An Introduction to Geometrization

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Preliminaries

Throughout this talk, a *manifold* is a connected, orientable, smooth manifold, possibly with boundary.

A 3-manifold M is *prime* if, whenever $M = M_1 \# M_2$, one of M_1 and M_2 is homeomorphic to S^3. M is *irreducible* if any embedded 2-sphere bounds a 3-ball. Irreducibility is slightly stronger than primeness; consider, for example, $S^2 \times S^1$.

An embedded 2-sphere is *incompressible* if it is not null-homotopic. An embedding of any other surface $\Sigma \hookrightarrow M$ is *incompressible* if it induces an injection on π_1. A surface is *essential* if it is incompressible and not isotopic to a boundary component.

Geometries

Let (X, G) be a manifold and a group of diffeomorphisms. A manifold M is *modelled on* X (or has an (X, G)-structure) if the interior of M is diffeomorphic to the quotient of X by a discrete subgroup of G acting freely and properly discontinuously.

A *geometry* (X, G) is a connected, simply connected manifold X so that:

1. there exists a complete Riemannian metric so that $G = \text{Isom}(X)$ acts transitively on X;

2. (X, G) has a compact model.

A 3-manifold is *geometric* if it is modelled on a geometry.

2-Dimensional Geometries

Recall 'the fundamental theorem of differential geometry'.

Theorem 1 *If X^n is a simply connected, complete Riemannian manifold with constant sectional curvature* $+1$, 0 *or* -1 *then X is isometric to S^n, $\mathbb{E}^n$ or $\mathbb{H}^n$ respectively.*

In particular the only homogenous, simply connected Riemannian surfaces are S^2, $\mathbb{E}^2$ and $\mathbb{H}^2$; these all have compact quotients, so they are the three 2-dimensional geometries.

The interior of every compact surface admits such a geometric structure: using the Gauss-Bonnet theorem, the structure is spherical, Euclidean or hyperbolic depending on whether the Euler characteristic is positive, zero or negative.

Some compact 3-manifolds

Example 2 *Let* Σ *be a closed hyperbolic surface. Consider* $M = U\Sigma$*, the unit-circle bundle in the tangent bundle of* Σ*.*

Example 3 *Consider a solid dodecahedron* D*. Let* M *be the 3-manifold obtained by identifying opposite sides, after a rotation of* $3\pi/5$*. The result is called 'Seifert-Weber dodecahedral space'.*

Example 4 *Consider the 2-torus* T*. For* $\phi \in SL_2(\mathbb{Z})$ *the mapping class group of* T*, let* $M = M_\phi$ *be the corresponding mapping torus.*

The 3-dimensional geometries

Of course, the first examples of 3-dimensional geometries are the isotropic ones: S^3, $\mathbb{E}^3$ and $\mathbb{H}^3$ (with the usual metrics). Seifert-Weber dodecahedral space is an example of a compact hyperbolic manifold.

The simplest examples of non-isotropic geometries are $S^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$, which are modelled by trivial circle bundles over surfaces.

By contrast, $U\Sigma$ is a non-trivial circle bundle over Σ. It's clearly a quotient of $U\mathbb{H}^2$ by a discrete group action. Since $U\mathbb{H}^2 \equiv PSL_2(\mathbb{R})$, it's modelled on the universal cover $\widetilde{SL_2}(\mathbb{R})$

The remaining geometries cover mapping tori of the torus.

Nil-**geometry**

Let Nil be the Heisenberg group, namely the space of all matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

for $x, y, z \in \mathbb{R}$. Nil expresses $\mathbb{R}^3$ as a non-trivial line bundle over $\mathbb{R}^2$. If $A \in SL_2(\mathbb{Z})$ is conjugate to a shear, then M_A has a Nil-structure.

Sol-**geometry**

Sol is the Lie group structure on $\mathbb{R}^3$ given by the multiplication

$$(x, y, t)(x', y', t') = (x + e^{-t}x', y + e^t y', t + t').$$

If $A \in SL_2(\mathbb{Z})$ is diagonalizable with distinct positive eigenvalues then M_A has a Sol-structure.

Thurston's classification

Thurston proved that the eight geometries listed above are all there are. Here is the outline of the proof, in three cases depending on S, the identity component of the (orientation-preserving) stabilizer of a point.

1. $S = SO(3)$. Then X is isotropic, and by 'the fundamental theorem of differential geometry', X is one of S^3, $\mathbb{E}^3$ or $\mathbb{H}^3$ with the usual group of isometries.

2. $S = SO(2)$. Then X fibres over a surface. If the base has positive curvature then X is either $S^2 \times \mathbb{R}$ or S^3. If the base is flat, X is either $\mathbb{R}^3$ or Nil. If the base is negatively curved, X is either $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{SL2}(\mathbb{R})$.

3. $S = 1$. Then X is identified with the identity component of G, so is a Lie group. A classification of 3-dimensional simply connected Lie groups gives that all the ones with compact quotients are subgroups of SU_2, $\mathbb{R}^3$, $\widetilde{SL_2(\mathbb{R})}$, Nil or Sol.

The Kneser-Milnor Prime Decomposition

We would like to be able to conjecture that every compact 3-manifold has a geometric structure, but there are some obvious counter-examples: almost no connected sums can be geometric. To get round this, we appeal to a result of Kneser and Milnor.

Theorem 5 *Every 3-manifold M has a decomposition as*

$$M = M_1 \# \ldots \# M_n \# S^1 \times S^2 \# \ldots \# S^1 \times S^2$$

where each M_i is irreducible. The decomposition is unique up to permutations of the pieces.

The proof relies on the theory of normal surfaces.

Another counter-example

There's another obstruction to geometric structures: neither hyperbolic nor spherical manifolds admit essential tori. For example, consider N the complement of a tubular neighbourhood of the figure-eight knot, a simple example of a hyperbolic manifold with a 'cusp'; that is, a boundary torus with Euclidean structure.

Let M be the double of N along the boundary. Then M is irreducible, and clearly can't admit any non-hyperbolic structures. But it also can't admit a hyperbolic structure, since the boundary torus of N is essential in M.

Seifert-fibred 3-manifolds

M is *Seifert-fibred* if it is foliated by circles, and each leaf has the following local structure.

- A generic leaf has a tubular neighbourhood homeomorphic to a solid torus with the obvious foliation by circles.

- A singular leaf has a tubular neighbourhood homeomorphic to a solid torus, with the foliation induced by cutting along a disc and gluing again after a twist through $2\pi p/q$ (for p, q coprime).

Seifert-fibred manifolds should be thought of as examples of manifolds with lots of essential tori.

The best way of thinking about Seifert-fibred manifolds is as circle bundles over orbifolds. Let M be Seifert-fibred, and let Σ be the leaf space. Then because of the local structure of the leaves of M, Σ has the topological type of a surface; generic leaves correspond to ordinary points, while singular leaves of type (p, q) correspond to $\mathbb{Z}_q$ cone points. That is, Σ naturally inherits an orbifold structure.

This gives two important invariants of M: χ, the *orbifold Euler characteristic* of Σ; and e, the Euler class of the bundle structure on M.

It's now easy to see the essential tori in M. Just pick an essential curve in Σ. Then its preimage in M is an essential torus. Note that, in general, essential tori may intersect essentially.

The JSJ decomposition

To remove the problem of essential tori, we appeal to a theorem of Jaco–Shalen and Johannson. A 3-manifold is *atoroidal* if it contains no essential tori.

Theorem 6 *In a closed 3-manifold M there exists a finite collection of disjoint essential tori $T_1, \ldots, T_n$, unique up to isotopy, so that every component of*

$$M - \bigcup_i T_i$$

is either Seifert-fibred or atoroidal.

In contrast to the case of the prime decomposition, there is no canonical way of closing off the resulting toral boundary components.

There is a remarkably constructive proof of the existence of this decomposition, due to Neumann and Swarup. An essential torus is *canonical* if any other embedded torus can be isotoped off it. The JSJ decomposition is defined simply to be a maximal collection of canonical tori.

Let N be a piece of the complement that isn't atoroidal. Cut it along some maximal collection of disjoint essential tori, $S_1, \ldots, S_m$. Since no S_i is canonical, for each i there exists an essential torus in N that can't be homotoped off S_i. Therefore, every component of

$$N - \bigcup_i S_i$$

contains an essential annulus. A case-by-case analysis of how the annulus intersects the boundary tori now shows that each component is Seifert-fibred. Finally, you observe that the Seifert structures on each component match up to give a Seifert-fibred structure for N.

The Geometrization Conjecture

We are now in a position to state Thurston's conjecture.

Conjecture 7 *If M is a closed 3-manifold, let $M_1, \ldots, M_n$ be the manifolds obtained by applying first the Kneser-Milnor decomposition, then the JSJ decomposition. Then each M_i is geometric.*

The Easy Pieces

Fortunately, the non-spherical and non-hyperbolic geometries are well understood.

Theorem 8 *Let M be a closed 3-manifold.*

1. *M has a* Sol*-structure if and only if M is finitely covered by the mapping torus of a hyperbolic torus automorphism.*

2. *If M is Seifert-fibred then M is modelled on one of S^3, $\mathbb{E}^3$, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$,* Nil *or $\widetilde{SL_2}(\mathbb{R})$.*

The geometry of a Seifert-fibred manifold is, as you would expect, determined by χ and e.

The Hard Pieces

The problematic pieces are the atoroidal ones. In this case, the conjecture splits into two.

Conjecture 9 (Elliptization) *Let M be a closed 3-manifold with $\pi_1(M)$ finite. Then M has a spherical structure.*

Conjecture 10 (Hyperbolization) *Let M be a compact atoroidal 3-manifold with $\pi_1(M)$ infinite. Then M has a hyperbolic structure.*

Elliptization in turn is equivalent to two famous conjectures, both very poorly understood.

Conjecture 11 (Poincaré) *Every closed simply connected 3-manifold is homeomorphic to S^3.*

Conjecture 12 (Smale) *Every finite group action on S^3 by diffeomorphisms is conjugate to an action by isometries.*

Much more progress has been made with hyperbolization. A 3-manifold is *Haken* if it contains an incompressible surface.

Theorem 13 (Thurston) *Every atoroidal Haken 3-manifold is geometric.*

Conjecturally (the Waldhausen conjecture), every 3-manifold is finitely covered by a Haken 3-manifold. The Waldhausen conjecture is one of the few big conjectures in 3-manifolds not implied by geometrization.