\mathrm{Axis}(\gamma) \subset T$$

on which γ *acts as translation by* $l(\gamma)$*. In particular,* γ *is hyperbolic.*

Remark 2.3 *It is surprisingly easy to find an axis for* γ*: it suffices to find* $x \in T$ *such that*

$$d(x, \gamma^2 x) = 2d(x, \gamma x).$$

It is then clear that the γ*-translates of* $[x, \gamma x]$ *form a* γ*-invariant line. Furthermore, note that if such a line* L *exists then any* γ*-invariant subtree must contain* L*, so* γ *is hyperbolic and* L *is an axis and the unique* γ*-invariant line.*

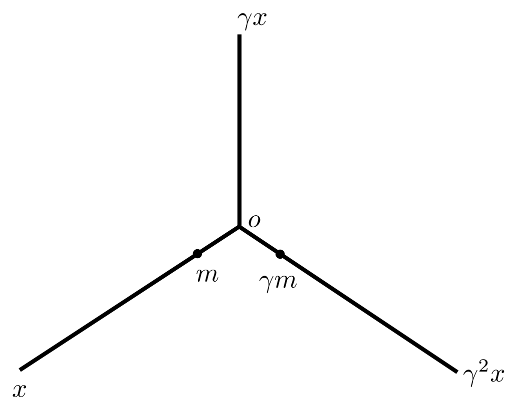

Figure 1: the mid-point is moved less far.

The intuition behind the proof of the lemma arises from the observation that, if m is the mid-point of $[x, \gamma x]$, then

$$d(x, \gamma x) \geq d(m, \gamma m).$$

So this mid-point looks like a good place to look for the axis.

Proof of lemma 2.2: Consider the tripod $[x, \gamma x, \gamma^2 x]$. Let o be its crux, and let m be the mid-point of $[x, \gamma x]$. It's clear that if $d(m, x) \geq d(o, x)$ then γ fixes m, contradicting the assumption that γ is not elliptic. Therefore, $d(o, x) > d(m, x)$.

By remark 2.3, it now suffices to show that $d(m, \gamma^2 m) = 2d(m, \gamma m)$. But $o \in [m, \gamma m]$ and $\gamma o \in [\gamma m, \gamma^2 m]$, so we only need to show that $d(o, \gamma o) = 2d(o, \gamma m)$.

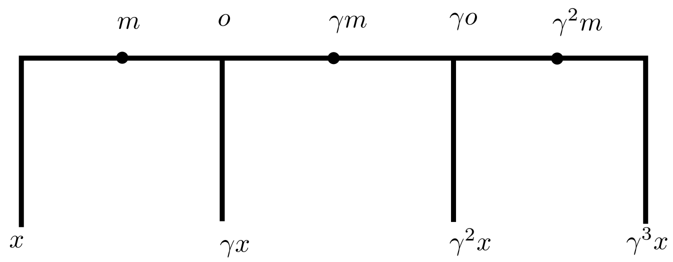

Figure 2: The axis of a hyperbolic isometry.

Since $o \in [m, \gamma m]$ and likewise $\gamma(o) \in [\gamma m, \gamma^2 m]$, we only need to show that $d(o, \gamma o) = 2d(o, \gamma m)$. But

$$\begin{aligned} d(o, \gamma o) &= d(\gamma x, \gamma^2 x) - 2d(o, \gamma x) \\ &= d(x, \gamma x) - 2(\frac{1}{2} d(x, \gamma x) - d(o, \gamma m)) \\ &= 2d(o, \gamma m) \end{aligned}$$

as required. *QED*

In summary, a hyperbolic isometry γ has a unique invariant line $\mathrm{Axis}(\gamma)$, whereas an elliptic isometry γ has a fixed point set $\mathrm{Fix}(\gamma)$. These are precisely the subtrees on which the function $x \mapsto d(x, \gamma x)$ attains its infimum, so are sometimes collectively denoted $\mathrm{Min}(\gamma)$.

3 Composition of isometries

To understand the structure of groups acting on trees, we need to know how these isometries compose. It's clear that, if $\gamma, \delta \in \mathrm{Isom}(T)$ are elliptic and $\mathrm{Fix}(\gamma) \cap \mathrm{Fix}(\delta) \neq \varnothing$, then $\gamma \circ \delta$ is elliptic. The next lemma generalizes this observation.

Lemma 3.1 *Let $\gamma, \delta \in \mathrm{Isom}(T)$.*

1. *If γ, δ are elliptic with disjoint fixed-point sets then $\gamma\delta$ is hyperbolic with*

$$l(\gamma\delta) = 2d(\mathrm{Fix}(\gamma), \mathrm{Fix}(\delta)).$$

2. *If γ, δ are hyperbolic with disjoint axes then $\gamma\delta$is hyperbolic with*

$$l(\gamma\delta) = l(\gamma) + l(\delta) + 2d(\mathrm{Axis}(\gamma), \mathrm{Axis}(\delta))$$

and, furthermore, $\mathrm{Axis}(\gamma\delta)$ *intersects* $\mathrm{Axis}(\gamma)$ *and* $\mathrm{Axis}(\delta)$*.*

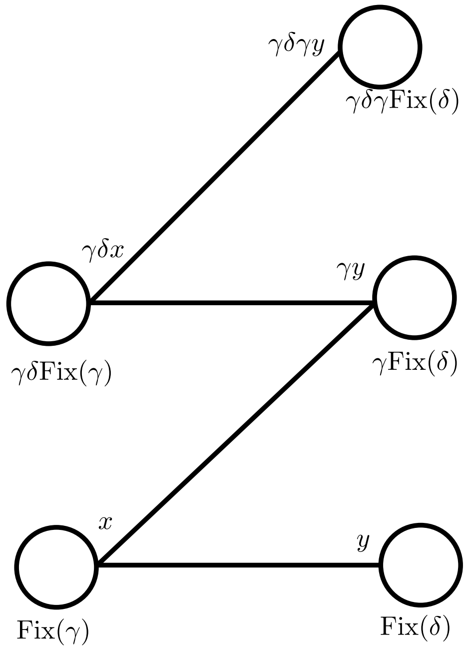

Figure 3: The axis of the composition of two elliptic isometries with disjoint fixed-point sets.

Proof: Suppose γ, δ are elliptic with disjoint fixed-point set. Note that $\mathrm{Fix}(\gamma)$ and $\mathrm{Fix}(\delta)$ are closed subtrees of T. Let $x \in \mathrm{Fix}(\gamma)$ be the unique point closest to $\mathrm{Fix}(\delta)$, and let $y \in \mathrm{Fix}(\delta)$ be the unique point closest to $\mathrm{Fix}(\gamma)$. Then unique geodesic from y to $\gamma\delta y$ is $[x, y] \cup \gamma[x, y]$ so

$$d(y, \gamma\delta y) = d(y, \gamma y) = 2d(x, y)$$

because no points in the interior of $[x, y]$ are fixed by γ. Likewise the geodesic from y to $(\gamma\delta)^2 y$ is

$$[x, y] \cup \gamma[x, y] \cup \gamma\delta[x, y] \cup \gamma\delta\gamma[x, y]$$

so

$$d(y, (\gamma\delta)^2 y) = 4d(x, y) = 2d(y, \gamma\delta y)$$

as required. See figure 3.

Now suppose γ, δ are hyperbolic isometries with disjoint axes. Let x be the unique point of $\mathrm{Axis}(\gamma)$ closest to $\mathrm{Axis}(\delta)$, and y the unique point of $\mathrm{Axis}(\delta)$ closest to $\mathrm{Axis}(\gamma)$. Then the geodesic from x to $\gamma\delta y$ is

$$[x, \gamma x] \cup \gamma[x, y] \cup \gamma[y, \delta y] \cup \gamma\delta[y, x]$$

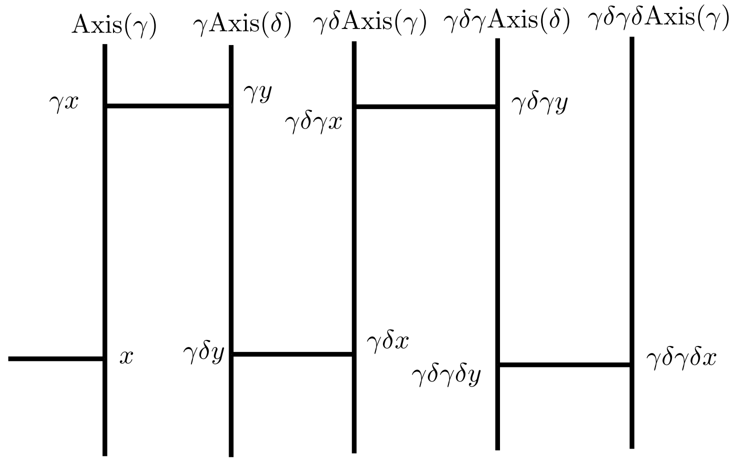

Figure 4: The axis of the composition of two hyperbolic elements with disjoint axes.

so

$$d(y, \gamma\delta y) = l(\gamma) + l(\delta) + 2d(x, y).$$

Likewise, the geodesic from x to $(\gamma\delta)^2 x$ is

$$[x, \gamma x] \cup \gamma[x, y] \cup \gamma[y, \delta y] \cup \gamma\delta[y, x] \cup \gamma\delta[x, \gamma x] \cup \gamma\delta\gamma[x, y] \cup \gamma\delta\gamma[y, \delta y] \cup \gamma\delta\gamma\delta[y, x]$$

and so

$$d(y, (\gamma\delta)^2 y) = 2l(\gamma) + 2l(\delta) + 4d(x, y) = d(y, \gamma\delta y)$$

as required. See figure 3. *QED*

4 Minimal trees

If a group G acts by isometries on a tree T, then T is called a *G-tree*. T is *trivial* if there exists a point of T fixed by all of G. If T doesn't contain any proper G-invariant subtrees, then T is called *minimal*.

The idea behind these definitions is that arbitrarily complicated trees and their G-translates can be glued to a G-tree, so only the minimal invariant subtree contains information about G. In particular, in the simplicial case a trivial G-tree corresponds to a trivial splitting of G in Bass-Serre theory.

The aim of this section is to prove the following proposition.

Proposition 4.1 *If G is finitely generated and T is a non-trivial G-tree then T contains a unique minimal G-invariant subtree, which is a countable union of lines.*

First we need a lemma.

Lemma 4.2 *Let $\gamma_1, \ldots, \gamma_n$ be elliptic isometries of T, and suppose*

$$\mathrm{Fix}(\gamma_i) \cap \mathrm{Fix}(\gamma_j) \neq \varnothing$$

for all i, j. Then

$$\bigcap_i \mathrm{Fix}(\gamma_i) \neq \varnothing.$$

Proof: The proof is by induction on n. The case $n = 2$ is trivial. Now suppose $n > 2$. By induction, there exist

$$x_k \in T_k = \bigcap_{i \neq k} \mathrm{Fix}(\gamma_i)$$

whenever $1 \leq k \leq n$. Consider $[x_1, x_2] \subset \mathrm{Fix}(\gamma_k)$ for each $k > 2$. But $x_1 \in \mathrm{Fix}(\gamma_2)$ and $x_2 \in \mathrm{Fix}(\gamma_1)$, so $[x_1, x_2]$ must pass through the (non-empty) intersection $\mathrm{Fix}(\gamma_1) \cap \mathrm{Fix}(\gamma_2)$. So there exists some $x \in [x_1, x_2]$ that lies in all the fixed-point sets. *QED*

The proof of the proposition is now a simple construction.

Proof of proposition 4.1: Let T be a G-tree. We first aim to show that if T is non-trivial then G contains a hyperbolic element. Let $\gamma_1, \ldots, \gamma_n$ be a finite generating set for G, and suppose all γ_i are elliptic. Then either every pair of generators has intersecting fixed-point sets, in which case T is trivial by lemma 4.2, or some pair γ_i, γ_j has disjoint fixed-point set. In the second case, $\gamma_i \gamma_j$ is hyperbolic by part 1 of lemma 3.1.

Now consider the subspace $T' \subset T$ defined as the union of the axes of all the hyperbolic elements of G. Then T' is connected by lemma 3.1. Since, for hyperbolic $\gamma \in G$, any γ-invariant subtree contains $\mathrm{Axis}(\gamma)$, it is clear that T' is contained in any G-invariant subtree. Finally, T' is itself G-invariant, since

$$\delta \mathrm{Axis}(\gamma) = \mathrm{Axis}(\delta \gamma \delta^{-1})$$

for hyperbolic γ and arbitrary δ. *QED*

5 The boundary of a tree

This construction of a minimal G-tree leads naturally to a coarse classification. First, though, we need the notion of the boundary at infinity of a tree.

Definition 5.1 *Let T be a metric tree. Let Y be the set of geodesic rays*

$$[0, \infty) \to T.$$

Then the boundary at infinity *of T is defined to be the quotient of Y by the equivalence relation equating $r, s \in Y$ if and only if the function*

$$t \mapsto d(r(t), s(t))$$

is bounded. The boundary is denoted $\partial_\infty T$.

Note that an action of G by isometries on T extends to an action on $\partial_\infty T$.

6 A coarse classification

The results so far give a classification of G-trees, comparable to our classification of isometries as elliptic or hyperbolic.

Theorem 6.1 *Let G be a finitely generated group and T a minimal G-tree. Then T is one of the following.*

1. ***Elliptic:*** *T is a point.*
2. ***Linear:*** *T is a line.*
3. ***Parabolic:*** *G fixes a point of $\partial_\infty T$.*
4. ***Hyperbolic:*** *There exists $\gamma, \delta \in G$, hyperbolic on T, whose axes intersect in a compact subset.*

Proof: From proposition 4.1 it follows that T is the union of the hyperbolic elements of G. Then every point of $\partial_\infty T$ arises as the end of an axis of a hyperbolic element of G.

It follows immediately that $\partial_\infty T = \varnothing$ if and only if G is elliptic (so T is a point) and that $|\partial_\infty G| = 2$ if and only if G is a line.

Now suppose that $|\partial_\infty T| > 2$. If $\gamma \in G$ is hyperbolic on T and $[r] \in \partial_\infty T$ doesn't correspond to an end of $\mathrm{Axis}(\gamma)$, then $d(r(t), \mathrm{Axis}(\gamma)) \to \infty$ as $t \to \infty$. But

$$d(r(t), \gamma r(t)) = 2d(r(t), \mathrm{Axis}(\gamma)) + l(\gamma)$$

which also tends to infinity with t. So γ doesn't fix $[r]$. In conclusion, hyperbolic γ fixes a point of $\partial_\infty T$ if and only if that point corresponds to an end of $\mathrm{Axis}(\gamma)$.

Assume that every pair of hyperbolic elements have axes that intersect in non-compact sets, and yet G doesn't fix a point of $\partial_\infty T$. Then there exist hyperbolic $\gamma_1, \gamma_2, \gamma_3$ with axes the three sides of an infinite tripod. But now (possibly after replacing γ_2 by its inverse) $\gamma_1^{\gamma_2}$ is a hyperbolic element, and

$$\mathrm{Axis}(\gamma_1^{\gamma_2}) \cap \mathrm{Axis}(\gamma_3) = \gamma_2 \mathrm{Axis}(\gamma_1) \cap \mathrm{Axis}(\gamma_3)$$

is compact, a contradiction. See figure 6. *QED*

7 The ping-pong lemma

This famous result of Tits shows that only certain groups can act hyperbolically.

Lemma 7.1 *Suppose γ, δ are hyperbolic elements whose axes intersect in a compact set. Then there exists n such that γ^n, δ^n generate a free group.*

Proof: Consider a non-trivial word in γ and δ, and let $K = \mathrm{Axis}(\gamma) \cap \mathrm{Axis}(\delta)$. For $\phi = \gamma^{\pm 1}, \delta^{\pm 1}$ let $C(\phi)$ be the component of $T - K$ such that $\phi(C(\phi)) \subset C(\phi)$. Now choose n large enough that $\phi^n(K) \subset C(\phi)$ for all ϕ. Note that now

$$\gamma^{\pm n}(C(\delta^{\pm 1})) \subset C(\gamma^{\pm 1})$$

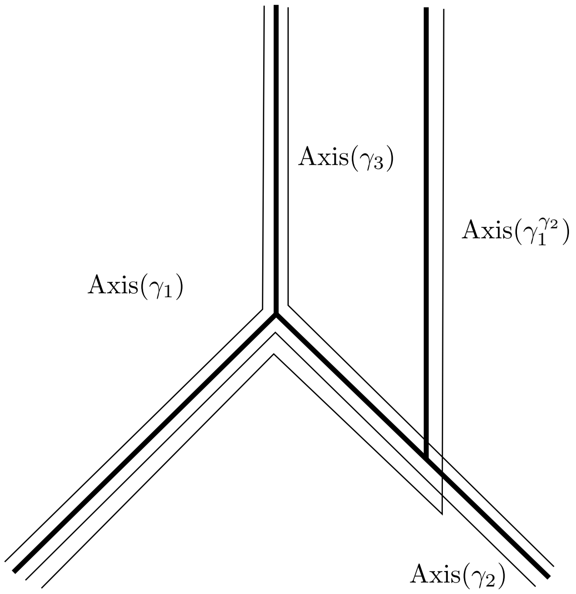

Figure 5: The axes of $\gamma_1, \gamma_2, \gamma_3$ and $\gamma_1^{\gamma_2}$.

and similarly for $\delta^{\pm 1}$.

Let w be a non-trivial word in γ^n and δ^n. The result follows from the claim that, for all ϕ, if w begins with ϕ^n then there exist three components C of $T-K$ such that $w(C) \subset C(\phi)$.

The claim is immediate if w is of length 1. Otherwise, $w = \phi^n w'$ for w' some word that doesn't begin with ϕ^{-n}. But the claim is true for w' by induction on word length, and so follows immediately for w. *QED*

In particular, if T is a hyperbolic G-tree then G contains a free subgroup.