Subgroup separability and limit groups

Henry Wilton

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1 Definitions and examples

Let $\mathcal{P}$ be a class of groups. A group G is *residually* $\mathcal{P}$ if, whenever $g \in G \smallsetminus 1$, there exists a homomorphism

$$f : G \to Q$$

where $Q \in \mathcal{P}$ and $f(g) \neq 1$.

More generally, a subgroup $H \subset G$ is *closed in the pro-*$\mathcal{P}$ *topology* if whenever $g \in G \smallsetminus H$ there exists a homomorphism $f : G \to Q$ with $Q \in \mathcal{P}$ and

$$f(g) \notin f(H).$$

We will, at first, be most interested in the case where $\mathcal{P}$ is the set of finite groups. If $H \subset G$ is closed in the pro-finite topology then we call H *separable*. The group G is *subgroup separable* if any finitely generated subgroup is separable. (Such groups are also called *LERF*.)

Example 1.1 *Abelian groups are subgroup separable.*

Example 1.2 (M. Hall '49) *Free groups are subgroup separable.*

Example 1.3 (Burns '71, Romanovskii '69) *Free products of subgroup separable groups are subgroup separable.*

Example 1.4 (P. Scott '78) *Surface groups are subgroup separable.*

However, subgroup separability fails to be preserved by many natural operations.

Example 1.5 *If F is a finitely generated free group, $F \times F$ is not subgroup separable.*

So it is not closed under direct products.

Example 1.6 (Burns, Karrass & Solitar '87) *Fix a basis a, b for $\mathbb{Z}^2$. Let G be the HNN-extension of $\mathbb{Z}^2$ corresponding to the map sending $a \mapsto b$. Then G is not subgroup separable.*

So you have to be careful when taking HNN-extensions or amalgamated products.

Here are some elementary properties of subgroup separability.

Proposition 1.7 *Let G be a subgroup separable group.*

1. *If $H \subset G$ is a subgroup then H is subgroup separable.*
2. *If G is a finite-index subgroup of G' then G' is subgroup separable.*

The first property is immediate from the definition. The second is a short exercise.

2 Some topology

I will explain how to prove Hall's Theorem using the topology of graphs. We will need some reformulations of separability.

Lemma 2.1 *Let G be a group and $H \subset G$ a subgroup. Let X be a complex with $G = \pi_1(X)$ and $X^H \to X$ the covering corresponding to H. The following are equivalent.*

1. *H is separable.*
2. *For any $g \in G \setminus H$ there exists a finite-index subgroup $K \subset G$ so that $H \subset K$ and $g \notin K$.*
3. *Whenever $\Delta \subset X^H$ is a finite subcomplex there exists an intermediate, finite-sheeted covering*
$$X^H \to \hat{X} \to X$$
so that Δ embeds in $\hat{X}$.

A map of graphs $f : \Delta \to \Gamma$ is an *immersion* if it is a local embedding; that is, an embedding in a neighbourhood of every vertex. If f is in fact a local isomorphism then f is a *covering*. An immersion looks like a piece of a covering. Remarkably, in the case of graphs every immersion arises in this way.

Proposition 2.2 *Let Δ and Γ be finite graphs and $\Delta \to \Gamma$ an immersion. Then $\Delta \to \Gamma$ extends to a finite-sheeted covering $\hat{\Gamma} \to \Gamma$.*

This is most easily seen when Γ is a rose. Fix an orientation and a colouring on the edges of Γ. This lifts to an orientation and a colouring on Δ. In fact, immersions $\Delta \to \Gamma$ correspond precisely to colourings on Δ.

Proof. For each colour, simply count how many edges of Δ are 'missing' going into a vertex, and how many are 'missing' going out. Summing over all vertices, we get the same number!

Pairing them up arbitrarily and filling in the missing edges gives the required cover $\hat{\Gamma}$. $\square$

We can exploit this fact to prove Hall's Theorem, because free groups are precisely the fundamental groups of graphs.

Corollary 2.3 *Free groups are subgroup separable.*

Proof. Let Γ be a finite rose and H a finitely generated subgroup of $F = \pi_1(\Gamma)$. Let $\Gamma^H \to \Gamma$ be the covering corresponding to H, and consider a finite subcomplex $\Delta \subset \Gamma^H$. Because H is finitely generated, we can enlarge Δ to assume that $\pi_1(\Delta) = H$. But $\Delta \to \Gamma$ is an immersion, so can be completed to a finite-sheeted covering $\hat{\Gamma} \to \Gamma$. $\square$

Versions of this topological proof were given independently by J. Hempel, P. Scott and J. Stallings. Stallings went on to show how to use the topology of graphs to easily deduce many properties of free groups.

The stronger theorem originally proved by Hall also follows from this proof.

Corollary 2.4 *Let F be a free group and $H \subset F$ a finitely generated subgroup. Then there exists a finite-index subgroup F' containing H so that*

$$F' = H * F''.$$

3 Limit groups

We now turn our attention to the case where $\mathcal{P}$ is the set of free groups. Recall that a group G is *residually free* if, whenever $g \in G \smallsetminus 1$, there exists a homomorphism $f : G \to F$ so that $f(g) \neq 1$.

A group G is *ω-residually free* if, for any finite subset $X \subset G$, there exists a homomorphism $f : G \to F$ so that $f|X$ is injective.

Remark 3.1 *Being ω-residually finite is just the same as being residually finite, because a direct product of finite groups is finite.*

A finitely generated, ω-residually free group is called a *limit group*.

Here are some examples of limit groups

Example 3.2 *Free groups are limit groups.*

Example 3.3 *Free abelian groups are limit groups.*

Example 3.4 *If Σ is a closed surface and $\chi(\Sigma) < -1$ then $\pi_1(\Sigma)$ is a limit group.*

Here are some properties of limit groups.

Proposition 3.5 *Let G be a limit group.*

1. *G is torsion-free.*
2. *Every finitely generated subgroup of G is a limit group.*
3. *G is* commutative transitive*—that is, for $x, y, z \in G$, if $[x, y] = [y, z] = 1$ then $[x, z]=1$.*
4. *Abelian subgroups of G are finitely generated.*
5. *G is finitely presented.*
6. *There exists a finite $K(G, 1)$.*

Limit groups are historically of interest because of their connection to the logic of groups. Given a group G, the *elementary theory* of a group is the set of sentences in first-order logic that are true in G. For example, if G is abelian then the sentence

$$\forall x \; \forall y. \; ([x, y] = 1)$$

is in the elementary theory of G.

Question 3.6 (Tarksi) *Classify the finitely generated groups with the same elementary theory as F_2, the free group of rank 2.*

The *existential theory* of G is the set of all sentences that use just one quantifier $\exists$.

Theorem 3.7 (V. N. Remeslennikov '89) *The set of finitely generated groups with the same existential theory as F_2 is precisely the set of limit groups.*

Attempts to solve Tarski's problem have led to a structure theory for limit groups.

Definition 3.8 Tower spaces *are defined inductively on height. A tower X_0 of height 0 is a compact, connected one-point union of tori, graphs and hyperbolic surfaces of Euler characteristic < -1. A tower X_h of height h is built from a tower X_{h-1} of height $h-1$ by attaching one of two sorts of blocks.*

1. ***Quadratic block.*** *X_h is obtained by gluing a compact hyperbolic surface Σ along its boundary components to X_{h-1}.*
2. ***Abelian block.*** *X_h is obtained by gluing a coordinate circle of an n-torus T to a non-trivial loop in X_{h-1}.*

It is also required that there exists a retraction $\rho : X_h \to X_{h-1}$ satisfying certain conditions.

The following theorem provides a structure theory for limit groups.

Theorem 3.9 (Z. Sela '03) *A group is a limit group if and only if it is a finitely generated subgroup of the fundamental group of some tower* X_h.

O. Kharlampovich and A. Myasnikov also proved an essentially equivalent theorem. Sela went on to provide an answer to Tarski's Problem. A tower space is *hyperbolic* if no tori are used in its construction.

Theorem 3.10 (Sela) *A finitely generated group has the same elementary theory as* F_2 *if and only if it is the fundamental group of a hyperbolic tower.*

We call such groups *elementarily free.*

It seems natural to ask what the relationship is between the two residual properties of being subgroup separable and being ω-residually free.

Theorem 3.11 (W.) *Elementarily free groups are subgroup separable.*

The proof is inspired by the topological proof of Hall's Theorem. I will try to explain some ways of simplifying the tower to make this problem more tractable.

4 Simplifying the tower

The aim is to pass to a finite-sheeted covering of the tower in which the gluing maps are of a nice form.

Lemma 4.1 *If X_h and $\gamma : S^1 \to X_h$ is a non-trivial curve then after a homotopy of γ there exists a finite-sheeted covering $\hat{X} \to X_h$ so that γ lifts to $\hat{X}$ and furthermore:*

1. *$\gamma_* H_1(S^1, \mathbb{Z})$ is an infinite direct factor in $H_1(\hat{X})$;*
2. *$\gamma : S^1 \to \hat{X}$ is injective.*

Applying this lemma successively to X_{h-1} and pulling the resultant cover back along the retraction, we obtain a finite-sheeted covering of X_h in which all the gluing maps have the properties of the lemma.

First, I'll show how to obtain the homological condition.

Fixing a base-point, the curve γ defines an element of $\pi_1(X_h)$. Since $L_h = \pi_1(X_h)$ is residually free, there exists a homomorphism

$$f : L_h \to F$$

so that $f(\gamma) \neq 1$. By Hall's Theorem, there exists a finite-index subgroup $F' \subset F$ so that $\langle f(\gamma) \rangle$ is a free factor in F'. In particular, $\langle f(\gamma) \rangle$ is a direct factor of $H_1(F')$. Set $L' = f^{-1}(F')$. Then $H_1(\langle \gamma \rangle)$ is an infinite direct factor of $H_1(L')$.

Forcing the gluing maps to be injective requires a little geometry. It also uses that cyclic subgroups are separable, which is true by induction, or by a rather neat argument given later.

Towers can be endowed with metrics so that they are non-positively curved. In a non-positively curved space, a loop γ is freely homotopic to a (unique) local geodesic. Very roughly, if γ is not an embedding then at a singular point γ can be decomposed as $\gamma = \gamma_1\gamma_2$ with γ a closed loop. But γ is the shortest loop in $\langle\gamma\rangle$ and γ_1 is shorter than γ, so $\gamma_1 \notin \langle\gamma\rangle$. Because cyclic subgroups are separable, there exists a finite-index subgroup $L' \subset L$ so that $\gamma \in L$ but $\gamma_1 \notin L'$. Repeatedly desingularizing in this way, we eventually arrive at a covering in which γ is an embedding.

Here is a different simplification that uses related ideas. We call a tower *positive-genus* if every surface used in its construction is of positive genus.

Theorem 4.2 (Bridson, Tweedale & W.) *Every limit group is virtually a subgroup of the fundamental group of a positive-genus tower. Every elementarily free group is virtually a subgroup of the fundamental group of a positive-genus, hyperbolic tower.*

Again, the idea behind this is to find a nice covering of X_{h-1} and then pull back along the retraction ρ.

5 The pro-free topology

By Hall's theorem, it is immediate that the pro-free topology is stronger than the pro-finite topology. (That is, if a subgroup is closed in the pro-free topology then it is closed in the pro-finite topology.)

Question 5.1 *Which subgroups of limit groups are closed in the pro-free topology?*

Abelian subgroups are easily understood.

Lemma 5.2 *If G is a limit group and $A \subset G$ is maximal abelian then A is closed in the pro-free topology.*

Proof. Suppose $g \in G \setminus A$ and fix a basis $a_1, \ldots, a_n$ for G. Then $[g, a_i] \neq 1$ for all i. So there exists $f : G \to F$ so that $f([g, a_i]) \neq 1$ for all i. So $f(g) \notin f(A)$. $\square$