

# Bass-Serre Theory

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## Elementary splittings of groups

**Definition 1** Let  $A, B, C$  be groups and  $\alpha : C \rightarrow A, \beta : C \rightarrow B$  injective homomorphisms. The amalgamated free product of  $A$  and  $B$  over  $\alpha$  and  $\beta$  is the direct limit of this diagram of groups. We abuse notation and denote this

$$A *_C B.$$

**Definition 2** Now suppose  $A = B$ . Consider  $\{A_i | i \in \mathbb{Z}\}$  a collection of copies of  $A$ ,  $\{C_j | j \in \mathbb{Z}\}$  a collection of copies of  $C$ , and homomorphisms  $\alpha_i : C_i \rightarrow A_i$  coinciding with  $\alpha$ ,  $\beta_i : C_i \rightarrow A_{i+1}$  coinciding with  $\beta$ . Let  $H$  be the direct limit of this system, and  $u : H \rightarrow H$  the shift automorphism mapping  $G_i \rightarrow G_{i+1}$ . The semidirect product  $H \rtimes_u \mathbb{Z}$  is called the HNN-extension of  $A$  over  $\alpha$  and  $\beta$ , and abusively denoted

$$A *_C .$$

A more concrete picture of these constructions is given by their presentations. Let  $A = \langle G|R \rangle, B = \langle H|S \rangle$ . Then it is easy to write down a presentation for  $A *_C B$ , namely

$$\langle G, H|R, S, \{\alpha(c)\beta(c^{-1})|c \in C\} \rangle.$$

$A *_C B$  is the freest group into which  $A, B$  inject and the images of  $C$  are identified.

In the case where  $A = B = \langle G|R \rangle$ , the HNN-extension  $A *_C$  has presentation

$$\langle G, t|R, \{t\alpha(c)t^{-1}\beta(c^{-1})|c \in C\} \rangle.$$

$A *_C$  is the freest group into which  $G$  injects and the isomorphism between the images of  $C$  is realized as conjugation by an element. The element  $t$  is known as the *stable letter*.

**Example 3** The (non-abelian) free group of rank 2,  $F$ , can be decomposed in either way. It can be written as the amalgamated free product of two copies of  $\mathbb{Z}$  over the trivial group:

$$F = \mathbb{Z} *_1 \mathbb{Z} = \mathbb{Z} * \mathbb{Z}.$$

It is also the HNN-extension of  $\mathbb{Z}$  over the trivial group:

$$F = \mathbb{Z} *_1 .$$

**Example 4** Free abelian groups only decompose in one way, namely as an HNN-extension of a codimension-one subgroup by itself:

$$\mathbb{Z}^n = \mathbb{Z}^{n-1} *_\mathbb{Z}{}^{n-1} .$$

The homomorphisms  $\alpha, \beta : \mathbb{Z}^{n-1} \rightarrow \mathbb{Z}^{n-1}$  are both taken to be the identity.

By the end of this talk we will see many more less trivial examples.

## The topological perspective

Van Kampen's theorem provides the connection between these decompositions and topological decomposition.

**Theorem 5 (Van Kampen)** *Let  $X, Y, Z$  be path-connected topological spaces and  $\alpha : Z \rightarrow X$ ,  $\beta : Z \rightarrow Y$   $\pi_1$ -injective continuous maps. Then the fundamental group of the double mapping cylinder of  $\alpha, \beta$  is canonically isomorphic to the amalgamated free product*

$$\pi_1(X) *_{\pi_1(Z)} \pi_1(Y).$$

*If  $X = Y$  then the fundamental group of the mapping torus of  $\alpha \circ \beta^{-1}$  is canonically isomorphic to the HNN-extension*

$$\pi_1(X) *_{\pi_1(Z)} .$$

This gives many more examples of elementary splittings. The easiest are for surface groups.

**Example 6** *The fundamental group of a surface has many elementary splittings, which can be seen by cutting along simple closed curves. (If the surface has boundary, the curves shouldn't be boundary-parallel.)*

*If the curve is separating, the result is an amalgamated free product. If the curve is non-separating, the result is an HNN-extension.*

## Reduced words

The main technical tool for dealing with elementary splittings is the notion of a reduced word. Consider  $G = A *_C B$ . A word in the elements of  $A$  and  $B$  is *reduced* if it is of the form

$$a_1 b_1 a_2 \dots a_n b_n$$

where  $a_i \in A$ ,  $b_i \in B$ , and moreover  $a_i \notin C$  for  $i > 1$  and  $b_i \notin C$  for  $i < n$ .

**Theorem 7** *Every element of  $G = A *_C B$  can be represented as a reduced word. Moreover, the number  $n$  is unique.*

Here is a sketch of the proof

Choose sets of coset representatives for  $A/C$  and  $B/C$ . Let  $X$  be the set of reduced words of the form  $ca_1b_1a_2\dots a_nb_n$  where  $c \in C$  and  $a_i, b_i$  are chosen coset representatives. Then  $G$  acts on  $X$  by left-multiplication, and the resultant map  $G \rightarrow X$  defined by

$$g \mapsto g.e$$

provides a left-inverse to the natural map  $X \rightarrow G$ ; this gives the surjectivity of  $X \rightarrow G$  and the uniqueness of the decomposition.

For the HNN-extension  $G = A *_C$  a reduced word is of the form

$$a_0 t^{\epsilon_1} a_1 \dots a_{n-1} t^{\epsilon_n} a_n$$

where  $\epsilon_i = \pm 1$ ; furthermore, if  $\epsilon_i = -\epsilon_{i+1}$  then  $a_i \notin C$ . A similar theorem holds.

## Graphs of groups

**Definition 8** Let  $\Gamma$  be an oriented connected finite graph. For each vertex  $v$  let  $G_v$  be a group. For each edge  $e$  let  $G_e$  be a group, and let

$$\partial_e^{+1} : G_e \rightarrow G_{t(e)}$$

and

$$\partial_e^{-1} : G_e \rightarrow G_{s(e)}$$

be injective homomorphisms. The data

$$(\Gamma, G, \partial^{+1}, \partial^{-1})$$

defines a graph of groups. We often abusively denote it just by  $\Gamma$ .

**Example 9** The simplest non-trivial examples occur when  $\Gamma$  has just one edge. There are two cases:  $\Gamma$  has one vertex, and  $\Gamma$  has two vertices.

**Example 10** Let  $T$  be an oriented graph on which a group  $G$  acts simplicially and cocompactly, without edge inversions. Let  $\Gamma$  be the quotient topological space, an oriented finite graph. For each vertex  $v$  of  $\Gamma$  choose a lift  $\tilde{v} \in T$ ; set  $G_v = \text{Stab}_G(\tilde{v})$ . This choice is well defined up to isomorphism, because  $\text{Stab}_G(g.\tilde{v}) = g\text{Stab}(\tilde{v})g^{-1}$ .

For each edge  $e$  of  $\Gamma$ , choose a lift  $\tilde{e}$  in  $T$ ; set  $G_e = \text{Stab}_G(\tilde{e})$ ; again, this is well defined up to isomorphism. Moreover, the inclusions

$$\text{Stab}_G(\tilde{e}) \hookrightarrow \text{Stab}_G(t(\tilde{e})), \text{Stab}_G(s(\tilde{e}))$$

induce injective homomorphisms

$$G_e \rightarrow G_{t(e)}, G_{s(e)}$$

which are well defined up to conjugation by an element of the vertex group; these are taken to define  $\partial_e^{+1}, \partial_e^{-1}$ , respectively.

The resultant graph of groups  $\Gamma$  is called the quotient graph of groups; write  $\Gamma = T/G$ .

## The fundamental group

Let  $\Gamma$  be a graph of groups. Consider the group  $F(\Gamma)$  generated by the vertex groups  $\{G_v\}$ , and the set of edges  $\{e\}$  of  $\Gamma$ , subject to the following relations: if  $e$  is an edge and  $g \in G_e$  then

$$e\partial_e^{+1}(g)e^{-1} = \partial_e^{-1}(g).$$

Let  $c$  be a path in  $\Gamma$ , combinatorially represented by the string

$$e_1^{\epsilon_1} \dots e_n^{\epsilon_n}.$$

A *word of type*  $c$  is an element of  $F(\Gamma)$  of the form

$$g_0 e_1^{\epsilon_1} g_1 \dots g_{n-1} e_n^{\epsilon_n} g_n$$

where  $\epsilon_i = \pm 1$  and, furthermore, if  $\epsilon_i = +1$  then  $g_i \in G_{t(e)}$  and if  $\epsilon_i = -1$  then  $g_i \in G_{s(e)}$ .

Fix a vertex  $v_0$ . Then  $\pi_1(\Gamma, v_0)$  is the subgroup of  $F(\Gamma)$  consisting of words whose type is a loop based at  $v_0$ .

Graphs of groups generalize elementary splittings.

**Example 11** *Let  $\Gamma$  be a graph of groups with one edge and two vertices; let  $A, B$  be the vertex groups and  $C$  the edge group. Then*

$$\pi_1(\Gamma) \cong A *_C B.$$

*The isomorphism is given by forgetting the edge elements.*

**Example 12** *Let  $\Gamma$  be a graph of groups with one edge and one vertex; let  $A$  be the vertex group and  $C$  the edge group. Then*

$$\pi_1(\Gamma) \cong A *_C.$$

*The isomorphism is given by mapping the edge element to the stable letter of the HNN-extension.*

## An alternative description

It will be useful to be able to think about the fundamental group in a second way. Fix  $\Theta$  a maximal tree of  $\Gamma$ ; let  $E$  be the set of edges in  $\Gamma - \Theta$ . Then the *fundamental group of  $\Gamma$  relative to  $\Theta$* , denoted  $\pi_1(\Gamma, \Theta)$ , is the group generated by the vertex groups  $\{G_v\}$  and the edges in  $E$ , with the following additional relations: if  $e \in E$  and  $g \in G_e$  then

$$e\partial_e^{+1}(g)e^{-1} = \partial_e^{-1}(g).$$

Note that there is a natural map  $F(\Gamma) \rightarrow \pi_1(\Gamma, \Theta)$  given by mapping edges in  $\Theta$  to the identity.

**Lemma 13** *Let  $\Gamma$  be a graph of groups, let  $v_0$  be a vertex, and let  $\Theta$  be a maximal tree. Then the induced map*

$$\pi_1(\Gamma, v_0) \rightarrow \pi_1(\Gamma, \Theta).$$

*is an isomorphism.*

## Dévissage

The first aim is to prove a reduced word theorem for graphs of groups. Here is a very useful technical lemma.

**Lemma 14** *Let  $\Gamma$  be a graph of groups, and let  $\Gamma'$  a subgraph. Let  $\Delta$  be the graph of groups defined by contracting  $\Gamma'$  to a single vertex  $v$  and setting  $G_v = \pi_1(\Gamma')$ . Then the natural map*

$$F(\Gamma) \rightarrow F(\Delta)$$

*is an isomorphism.*

Using dévissage allows results about graphs of groups to be proved by induction.

## Reduced words in graphs of groups

A word of type  $c$  as above, of the form

$$g_0 e_1^{\epsilon_1} g_1 \dots g_{n-1} e_n^{\epsilon_n} g_n,$$

is *reduced* if the following two conditions hold:

- if  $n = 0$  then  $g_0 \neq 1$ ;
- if  $n > 0$ , whenever  $e_i = e_{i+1}$  and  $\epsilon_i = -\epsilon_{i+1}$ ,  $g_i \notin \partial_e^{\epsilon_i}(G_e)$ .

Note that this notion of being reduced coincides with the notions of being reduced in the one-edge cases. Note also that every element of  $\pi_1(\Gamma)$  can be represented by a reduced word.

**Theorem 15** Consider a reduced word  $w$  in  $F(\Gamma)$  of type  $c$ , where  $c$  is a path. Then  $w$  is not the identity.

**Sketch of proof:** The theorem follows from the observations that the inclusion map

$$F(\Gamma') \rightarrow F(\Gamma)$$

and the dévissage map

$$F(\Gamma) \rightarrow F(\Delta)$$

preserve the property of being reduced.

We already know the special case of amalgamated free products; dévissage upgrades this to trees of groups, by induction.

We also already know the special case of HNN-extensions; dévissage upgrades this to roses of groups, by induction.

Using dévissage to contract a maximal tree in an arbitrary graph of groups gives a rose of groups; the theorem follows.

## Developing graphs of groups

**Theorem 16** *Let  $\Gamma$  be a graph of groups, let  $\Theta$  be a maximal tree, and set  $G = \pi_1(\Gamma, \Theta)$ . Then there exists a tree  $T$  on which  $G$  acts without edge inversions, and*

$$\Gamma = T/G.$$

It's not hard to see what the vertices and edges of this tree must be. Let  $q : T \rightarrow \Gamma$  be the quotient map, and suppose  $\tilde{x}$  is a vertex or an edge of  $T$ . Then

$$\text{Stab}_G(\tilde{x}) = gG_{q(\tilde{x})}g^{-1}$$

for some  $g \in G$ , by construction. Therefore take as vertices the set

$$\bigsqcup_v G_v \setminus G$$

of left-cosets of vertex groups. Likewise take as edges the set

$$\bigsqcup_e G_e \setminus G$$

of left-cosets of edge groups.

The graph  $T$  is defined by attaching the edges to the vertices according to the following formulae:

$$t(gG_e) = geG_{t(e)}$$

and

$$s(gG_e) = gG_{s(e)}.$$

We are thinking of  $G$  as the fundamental group relative to a maximal tree  $\Theta$ ; in particular, the element  $e$  is the identity if  $e$  lies in  $\Theta$ .

There is an obvious map  $T \rightarrow \Gamma$  given by

$$gG_x \mapsto x$$

for  $x$  a vertex or an edge. This descends to an isomorphism of graphs; the edge groups and vertex groups are isomorphic, and the  $\partial$ -maps are equivalent up to conjugation by an element of a vertex group. It remains to see that  $T$  is a tree.

First we show  $T$  is connected. For every edge  $e$  in  $\Theta$ , there is an edge in  $T$  joining the vertices  $G_{t(e)}$  and  $G_{s(e)}$ ; therefore  $\Theta$  lifts to  $T$ , and for any pair of vertices  $u, v$  of  $\Gamma$ ,  $G_u$  and  $G_v$  are joined in  $T$ .

Consider a vertex of the form  $gG_v$  of  $T$ , for  $g \in G_u$ . Since  $G_v$  is joined to  $G_u$  it follows that  $gG_v$  is joined to  $gG_u = G_u$ .

Now consider a vertex of the form  $eG_v$  for  $e$  an edge. Then  $eG_v$  is joined to  $eG_{t(e)} = t(G_e)$ , and so to  $G_{s(e)}$ .

But the elements of the vertex groups and the edges generate  $\pi_1(\Gamma, \Theta)$ ; applying this argument inductively gives the connectedness of  $T$ .

We now show that  $T$  is simply-connected. Consider a path in  $T$ , beginning at  $G_{v_0}$ . It first crosses an edge  $g_0 G_{e_1}$  (for some  $g_0 \in G_{v_0}$ ,  $s(e_1) = v_0$ ) to  $g_0 e_1^{\epsilon_1} G_{v_1}$  where  $v_1 = t(e_1)$ . Repeating this process inductively gives a representation of the final vertex of the loop as the coset

$$g_0 e_1^{\epsilon_1} g_1 \dots e_n^{\epsilon_n} G_{v_n}.$$

The loop backtracks if and only if, for some  $i$ ,

$$g_0 e_1^{\epsilon_1} g_1 \dots e_i^{\epsilon_i} G_{v_i} = g_0 e_1^{\epsilon_1} g_1 \dots e_i^{\epsilon_i} g_i e_{i+1} G_{v_{i+1}}.$$

This only happens if  $e_i = e_{i+1}$ ,  $\epsilon_i = -\epsilon_{i+1}$ , and  $g_i \in \partial_{e_i}^{\epsilon_i} G_{e_i}$ . In other words, backtracking occurs if and only if the word is not reduced.

Now the claim that the path is a loop is equivalent to asserting that

$$g_0 e_1^{\epsilon_1} g_1 \dots e_n^{\epsilon_n} = g_n \in G_{v_0}$$

or equivalently

$$g_0 e_1^{\epsilon_1} g_1 \dots e_n^{\epsilon_n} g_n^{-1} = 1.$$

But this is a reduced word, so that can't happen.

**Application:**  $SL_2(\mathbb{Z}) \cong \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$

Recall that  $\text{Isom}(\mathbb{H}^2) \cong PSL_2(\mathbb{R})$ , so  $SL_2(\mathbb{Z})$  acts on  $\mathbb{H}^2$  in a natural way, with kernel  $\{\pm 1\}$ .  $G = PSL_2(\mathbb{Z})$  is generated by

$$z \mapsto z + 1$$

and

$$z \mapsto \frac{-1}{z}$$

and a fundamental domain for the action is given by

$$\{|z| \geq 1, |\operatorname{Re} z| \leq \frac{1}{2}\}.$$

The translates of the segment  $[i, e^{\frac{i\pi}{3}}]$  form a tree, on which  $G$  acts without edge inversions. It is easy to check that  $\text{Stab}_G(i)$  is generated by  $z \mapsto \frac{-1}{z}$  and is of order 2, while  $\text{Stab}_G(e^{\frac{i\pi}{3}})$  is generated by  $z \mapsto 1 - \frac{1}{z}$  and is of order 3.

This action on the tree gives a decomposition of  $G$  as  $\mathbb{Z}/2 * \mathbb{Z}/3$ , and of  $SL_2(\mathbb{Z})$  as  $\mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$ .