Algebraic Geometry over Groups

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Equations over free groups

Fix $\mathbb{F}$ a free (non-abelian) group of rank at least 2, and consider a finite set Φ of equations

$$w_i(x_1, \ldots, x_n) = 1$$

in n unknowns. Let $G = G(\Phi)$ be the group with presentation

$$\langle x_1, \ldots, x_n | w_i(x_1, \ldots, x_n) \rangle.$$

A solution of Φ defines a homomorphism

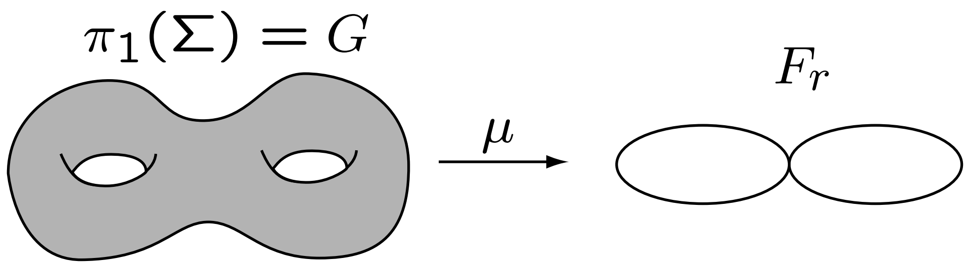
$$G \to \mathbb{F},$$

and, conversely, such a homomorphism defines a solution of Φ. So the 'variety' associated to Φ is really just $\mathrm{Hom}(G, \mathbb{F})$. This is the object we shall attempt to describe.

First examples

- $G = F_r$ the free group of rank r. Then
$$\text{Hom}(G, \mathbb{F}) \equiv \mathbb{F}^r.$$

- $G = \mathbb{Z}^r$ the free abelian group of rank r. Let $\mu : G \to \mathbb{Z}$ be projection onto the first factor. Any homomorphism $f : G \to \mathbb{F}$ decomposes as
$$G \xrightarrow{\alpha} G \xrightarrow{\mu} \mathbb{Z} \to \mathbb{F}$$
for some automorphism α. So we have an epimorphism
$$GL_r(\mathbb{Z}) \times \mathbb{F} \to \text{Hom}(G, \mathbb{F}).$$



- $G = \pi_1(\Sigma)$ the fundamental group of a closed orientable surface of genus $g > 1$, and let $\mu : G \to F_r$ be the homomorphism induced by the inclusion of Σ as the boundary in the handlebody of genus r. Then every homomorphism $G \to \mathbb{F}$ decomposes as

$$G \xrightarrow{\alpha} G \xrightarrow{\mu} F_r \to \mathbb{F}$$

for some automorphism α of G arising from an automorphism of Σ. So we have an epimorphism

$$\mathrm{Aut}(\Sigma) \times \mathbb{F}^r \to \mathrm{Hom}(G, \mathbb{F}).$$

Makanin-Razborov Diagrams

A general description of Hom$(G, \mathbb{F})$ along these lines was first given by Makanin and Razborov.

Theorem 1 (Makanin, Razborov) *To every finitely generated group G there is associated a finite tree of homomorphisms from G to $\mathbb{F}$, called a* Makanin-Razborov *diagram. Each group in the tree is a* limit group, *and each homomorphism $G \to \mathbb{F}$ factors through a branch of the diagram, after composing at each stage with automorphisms of the limit groups.*

Limit groups

There are many equivalent definitions of limit groups. This one will best suit our purposes.

Definition 2 *A group G is a* limit group *if, for any finite subset $S \subset G$, there exists a homomorphism $f : G \to \mathbb{F}$, such that $f|S$ is injective.*

Here are the simplest examples.

- Free groups

- Free abelian group

- Fundamental groups of closed surfaces of Euler characteristic less than -1

The rest of this talk is devoted to explaining the proof of theorem 1 (skating over some details). Its principle assertions are about the finiteness of the tree. The next theorem shows that the tree is only finitely long.

Theorem 3 *Let*

$$G_1 \to G_2 \to G_3 \to \cdots$$

be a sequence of epimorphisms of finitely generated groups. Then the corresponding sequence of monomorphisms

$$\text{Hom}(G_1, \mathbb{F}) \leftarrow \text{Hom}(G_2, \mathbb{F}) \leftarrow \text{Hom}(G_3, \mathbb{F}) \leftarrow \cdots$$

eventually stabilizes.

The proof of theorem 3 makes use of a little classical algebraic geometry.

Theorem 4 (Hilbert's Basis Theorem) *If R is a Noetherian ring then the polynomial ring $R[x]$ is also Noetherian.*

In particular, every descending sequence of algebraic varieties

$$X_1 \supset X_2 \supset X_3 \supset \dots$$

eventually terminates.

Proof of theorem 3: Embed $\mathbb{F} \hookrightarrow SL_2(\mathbb{R})$. (For example, a hyperbolic metric on a punctured sphere gives an embedding $\mathbb{F} \hookrightarrow PSL_2(\mathbb{R})$. This lifts to $SL_2(\mathbb{R})$.) This induces an embedding

$$\text{Hom}(G, \mathbb{F}) \to \text{Hom}(G, SL_2(\mathbb{R})).$$

Fix a presentation

$$G = \langle g_1 \dots g_m | r_1, r_2, \dots \rangle.$$

A homomorphism $f : G \to SL_2(\mathbb{R})$ is just a choice of values for the $f(g_i)$ such that the relations $f(r_j)$ are satisfied. In other words,

$$\text{Hom}(G, SL_2(\mathbb{R})) \hookrightarrow SL_2(\mathbb{R})^m$$

as a subvariety. (I think Richard would rather I said sub-scheme.) By Hilbert's Basis Theorem, the resulting decreasing sequence of varieties eventually stabilizes. **QED**

The remainder of the proof of theorem 1 consists of showing that the diagram is finitely wide.

Definition 5 *Let G be a finitely generated group. A* factor set *is a finite set of proper quotients*

$$\{q_i : G \to L_i\}$$

such that any homomorphism $f : G \to \mathbb{F}$ factors as

$$G \xrightarrow{\alpha} G \xrightarrow{q_i} L_i \to \mathbb{F},$$

where α is a 'modular' automorphism of G.

I won't define modular automorphisms, but if G isn't a limit group then the group of modular automorphisms is trivial.

Theorem 6 *Every non-free finitely generated group has a factor set*

$$\{q_i : G \to L_i\}$$

with each L_i a limit group.

A nice reduction

There's a nice observation that reduces theorem 6 to the case of limit groups straight away. Suppose G is *not* a limit group. Then there exist elements $g_1, \ldots, g_n$ such that any homomorphism $f : G \to \mathbb{F}$ kills one of the g_i. Now

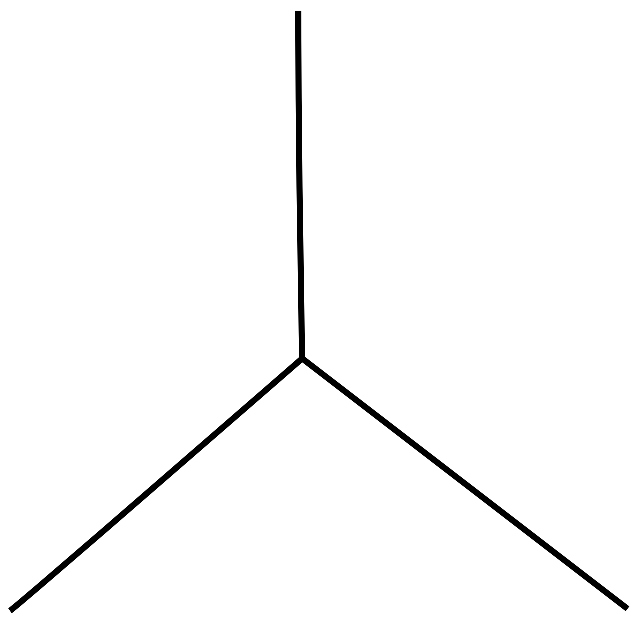
$$\{q_i : G \to L_i = G_i/\langle\langle g_i \rangle\rangle\}$$

is a factor set for G.

Metric trees

Metric trees (also known as $\mathbb{R}$-trees) generalize the usual (simplicial) notion of tree. A metric space is *geodesic* if every pair of points are joined by an isometrically embedded interval.

Definition 7 *A* metric tree *is a geodesic metric space* (T, d) *in which every geodesic triangle is isometric to a tripod.*



Simplicial trees are clearly metric trees. Here's a non-simplicial example.

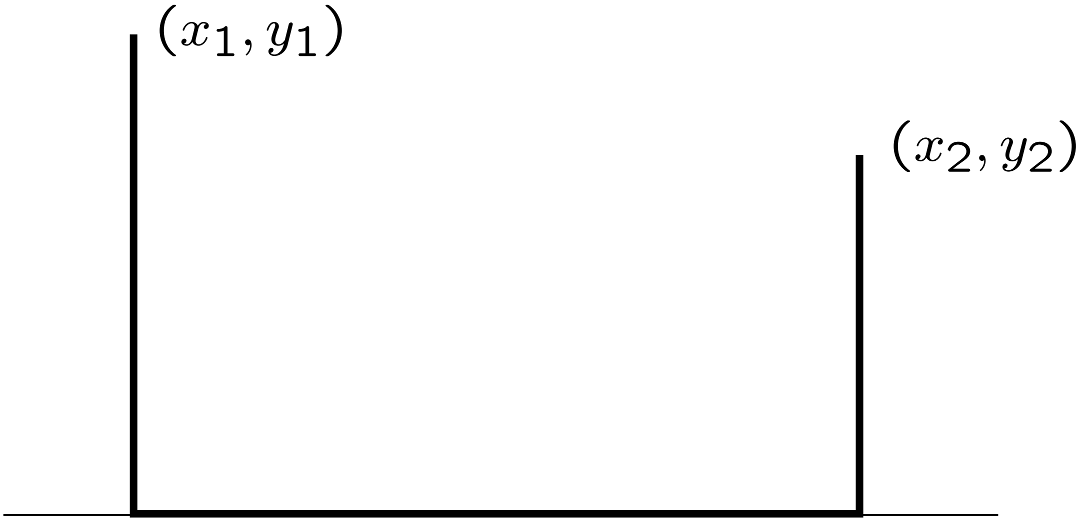
Example 8 (The SNCF metric) *Consider the metric on* $\mathbb{R}^2$ *given by*

$$d((x, y_1), (x, y_2)) = |y_1 - y_2|$$

and

$$d((x_1, y_1), (x_2, y_2)) = |y_1| + |x_1 - x_2| + |y_2|$$

for $x_1 \neq x_2$.



G-trees

A metric tree equipped with an action of a finitely generated group G by isometries is called a *G-tree*. Here we review a few of the basics of the theory of group actions on trees.

A G-tree T is *trivial* if there is a point of T fixed by G.

T is *minimal* if it contains no proper G-invariant subtrees.

Lemma 9 *Every non-trivial G-tree contains a unique minimal subtree, which is a countable union of lines.*

Cayley graphs

Let G be a group, and S a generating set. Then the *Cayley graph* of G with respect to S is the graph with vertex set G and an edge (g, h) if

$$h = gs$$

for some $s \in S$. The Cayley graph has a G-action inherited from left-multiplication by G, and a G-invariant metric given by counting the number of edges in the shortest path.

Example 10 *Loops in the Cayley graph correspond to relations between the generators. So a group has a Cayley graph which is a tree if and only if it's free.*

Fix a generating set for $\mathbb{F}$, such that its Cayley graph T is a tree. Then a homomorphism $f : G \to \mathbb{F}$ induces an action of G on T, where

$$g : t \mapsto f(g)t.$$

Denote the minimal G-invariant subtree of T by T_f.

The space of trees

Let $\mathcal{A}(G)$ be the set of non-trivial minimal G-trees. It can be endowed with a topology, known as *equivariant Gromov-Hausdorff topology*. I won't give details of this topology here.

Let $\mathbb{P}\mathcal{A}(G)$ be the quotient space arising from identifying (T, d) with $(T, \lambda d)$ for all $\lambda > 0$. The space of interest is

$$\mathcal{T}(G) \subset \mathbb{P}\mathcal{A}(G)$$

the closure of $\{T_f | f \in \text{Hom}(G, \mathbb{F})\}$, the subspace of G-trees arising from homomorphisms to $\mathbb{F}$.

Strategy

The strategy for proving theorem 6 is now approximately as follows.

1. Show that $\mathcal{T}(G)$ is compact.

2. Apply compactness to the open cover
$$\mathcal{U} = \{U(k) | k \in G - \{1\}\}$$
where $U(k) = \{T | k \in \ker T\}$.

The theorem would then follow; for by compactness, $\mathcal{T}(G)$ is covered by
$$U(k_1), \ldots, U(k_n).$$
In particular, each homomorphism $f : G \to \mathbb{F}$ factors through one of
$$q_i : G \to L_i = G/\langle\langle k_i \rangle\rangle.$$
The slickest way to show compactness uses a technique of non-standard analysis pioneered by Gromov.

Ultralimits

An *ultrafilter* ω is a finitely additive set function on $\mathbb{N}$, such that for every $S \subset \mathbb{N}$, $\omega(S) \in \{0, 1\}$. An ultrafilter is *principal* if any finite subset $S \subset \mathbb{N}$ has $\omega(S) = 1$.

Fix ω a non-principal ultrafilter (existence requires the axiom of choice). Let X be a topological space, and $x_n \in X$. Then $x = \lim_\omega x_n$ is the *ultralimit* of x_n if, for every open neighbourhood U of x,

$$\omega\{n \in \mathbb{N} | x_n \in U\} = 1.$$

Lemma 11 *If X is a compact space then every sequence has an ultralimit.*

Ultraproducts

Let (X_n, d_n, x_n) be a sequence of pointed metric spaces. Let

$$Y \subset \prod X_n$$

be the subspace consisting of sequences (y_n) with $d_n(x_n, y_n)$ bounded. Then Y inherits a pseudo-metric given by

$$D((y_n), (z_n)) = \lim_{\omega} d_n(x_n, y_n).$$

The *ultraproduct* of the sequence (X_n, d_n, x_n), denoted (X_ω, d_ω), is the associated metric space. It has the following useful properties.

Lemma 12 *Suppose all the X_n are geodesic. Then so is X_ω.*

Suppose T_n is a sequence of trees. Then so is T_ω.

If each T_n admits a G-action then the induced action on Y descends to T_ω. Furthermore, a sequence of G-trees converges to its ultralimit in the equivariant Gromov-Hausdorff topology.

It remains to show that T_ω is non-trivial: then we can pass to the minimal invariant subtree. This is done by carefully choosing the base-point and scale factor.

Fix a generating set S for G, and define $\sigma_n : T_n \to \mathbb{R}$ by

$$\sigma_n(x) = \max_{g \in S} d_n(x, gx).$$

Let $\delta_n = \inf_{x \in T} \sigma_n(x)$, and choose $x_n \in T_n$ to minimize σ_n. Now modify T_n by dividing the metric by δ_n. Let $t = [(t_n)] \in T_\omega$. For each t_n there exists $g \in S$ with

$$d_n(t_n, gt_n) \geq \sigma_n(x_n) = 1$$

so, by construction, for some $g \in S$,

$$d_\omega(t, gt) \geq 1.$$

Short automorphisms

The first part of the strategy is now complete. If the second part worked, then we could get away without modular automorphisms. The problem is that $\mathcal{U}$ doesn't cover $\mathcal{T}(G)$.

Fix a basis S for G. For $f : G \to \mathbb{F}$, define

$$|f| = \max_{g \in S} l(f(s))$$

where l is word length in $\mathbb{F}$. A homomorphism f is *short* if

$$|f| < |i_c \circ f \circ \alpha|$$

for all $c \in \mathbb{F}$ and modular automorphisms α. The key is the following tricky theorem of Sela.

Theorem 13 *For a sequence of short automorphisms* $f_n : G \to \mathbb{F}$ *with* T_{f_n} *converging to* T*, the limit action on* T *is not faithful.*

Part 2 of our strategy now works, after restricting attention to

$$\mathcal{T}'(G) \subset \mathcal{T}(G)$$

the closure of the set of G-trees arising from short homomorphisms to $\mathbb{F}$. This completes the proof of theorem 6, and so theorem 1.

Further directions

This technique has proved very open to generalization, particularly in describing Hom(G, H) for other groups H.

- Sela has extended his work to cover *word hyperbolic* groups: groups whose Cayley graphs have uniformly thin triangles.

- Alibegovic has constructed Makanin-Razborov diagrams relative to limit groups.

- Groves is working on a series of papers which would generalize both of these, extending Sela's techniques to groups that are *hyperbolic relative to their maximal abelian subgroups*.