Some Residual Properties of Groups

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1 Introduction

In this talk I'll describe the following results.

Theorem A *Limit groups are LERF*

Theorem B *Limit groups have* property LR*. That is, every finitely generated subgroup is a virtual retract.*

I'll start by saying what all the words mean, and then try to give an inkling of the proof.

NB For the purposes of this talk, all groups are finitely generated.

Definition 1 *Let $\mathcal{P}$ be a class of groups. A group G is* residually $\mathcal{P}$ *if, whenever $g \in G \smallsetminus 1$, there exists an epimorphism*

$$f : G \to Q$$

where $Q \in \mathcal{P}$ and $f(g) \neq 1$.

We'll be interested in strengthening this notion when $\mathcal{P}$ is the class of finite groups and the class of free groups.

2 LERF

Definition 2 *A group G is* LERF *if, whenever H is a finitely generated subgroup and $g \notin H$ there exists a homomorphism*

$$f : G \to Q$$

to a finite group such that $f(g) \notin f(H)$.

Example 3 *Abelian groups are LERF.*

Example 4 (M. Hall Jr.) *Free groups are LERF.*

Example 5 (P. Scott) *Surface groups are LERF.*

LERF has a very nice topological interpretation, related to lifting immersions to embeddings in a finite cover.

Lemma 6 *Let X be a complex. Then $\pi_1(X)$ is LERF if and only if, for every 'finitely generated' covering $X' \to X$ and every finite subcomplex $\Delta \subset X'$, the covering map $X' \to X$ factors through a finite-sheeted covering*

$$\hat{X} \to X$$

such that Δ embeds in $\hat{X}$.

From this point of view, M. Hall's theorem is a direct consequence of the following.

Theorem 7 (J. Stallings and others) *Every immersion of finite graphs can be completed to a finite-sheeted covering map.*

A whirlwind tour of some other examples of LERF groups.

- Free products of LERF groups are LERF (Burns; Romanovskii).
- Any amalgamated product of free groups along a cyclic group is LERF (Brunner, Burns and Solitar).
- If G is LERF, F is free and $f \in F$ is rootless then $G *_{\langle f \rangle} F$ is LERF (Gitik).
- Graphs of free groups over cyclic edge groups are LERF if and only if they don't contain Baumslag–Solitar subgroups (Tretkoff; Niblo; Wise).
- All-right Coxeter groups are QCERF. (Agol, Long & Reid; Haglund).

3 Property LR

Here is a notion related to LERF.

Definition 8 (Long & Reid) *A subgroup $H \subset G$ is a* virtual retract *if H is a retract of a finite-index subgroup $K \subset G$. (That is, $H \subset K$ and the inclusion map has a left-inverse.)*

The group G has local retractions *or* property LR *if every finitely generated subgroup is a virtual retract.*

The proof that free groups are LERF also shows that they have property LR.

4 Limit groups

Now consider the case where $\mathcal{P}$ is the set of free groups.

Definition 9 *A group G is* ω-residually free *if, for any finite subset $X \subset G \setminus 1$, there exists a homomorphism $f : G \to F$ so that $1 \notin f(X)$.*

Note that this notion is not interesting in the case of residual finiteness, because a direct products of finite groups is finite.

Definition 10 *A finitely generated, ω-residually free group is called a* limit group*.*

Limit groups are extremely closely related to free groups. Indeed, the set of limit groups is the closure of the set of free groups in the Gromov–Grigorchuk topology on the set of marked groups.

Here are some examples of limit groups

Example 11 *Free groups are limit groups.*

Example 12 *Free abelian groups are limit groups.*

Example 13 *If* Σ *is a closed surface and* $\chi(\Sigma) < -1$ *then* $\pi_1(\Sigma)$ *is a limit group.*

Limit groups are torsion-free (easy) and finitely presented (hard), as well as having many other nice properties.

Theorem 14 (W.) *Limit groups are LERF and have property LR.*

Several nice properties of limit groups follow, including:

Corollary 15 *Limit groups have solvable generalized word problem.*

Corollary 16 (F. Dahmani) *Finitely generated subgroups of limit groups are quasi-isometrically embedded.*

Corollary 17 (Bridson & Howie) *Every non-abelian subgroup of a limit group has finite index in its normalizer.*

5 Structure theory

I'll try to indicate some of the proof. The idea is to generalize Stallings' proof of Hall's theorem.

One of the highlights of recent work on Tarksi's problems by Z. Sela and O. Kharlampovich and A. Myasnikov has been the development of a structure theory for limit groups.

Definition 18 *Construct the class of* ICE *(*Iterated Centralizer Extension) *spaces inductively as follows. Start with all compact, connected graphs at level 0.*

Given a space X_n of level n, choose a loop $z \in \pi_1(X)$ such that $\langle z \rangle$ is maximal abelian. Construct a space X_{n+1} of level $n+1$ by gluing one end of a cylinder to z, and the other end to a coordinate circle in a k-torus.

Theorem 19 (Kharlampovich & Myasnikov) *A group is a limit group if and only if it is a finitely generated subgroup of the fundamental group of an* ICE *space.*

There is a different proof of this theorem due to Champetier and Guirardel.

Since LERF and property LR both clearly pass to subgroups, it suffices to prove the result for the fundamental groups of ICE spaces.