Limit Groups, Measure Equivalence and Subgroup Separability

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1 What are limit groups?

I'll start by trying to help everyone acclimatize to limit groups. There are many different equivalent definitions; I'll give three different points of view. The first is in a sense **topological**.

There exists a natural topology (Grigorchuk, Gromov–Hausdorff) on the set of finitely generated groups. Consider the set $\mathfrak{F}$ of finitely generated free groups, and let

$$\mathfrak{L} = \overline{\mathfrak{F}}$$

the closure of the free groups in this topology. Then $\mathfrak{L}$ is the set of limit groups.

It is therefore natural to look to generalize properties of free groups to limit groups.

Next we have an **algebro-geometric** perspective. Consider $\mathbb{F}$ a finitely generated free group and a system of equations in n variables

$$\Phi = (w_i(x_1, \ldots, x_n) = 1)_i$$

over $\mathbb{F}$. Taking the variables as generators and the equations as relations defines a group

$$G(\Phi) = \langle x_1, \ldots, x_n | w_1, w_2, \ldots \rangle.$$

Now solutions to Φ are in bijection with

$$\mathrm{Hom}(G(\Phi), \mathbb{F}).$$

Lemma 1.1 *Any finitely generated group G has a finite collection of epimorphisms to limit groups*

$$q_i : G \to L_i$$

such that every homomorphism $f : G \to \mathbb{F}$ factors through some q_i.

This gives a natural decomposition of $\mathrm{Hom}(G, \mathbb{F})$ as

$$\bigcup_i \mathrm{Hom}(L_i, \mathbb{F})$$

analogous to the usual decomposition of a variety into irreducible pieces.

Finally, I will give you a **group-theoretic** definition. A finitely generated group G is a *limit group* if, for any finite subset $S \subset G$, there exists a homomorphism

$$f : G \to \mathbb{F}$$

so that the restriction of f to S is injective.

Example 1.2 *Every finitely generated free group embeds into* $\mathbb{F}$*, so all free groups are limit groups. (In particular, the choice of* $\mathbb{F}$ *doesn't matter.)*

Example 1.3 *Finite-rank free abelian groups are limit groups.*

Example 1.4 *If* Σ *is a closed surface and* $\chi(\Sigma) < -1$ *then* $\pi_1(\Sigma)$ *is a limit group.*

How do limit groups interact with other classes of groups?

- Every pair of elements in a limit group generates a free or free abelian group. So every solvable limit group is free abelian.

- Every non-abelian freely indecomposable limit group splits over $\mathbb{Z}$. So if M is a closed 3-manifold and $\pi_1(M)$ is a limit group then M is a direct sum of copies of T^3 and $S^2 \times S^1$.

- Limit groups are *CAT(0) with isolated flats* (Alibegovic and Bestvina). In other words, they are non-positively curved in a very nice sense.

Limit groups are easily seen to be torsion-free.

2 The structure of limit groups

Solutions to the Tarski Problem by Z. Sela and, independently, O. Kharlampovich and A. Myasnikov have led to a structure theory for limit groups. The set of *tower spaces* is defined recursively as follows.

A *tower space of height 0* X_0 is a compact one-point union of graphs, n-tori and hyperbolic surfaces with $\chi < -1$.

A *tower space of height h* X_h is built from a tower X_{h-1} of height $h-1$ by attaching one of two sorts of blocks.

1. **Quadratic block.** X_h is obtained by gluing a compact hyperbolic surface Σ along its boundary components to X_{h-1}.
2. **Abelian block.** X_h is obtained by gluing a coordinate circle of an n-torus T to a non-trivial loop in X_{h-1}.

It is also required that there exists a retraction $\rho : X_h \to X_{h-1}$ satisfying certain conditions.

Theorem 2.1 (Sela, Kharlampovich–Myasnikov) *A group is a limit group if and only it is a finitely generated subgroup of the fundamental group of a tower space.*

A tower space is called *hyperbolic* if no tori are used in its construction (equivalently, if its fundamental group is Gromov-hyperbolic).

Theorem 2.2 (Sela, Kharlampovich–Myasnikov) *A group has the elementary theory of a free group if and only if it is the fundamental group of a hyperbolic tower space.*

Such groups are called *elementarily free.*

3 Positive-genus towers

I want to describe a useful simplification one can make to the structure theory of limit groups.

Definition 3.1 *A surface* Σ *with Euler characteristic* χ *and* b *boundary components is* of positive genus *if*

$$\chi + b \leq 0.$$

A tower space is positive-genus *if every surface used in its construction is of positive genus.*

Theorem 3.2 (M. Bridson, M. Tweedale, W.) *Every limit group is virtually a subgroup of the fundamental group of a positive-genus tower. If the limit group is elementarily free then the tower can also be taken to be elementarily free.*

Corollary 3.3 *Consider a class of finitely generated groups $\mathcal{C}$ with the following properties.*

- *Free groups and surface groups lie in $\mathcal{C}$.*
- *$\mathcal{C}$ is closed under free products.*
- *$\mathcal{C}$ is closed under passing to subgroups.*
- *$\mathcal{C}$ is closed under passing to finite-index supergroups.*
- *If $G \in \mathcal{C}$ and Σ is a hyperbolic surface with a single boundary component then*

$$G *_{\langle \partial\Sigma \rangle} \pi_1(\Sigma) \in \mathcal{C}.$$

Then every elementarily free group lies in $\mathcal{C}$.

Proof. By the theorem, it suffices to prove that $\mathcal{C}$ contains the fundamental groups of positive-genus towers. Cutting around a certain simple closed curve exhibits a positive-genus tower as a free product amalgamated with a hyperbolic surface with one boundary component. □

I will now try to give the idea of the proof of the theorem.

Suppose Σ is a sphere with $b(\Sigma) > 2$ boundary components. Consider a map

$$\phi : \pi_1(\Sigma) \to \mathbb{Z}/p\mathbb{Z}$$

for some prime p that maps all the boundary cycles non-trivially. Consider the p-sheeted covering

$$\hat{\Sigma} \to \Sigma$$

such that $\pi_1(\hat{\Sigma}) = \ker(\phi)$. Let δ be a boundary curve of Σ and $\hat{\delta}$ a component of its preimage in $\hat{\Sigma}$. Then $\hat{\delta} \to \delta$ is also a p-fold covering (otherwise $o(\phi(\delta)) < p$) so the preimage of δ in $\hat{\Sigma}$ only has one component. Therefore $b(\hat{\Sigma}) = b(\Sigma)$ and

$$b(\hat{\Sigma}) + \chi(\hat{\Sigma}) = b(\Sigma) + p\chi(\Sigma)$$

which is non-positive as longs as $p > 2$.

We use that $\pi_1(X_h)$ is a limit group to construct a suitable map $\pi_1(X_h) \to \mathbb{Z}/p\mathbb{Z}$ that respects the tower structure.

4 Measure Equivalence

Definition 4.1 (M. Gromov) *Finitely generated groups* G_1, G_2 *are* quasi-isometric *if they admit commuting, proper, cocompact actions by isometries on a metric space* X*.*

A group is quasi-isometric to $\mathbb{F}$ if and only if it's virtually free (J. Stallings).

Definition 4.2 (Gromov) *Groups* G_1, G_2 *are* measure equivalent *if they admit commuting measure-preserving actions on a measure space* X *with finite-measure fundamental domains.*

For example, any surface group is measure-equivalent to $\mathbb{F}$: consider actions by left- and right-multiplication on $PSL_2(\mathbb{R})$ with Haar measure. The measure-equivalence class of the free groups is still poorly understood.

Here is a summary of some properties of measure equivalence, mostly due to D. Gaboriau.

- If $G \subset H$ is a finite-index subgroup then $G \sim H$. (Again, consider left- and right-actions on H with Haar measure.)
- If $G \sim \mathbb{F}$ and $H \subset G$ then H is measure-equivalent to some free group.
- If $G_1, G_2 \sim \mathbb{F}$ then $G_1 * G_2 \sim \mathbb{F}$.
- If Σ is a surface with one boundary component and $G \sim \mathbb{F}$ then

$$G *_{\langle \partial \Sigma \rangle} \pi_1(\Sigma) \sim \mathbb{F}.$$

Question 4.3 (Gaboriau) *Are limit groups measure equivalent to free groups?*

It follows immediately from the above properties and the corollary that all elementarily free groups are measure equivalent to free groups. But for even the simplest limit groups that aren't elementarily free the answer is unknown. For example, if

$$G = \mathbb{F} *_Z \mathbb{Z}^2$$

where Z is maximal cyclic then we don't know if $G \sim \mathbb{F}$.

5 Subgroup separability

Let G be a group and H a subgroup. Then $H \subset G$ is *separable* if, whenever $g \notin H$, there exists a finite-index subgroup

$$H \subset \hat{H} \subset G$$

so that $g \notin \hat{H}$. If every finitely generated subgroup is separable then G is called *subgroup separable* (or *LERF*).

Example 5.1 (M. Hall '49) *Free groups are subgroup separable.*

Example 5.2 (Burns '71, Romanovskii '69) *Free products of subgroup separable groups are subgroup separable.*

Example 5.3 (P. Scott '78) *Surface groups are subgroup separable.*

I will explain Stallings' proof of Hall's Theorem using the topology of graphs. A map of graphs $f : \Delta \to \Gamma$ is an *immersion* if it is a local embedding; that is, an embedding in a neighbourhood of every vertex. If f is in fact a local isomorphism then f is a *covering*. An immersion looks like a piece of a covering. Remarkably, in the case of graphs every immersion arises in this way.

Proposition 5.4 *Let* Δ *and* Γ *be finite graphs and* $\Delta \to \Gamma$ *an immersion. Then* $\Delta \to \Gamma$ *extends to a finite-sheeted covering* $\hat{\Gamma} \to \Gamma$*.*

This is most easily seen when Γ is a rose. Fix an orientation and a colouring on the edges of Γ. This lifts to an orientation and a colouring on Δ. In fact, immersions $\Delta \to \Gamma$ correspond precisely to orientations and colourings on Δ.

Proof. For each colour, simply count how many edges of Δ are 'missing' going into a vertex, and how many are 'missing' going out. Summing over all vertices, we get the same number!

Pairing them up arbitrarily and filling in the missing edges gives the required cover $\hat{\Gamma}$. $\square$

We can exploit this fact to prove Hall's Theorem, because free groups are precisely the fundamental groups of graphs.

Corollary 5.5 *Free groups are subgroup separable.*

Proof. Let Γ be a finite rose, H a finitely generated subgroup of $F = \pi_1(\Gamma)$ and $\gamma \notin H$. Let $\Gamma^H \to \Gamma$ be the covering corresponding to H. Because H is finitely generated, there exists some connected subgraph Δ such that $\pi_1(\Delta) = H$. Enlarging Δ, we can assume it contains the image of the lift γ' of γ to Γ^H. But $\Delta \to \Gamma$ is an immersion, so can be completed to a finite-sheeted covering $\hat{\Gamma} \to \Gamma$. This gives $\hat{H} = \pi_1(\hat{\Gamma})$ as required. $\square$

Question 5.6 (Sela) *Are limit groups subgroup separable?*

As in the case of measure equivalence, we have a convenient list of properties. Suppose G is subgroup separable.

- If $H \subset G$ then H is subgroup separable.
- If $G \subset G'$ is a finite-index subgroup then G' is subgroup separable.
- If Σ is a hyperbolic surface with one boundary component then

  $$G *_{\langle \partial \Sigma \rangle} \pi_1(\Sigma)$$

  is subgroup separable (R. Gitik).

So we can deduce:

Theorem 5.7 *Elementarily free groups are subgroup separable.*

Unfortunately this route doesn't help much with the general case. Nevertheless, further development of Stallings' ideas leads to:

Theorem 5.8 (W.) *Limit groups are subgroup separable.*