Notes from Sela's Fall 2007 course at MSRI

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November 4, 2007

1 Introduction to model theory over groups

1.1 Basic logical notions

We begin with first order theory over groups. The first order theory over a group is composed of well formed sentences in the language of groups:

- Variables, such as x, y, z, a, b, c, or finite tuples of variables: $\underline{x} = (x_1, \ldots, x_n)$
- Concatenation of variables representing multiplication: xy
- Quantifiers: $\forall$, $\exists$
- Logical operations: $\wedge$ and $\vee$
- Parentheses, equals, doesn't equal: (and), $=$, and $\neq$.
- And if sentences are to be interpreted over a fixed group G, elements of the group: $g \in G$. If no such terms appear then the sentence is *coefficient free*.
- Since every group has an identity element, we reserve the symbol '1' to indicate it.

For example, the following are sentences in the language of groups:

- $\forall\, x, y\, [x, y] = 1$
- $\exists\, \underline{x}(w(\underline{x}) = 1) \vee (v(\underline{x}) \neq 1)$

and in general

$$\exists \underline{x}_1 \forall \underline{x}_2 \exists \underline{x}_3 \cdots (\Sigma_1(\underline{x}_i) = 1 \wedge \Psi_1(\underline{x}_i) \neq 1) \vee \cdots$$

Definition 1.1 (Definable sets)**.** If a variable x is not bound to a quantifier then it is *free*. Let Q be a sentence, G the group Q is in the language of, and let $\underline{p}$ be the tuple of free variables appearing in Q. Then the definable set associated to Q is the set of values in $G^{|\underline{p}|}$ such that $Q(\underline{p})$ is true.

The following are examples of formulae with free variables which give definable sets:

$$Q(p) \equiv \exists x, y(p = [x, y])$$
$$Q(p) \equiv (\exists x_1, \ldots, x_4(p = [x_1, x_2]\,[x_3, x_4])) \wedge (\forall y_1, y_2(p \neq [y_1, y_2]))$$

The previous formulae are coefficient-free.

Our goal in this course is to say something about the geometric structure of definable sets.

1.2 Basic definable sets

Basic sets are *varieties*, and correspond to solutions to systems of equations, e.g.,

$$Q(\underline{p}) \equiv (\Sigma(\underline{p}, \underline{a}) = 1)$$

Where the tuple $\underline{a}$ is a collection of constants from G.

Over an algebraically closed field k, we have the following theorem due to Tarski and Seidenberg(sp?):

Theorem 1.1 (Tarski–Seidenberg[**?**])**.** *Definable sets over k are in the boolean algebra of basic sets.*

Theorem 1.1 is *not* true in general.

Let Ω be a collection of words in the variables $\underline{x}$ and their inverses. Then, tautologically, there is a one-to-one correspondence between solutions to Ω and homomorphisms from $F(\underline{x})/\langle\langle\Omega\rangle\rangle$.

Let p_1, p_2 be elements of $\pi_1(\Sigma_g)$, the orientable surface of genus g. Now consider the definable set determined by

$$Q(p_1, p_2) \equiv \exists x_1 \cdots, x_{2g}(p_1 = \omega_1(\underline{x})) \wedge (p_2 = \omega_2(\underline{x})) \wedge ([x_1, x_2] \cdots [x_{2g-1}, x_{2g}] = 1)$$

Then $Q(p_1, p_2)$ is likely not in the boolean algebra of basic sets if ω_1 and ω_2 are sufficiently complicated, and the Zariski closure of Q *is* $\mathrm{Hom}(\Sigma_g, F)$.

1.3 Aims of the course

1. Properties of definable sets and their families.

   Remark on model theory: Model theorists: totally abstract, how to abstract notions of logic correspond to concrete objects.

   Model theory: Theory, in "nature" .... study models like those of free groups. Then perhaps other objects have the same theories, e.g. $\mathbb{F} \to \cdots$.

2. Are certain sets definable or not? Example:

   $$\mathrm{Gen}(\mathbb{F}_k) = \{(f_1, \cdots, f_k) | \langle f_1, \cdots, f_k \rangle = \mathbb{F}_k\}$$

   The $\mathrm{Aut}(\mathbb{F}_k)$ orbit of a standard basis.

   Question [Malcev] Is $\mathrm{Gen}(\mathbb{F}_k)$ definable? If $k = 2$ then the answer is yes. f_1 and f_2 generate if and only if $[f_1, f_2] \cong [a, b]$. If $k > 2$ then the answer is no, by work of Bestvina and Feighn[**?**].

   Produce properties of definable sets that $\mathrm{Gen}(\mathbb{F}_k)$ doesn't have.

   Question [Bestvina–Feighn]: What subgroups of F are definable? (f.g.?)

Question 1 (Open question)**.** *Is there an infinite field in the first order theory of the free group? A hyperbolic group?*

$$K \text{ a field} \leftrightarrow D \text{ a definable set}$$

and two functions

$$f_1 \colon D \times D \to D \text{ representing addition}$$
$$f_1 \colon D \times D \to D \text{ representing multiplication}$$

such that f_1 and f_2 are definable. Recall that a function is definable if its graph is definable.

1.4 Structure theory for varieties

Basic Sets
Σ
V_Σ

$\Sigma \to G_\Sigma$

Tautology: $V_\Sigma \overset{1-1}{\longleftrightarrow} \mathrm{Hom}(G_\Sigma, F)$

Stallings' Example ('60s)

Theory for arbitrary system of equations. Interested in sequences of homomorphisms $\{h_k \colon G_\Sigma \to F\}$. A sequence is convergent, or stable, if

$$\forall g \in G_\Sigma\ \exists n_g (\forall n > n_g h_n(g) = 1) \vee (\forall n > n_g h_n(g) \neq 1)$$

The stable kernel of a stable sequence of homomorphisms is the normal closure of all elements which are eventually trivial:

$$\underrightarrow{\mathrm{Ker}}(h_k) = K_\infty = \{g \in G_\Sigma | h_n(g) \text{ is eventually trivial}\}$$

The quotient G_Σ/K_∞ is a *limit group*, as are all groups constructed in this manner.

Examples:

- $\mathbb{F}$
- $\mathbb{Z}^n$
- A double of a free group along a primitive element.
- Non-exceptional surface groups.

2 Lecture 2: Limit groups

2.1 Basic properties of limit groups

Let $\eta \colon G \to L = G/K_\infty$ be a limit group quotient of a finitely generated group G. The following exercises follow directly from the definitions.

1. L is finitely generated.
2. Every finitely generated subgroup of L is a limit group.
3. L is torsion-free.
4. Any pair of non-commuting elements of L generates a free subgroup.

5. If $x, y, z \in L \smallsetminus 1$ and $[x, y] = [y, z] = 1$ then $[x, z] = 1$. This property is called *commutative transitivity*.

6. If $A \subset L$ is a maximal abelian subgroup of L then A is malnormal. That is, L is *conjugacy-separated abelian* or *CSA*.

Other properties of limit groups are rather harder to prove.

7. L is finitely presented [Sel01, Gui04].

The idea behind this is as follows. Replacing L with a factor of its Grushko decomposition, we may assume that L is freely indecomposable. Any limit group has a (stable, very small) action on a real tree, and it follows by Rips theory that the (cyclic) JSJ decomposition of L is non-trivial, unless L is abelian or a surface group. We can therefore replace L with an arbitrary rigid vertex of the JSJ decomposition, and repeat. It is a theorem that this iterative process terminates.

8. L is hyperbolic relative to its maximal abelian subgroups [Ali05, Dah03].

9. L is $CAT(0)$ with isolated flats [AB06].

10. L is LERF [Wil].

2.2 Why do limit groups help us understand basic sets?

Remark 2.1. There is a partial order on the collection of limit quotients of a group G: $G \overset{\eta}{\twoheadrightarrow} L$: $L_1 \geq L_2$ if the quotient maps $\eta_1 \colon G \twoheadrightarrow L_1$ and $\eta_2 \colon G \twoheadrightarrow L_2$ can be completed to a commutative diagram with $\tau \colon L_1 \twoheadrightarrow L_2$:

$$\begin{array}{ccc} & & L_1 \\ & \overset{\eta_1}{\nearrow} & \downarrow \exists\tau \\ G & \xrightarrow{\eta_2} & L_2 \end{array}$$

There are finitely many maximal elements with respect to this partial order.

Theorem 2.1 ([Sel01], Lemma 5.5)**.** *Let G be a finitely generated group. There is a finite collection of maximal limit group quotients*

$$\eta_i \colon G \to L_i.$$

For every homomorphism $h \colon G \to F$ there exists some i and some $\nu_h \colon L_i \to F$ such that $h = \nu_h \circ \eta_i$.

We can define the Zariski topology on a variety $V \equiv \mathrm{Hom}(G, F)$, analogously to the definition in conventional algebraic geometry.

Definition 2.1. An *irreducible subvariety* of V is the image of the map

$$q^*\colon \mathrm{Hom}(L, F) \to V$$

induced by any limit group quotient $q\colon G \to L$. The closed sets of the *Zariski topology* on V are finite unions of irreducible subvarieties of V.

In this context, Theorem 2.1 can be thought of as decomposing the variety into finitely many irreducible pieces:

$$\mathrm{Hom}(G, F) = \bigcup_i \eta_i^* \mathrm{Hom}(L_i, F).$$

It follows from Guba's Theorem that this really is a topology.

In the light of Theorem 2.1, it suffices to understand $\mathrm{Hom}(L, F)$ where L is a limit group. In this context, Theorem 2.1 gives no information as there is a unique maximal limit group quotient, namely the identity homomorphism $L \to L$.

By Grushko's Theorem, we may assume that L is freely indecomposable. The next step is to consider the (cyclic) JSJ decomposition of L. The edge groups are cyclic. Some of the vertex groups are surface groups with edge groups glued to their boundary components (these vertices are sometimes called *quadratically hanging*), some vertices are abelian and the remainder are called *rigid*.

Associated to $\mathrm{JSJ}(L)$ we have a subgroup of $\mathrm{Aut}(L)$, denoted $\mathrm{Mod}(L)$ and generated by:

(i) Dehn twists in the edge groups of $\mathrm{JSJ}(L)$;

(ii) modular automorphisms of quadratically hanging vertices of $\mathrm{JSJ}(L)$ that fix the incident edge groups up to conjugacy;

(iii) linear automorphisms of abelian vertices of $\mathrm{JSJ}(L)$ that fix the incident edge groups.

(iv) Inner automorphisms.

As $\mathrm{JSJ}(L)$ is never trivial, it follows that $\mathrm{Mod}(L)$ is never trivial. The aim now is to understand elements of $\mathrm{Hom}(L, F)$, up to the action of $\mathrm{Mod}(L)$.

Theorem 2.2 ([Sel01], Proposition 5.6)**.** *Let L be a freely indecomposable limit group. There is a finite collection of limit group quotients*

$$\eta_j \colon L \to M_j$$

such that, for every homomorphism $h \colon L \to F$ there exists some j, some $\phi \in \mathrm{Mod}(L)$ and some $\nu_h \colon M_j \to F$ such that $h = \nu_h \circ \eta_j \circ \phi$.

We can iterate this procedure to construct a tree of limit group quotients such that every homomorphism to F factors through some branch of the tree. By Guba's Theorem and Koenig's lemma, this tree is finite and each leaf can be taken to be a free quotient. The tree is called a *Makanin–Razborov Diagram*, or MR *diagram*. A branch of the MR diagram is a *resolution*.

3 Lecture 3: Equations with parameters

Example 3.1 (One variable systems of equations)**.** Consider the following one variable system of equations:

$$\Sigma = \{\omega(x) = x^{\pm 1} c_1 \cdots x^{\pm 1} c_n = 1\}$$

The c_i's lie in the coefficient group F_k and the group G_Σ is given by the presentation

$$G_\Sigma = \langle F_k, x \mid \omega \rangle$$

Solutions to Σ correspond to homomorphisms $G_\Sigma \to F_k$ whose restrictions to the subgroup F_k are the identity. If G_Σ is not free of rank $k+1$ there are finitely many *sporadic* solutions $h_1, \ldots, h_s \colon G_\Sigma \twoheadrightarrow F_k$ and finite collection of limit group quotients

$$\eta_i \colon G_\Sigma \twoheadrightarrow F_k *_{\langle \lambda \rangle} (\langle \lambda \rangle \oplus \langle t \rangle)$$

through which every solution to Σ factors. The variable x can be solved for as $\alpha_i t \beta_i$ and one quickly sees that all non-sporadic solutions are of the form $\alpha_i \lambda^l \beta_i$.

Let

$$\Sigma(\underline{x}, \underline{p}) = \begin{cases} \omega_1(x_1, \ldots, x_n, p_1, \ldots, p_m) & = & 1 \\ & \vdots & \\ \omega_1(x_1, \ldots, x_n, p_1, \ldots, p_m) & = & 1 \end{cases}$$

be a system of equations with parameters. Define G_Σ and V_Σ as before. What is V_Σ for a fixed value of $\underline{p}$? Pictorially, we represent V_Σ as a bundle whose base space is the set of parameters $\{\underline{p}\}$, and for each fixed value $\underline{p}_0$ of $\underline{p}$, a fiber $V_{\Sigma(\underline{x},\underline{p}_0)}$, where

$$V_{\Sigma(\underline{x},\underline{p}_0)} = \{f \colon G_\Sigma \to F \mid f(\underline{\hat{p}}) = \underline{p}_0\}$$

For each specialization of the parameters $\underline{p}$ there is a Makanin-Razborov diagram encoding all solutions to Σ which map $\underline{p}$ to the given specialization. To understand solutions to Σ we need to see how the MR diagram changes for different values of $\underline{p}$. For this we introduce the *relative* MR *diagram*. Before we do this, the relative JSJ–decomposition and the relative modular group need to be introduced.

Let $L(\underline{x}, \underline{p})$ be a limit group with parameter subgroup $\underline{p}$. The relative JSJ decomposition is the JSJ–decomposition associated to the family of all abelian splittings of L in which the subgroup generated by the variables of $\underline{p}$ act elliptically. The relative modular group, $\mathrm{Mod}(L; \underline{p})$ is the group of modular automorphisms of L generated by all Dehn twists in one-edged splittings visible in the relative JSJ decomposition.

To a limit group L without parameters, a finite collection of *maximal shortening quotients* was constructed. This collection had the property that every homomorphism $L \to \mathbb{F}$ factored through an element of the collection, possibly after precomposition with a modular automorphism of L. In the relative case, we proceed as in the unrestricted case, constructing a set of maximal shortening quotients M_i of L such that every homomorphism $L \to \mathbb{F}$ factors through some M_i, possibly after precomposition with an element of $\mathrm{Mod}(L, \underline{p})$. Since the modular group used is not the full modular group, it is no longer possible to guarantee that the M_i are proper quotients of L. Such difficulties fall into two types:

- JSJ$(L, \underline{p})$ is trivial
- JSJ$(L, \underline{p})$ is nontrivial

Relative limit groups falling into the first class are called *rigid*, and those falling into the second are called *solid*.

We handle the second case first. Let $Rgd(x, p)$ be a rigid limit group. Then there exist finitely many *flexible quotients* $Flx_1(x, p), \ldots, Flx_m(x, p)$ such that for all homomorphisms $f \colon Rgd(x, p) \to \mathbb{F}$, where p is mapped to some fixed value of the parameter $\underline{p}$, f either factors through some $\eta_i \colon Rgd(x, p) \to Flx(x, p)$ or f is one of finitely many *sporadic solutions*. The number of sporadic solutions doesn't depend on the value of the parameter p. Moreover, the flexible quotients have nontrivial relative modular groups. If there are no flexible quotients, then for all values of p the number of sporadic solutions is bounded independently of p, and the fiber $V_{\Sigma(\underline{x},\underline{p}_0)}$ is a finite collection of points.

¡picture¿

Sets of solutions given by solid limit groups can be given a similar description. As before, there are finitely many flexible quotients $\eta_i\colon Sld(x,p) \twoheadrightarrow Flx(x,p)$ and finitely many families of sporadic solutions, i.e., every $f\colon Sld(x,p) \to \mathbb{F}$ either factors through some η_i after precomposition by an element of $\mathrm{Mod}(Sld(x,p);p)$ or is contained in one of the sporadic families of solutions. Family must be defined carefully, as can be seen in the example $\langle x_1, x_2; p = [x_1, x_2]\rangle$.... As before, the flexible quotients have nontrivial relative JSJ decompositions.

¡picture¿

With solid, rigid, and flexible limit groups and the above facts, the relative MR diagram can be constructed exactly as in the unrestricted case, bearing in mind that at each solid or rigid limit group appearing in the diagram, there are uniformly finitely many (families of) sporadic homomorphisms.

¡picture¿

One variable example...

4 Lecture 4: Merzlyakov's Theorem

4.1 Aside: the structure of definable sets

Relative Makanin–Razborov diagrams define a stratification of the set of parameters. A stratum is a set of parameters for which the relative Makanin–Razborov diagram looks the same. To be precise, parameters $\underline{p}_1$ and $\underline{p}_2$ are equivalent if

Make all this precise.

1. the number of exceptional solutions is the same;
2. the same resolutions exist; and
3. the same degenerations exist.

There are only finitely many strata.

Why?

Given a predicate $Q(\underline{p})$, there is a finite number of bundles such that the set defined by $Q(\underline{p})$ is a finite union of strata in these bundles. Each stratum is in the Boolean algebra of AE-sets, so it follows that $Q(\underline{p})$ is also in the Boolean algebra of AE-sets.

Bundles are varieties?

4.2 1-quantifier sets

A *universal sentence* is of the form

$$\exists\, \underline{x}\; \Sigma_i(\underline{x}) = 1 \wedge \Psi_j(\underline{x}) \neq 1$$

Makanin [Mak84] proved that the universal theory—the set of sentences with only one quantifier—of a free group is decidable. A *universal set* is defined by a predicate of the form

$$Q(\underline{p}) = \{\exists \underline{x} \mid \Sigma_i(\underline{x}, \underline{p}) = 1 \wedge \Psi_j(\underline{x}, \underline{p}) \neq 1\}.$$

Likewise, a *positive sentence* is one with no inequalities. Makanin also showed that the positive theory is decidable, by reducing it to the universal theory.

4.3 AE sentences

An *AE sentence* is of the form

$$\forall\, \underline{y}\; \exists\, \underline{x}\; \Sigma_i(\underline{x}, \underline{y}) = 1 \wedge \Psi_j(\underline{x}, \underline{y}) \neq 1.$$

To start with, we will restrict ourselves to the positive case:

$$\forall\, \underline{y}\; \exists\, \underline{x}\; \Sigma_i(\underline{x}, \underline{y}) = 1.$$

Merzlyakov's theorem is a sort of implicit function theorem that provides a proof that a given positive AE sentence is true. The truth of such a positive sentence can be rephrased in terms of the following extension problem. Let $G_\Sigma = \langle \underline{x}, \underline{y} \mid \Sigma_i(\underline{x}, \underline{y}) \rangle$ and let ν be the natural map $F(\underline{y}) \to G_\Sigma$. For every homomorphism $h : F(\underline{y}) \to F_k$ there exists $\hat{h} : G_\Sigma \to F_k$ such that $h = \hat{h} \circ \nu$.

$$\begin{array}{ccc} G_\Sigma & & \\ \nu \uparrow & \searrow^{\exists \hat{h}} & \\ F(\underline{y}) & \xrightarrow{\forall h} & F_k \end{array}$$

Such a sentence certainly exists if there exists a retraction $\eta : G_\Sigma \to F(\underline{y})$ (by a retraction we mean a left inverse to ν). Merzlyakov's theorem asserts that this is always the case.

Theorem 4.1 (Merzlyakov, '66)**.** *If the sentence*

$$\forall\, \underline{y}\; \exists\, \underline{x}\; \Sigma_i(\underline{x}, \underline{y}) = 1$$

is true then there exists $\eta : G_\Sigma \to F(\underline{y})$ such that $\eta \circ \nu$ is the identity on $F(\underline{y})$.

The theorem only holds for *positive* sentences. Consider the case with inequations:

$$\forall\, \underline{y}\, \exists\, \underline{x}\, \Sigma_i(\underline{x}, \underline{y}) = 1 \wedge \Psi_j(\underline{x}, \underline{y}) \neq 1.$$

We can consider the Ψ_j to be elements of G_Σ. The formal solution η fails to verify the sentence for the set of homomorphisms h such that $h \circ \eta(\Psi_k) = 1$ for some k.

Writing $\eta(\underline{x}) = \underline{x}(\underline{y})$, this set of unverified homomorphisms corresponds to the union of the varieties $V_{\Phi_k} = \{\underline{y} \mid \Phi_k(\underline{y}) = 1\}$ where $\Phi_k(\underline{y}) = \Psi_k(\underline{x}(\underline{y}), \underline{y})$. Therefore we now need to verify the sentence

$$\forall\, \underline{y} \in V_{\Phi_k}\, \exists\, \underline{x}\, \Sigma_i(\underline{x}, \underline{y}) = 1 \wedge \Psi_j(\underline{x}, \underline{y}) \neq 1.$$

So the question becomes 'Does Merzlyakov's theorem hold when $\underline{y}$ is restricted to some variety?'

Example 4.1. Consider a limit group defined as the amalgam of two rigid vertices that surjects a surface group Σ_k and thence the free group F_k:

$$L = U_1 *_{\langle w \rangle} U_2 \to \Sigma_k \to F_k.$$

Let $U_i = \langle \underline{y}_i \rangle$ and let $\Sigma_k = \langle \underline{z} \mid \prod_j [z_{2j}, z_{2j+1}] \rangle$. The homomorphism $L \to \Sigma_k$ can be pre-composed with a Dehn twist in w. One therefore has the sentence

$$\forall\, \underline{y}\, \exists\, t, \underline{z}\ ([t, w] = 1) \wedge (\underline{z} = \underline{z}(\underline{y}_1, t\underline{y}_2 t^{-1})) \wedge (\prod_j [z_{2j}, z_{2j+1}] = 1).$$

There is no implicit relation between the values of $\underline{y}_i$ in L and their values in Σ_k.

5 Lecture 5: Completions

5.1 Examples

Example 4.1 shows that one cannot expect a naive version of Merzlyakov's Theorem to hold. To state a version that does hold, we need to use the notion of a completion of a resolution. Rather than give a general definition here, we will give some examples that cover the main cases.

Example 5.1. Let $L = V_1 *_{\langle w \rangle} V_2$ where V_1 and V_2 are rigid. Consider a resolution

$$L \xrightarrow{\tau} F_k$$

and assume for simplicity that some $\tau(w)$ has no proper roots in F_k. Then the associated *completion* is defined to be

$$M = F_k *_{\langle \tau(w) \rangle} (\langle \tau(w) \rangle \times \langle t \rangle)$$

There is a natural map $L \to M$ defined by $V_1 \to \tau(V_1)$ and $V_2 \to t\tau(V_2)t^{-1}$. It is an easy exercise that L injects into M.

Example 5.2. Let $L = V_1 *_{\langle w_1 \rangle} \Sigma *_{\langle w_2 \rangle} V_2$ where V_1 and V_2 are rigid and Σ is the fundamental group of a surface with boundary. Consider a resolution

$$L \xrightarrow{\tau} F_k.$$

The associated *completion* M is constructed by amalgamating F_k with Σ along $\langle \tau(w_1) \rangle$ and $\langle \tau(w_2) \rangle$. Again, L injects into M.

For a general resolution

$$L_1 \to L_2 \to \ldots \to L_n \to F_k$$

one constructs the completion for L_1 inductively from the completion for L_2.

Remark 5.1. There is a subtlety relating to abelian vertices, as Merzlyakov's Theorem does not hold for abelian groups.

Definition 5.1. A limit group obtained as the completion of resolution is called an *(ω-residually free) tower.*

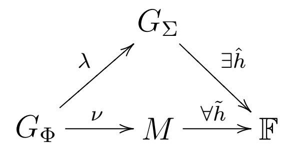
5.2 The Generalized Merzlyakov's Theorem

Using completions one can state a generalized version of Merzlyakov's Theorem. Again, we start by considering positive sentences, of the form

$$\forall\, \underline{y} \in V_\Phi\ \exists\, \underline{x}\ \Sigma_i(\underline{x}, \underline{y}) = 1$$

for $V_\Phi = \{\underline{y} \mid \Phi_k(\underline{y}) = 1\}$ a variety.

Just as before, we can rephrase the sentence in terms of an extension problem. As usual we can write $V_\Phi \equiv \mathrm{Hom}(G_\Phi, \mathbb{F})$ for $G_\Phi = \langle\ \underline{y} \mid \Phi(\underline{y})\ \rangle$. Consider the set of resolutions in the Makanin–Razborov diagram for G_Φ, and let $\{M\}$ be the (finite) set of completions of all these resolutions. So there is a collection of maps $\{\nu : G_\Phi \to M\}$ such that every homomorphism $G_\Phi \to \mathbb{F}$ factors through some ν. Consider the set of all $h \in V_\Phi$ that factor through some fixed ν—that is, those h for which there exists $\tilde{h} \in \mathrm{Hom}(M, \mathbb{F})$ such that $h = \tilde{h} \circ \nu$. Let $G_\Sigma = \langle\, \underline{x}, \underline{y} \mid \Sigma_i(\underline{x}, \underline{y}), \Phi_k(\underline{y}) \,\rangle$. We have the diagram Is this right?



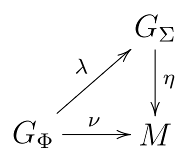
where λ is the natural map $G_\Phi \to G_\Sigma$. Once again, the truth of the sentence is equivalent to the assertion that for every $\tilde{h}$ there exists $\hat{h}$ making the diagram commute.

In particular, the extension $\hat{h}$ certainly always exists if there happens to exist a homomorphism $\eta : G_\Sigma \to M$ making the diagram commute. The Generalized Merzlyakov's Theorem asserts that such an η always exists when the sentence is true.

Theorem 5.1 (Generalized Merzlyakov's Theorem)**.** *If the sentence*

$$\forall\, \underline{y} \in V_\Phi\; \exists\, \underline{x}\; \Sigma_i(\underline{x}, \underline{y}) = 1$$

is true then there exists a homomorphism $\eta : G_\Sigma \to M_i$ such that the diagram



commutes.

Remark 5.2. Some points should be emphasized.

1. The statement of the theorem is not quite accurate. Because of the difficulty with abelian groups, one really needs to use the *closure* of the completion.

2. As before, we think of η as a 'formal solution'.

3. Note that η is not a retraction, in general.

4. The map η is in general not unique. However, one can describe the set of all η's, using a Makanin–Razborov diagram.

Let us now consider a general AE-sentence with inequalities, of the form

$$\forall\, \underline{y} \in V_\Phi\; \exists\, \underline{x}\; (\Sigma_i(\underline{x}, \underline{y}) = 1) \wedge (\Psi_j(\underline{x}, \underline{y}) \neq 1).$$

We can take the completion M to have a presentation of the form

$$M \cong \langle\, \underline{y}, \underline{z} \mid \Phi_k(\underline{y}), \Xi_l(\underline{x}, \underline{z}) \,\rangle.$$

The formal solution provided by the Generalized Merzlyakov's Theorem verifies the truth of the sentence except for on those h for which $\tilde{h} \circ \eta(\Psi_k) = 1$ (thinking of the words Ψ_k as elements of G_Σ) for some factorization $h = \tilde{h} \circ \nu$. Writing $\eta(\underline{x}) = \underline{x}(\underline{y}, \underline{z})$, the unverified set corresponds to

$$D = \{\underline{y} \in V_\Phi \mid \exists\, \underline{z}\ \Psi(\underline{x}(\underline{y}, \underline{z}), \underline{y}) = 1\}.$$

It therefore remains to verify the sentence

$$\forall\, \underline{y} \in D\ \exists\, \underline{x}\ (\Sigma_i(\underline{x}, \underline{y}) = 1) \wedge (\Psi_j(\underline{x}, \underline{y}) \neq 1)$$

where D is a Diophantine set.

5.3 Diophantine sets

Recall that a variety is a set of the form

$$V_\Phi = \{\underline{y} \in \mathbb{F}^n \mid \Phi(\underline{y}) = 1\}.$$

A *Diophantine set* is of the form

$$D_\Sigma = \{\underline{y} \in V_\Phi \mid \exists\, \underline{z}\ \Sigma(\underline{y}, \underline{z}) = 1\}.$$

Let $G_\Phi = \langle\, \underline{y} \mid \Phi(\underline{y}) \,\rangle$ as usual, and let $G_\Sigma = \langle\, \underline{y}, \underline{z} \mid \Phi(\underline{y}), \Sigma(\underline{y}, \underline{z}) \,\rangle$. There is a natural map $G_\Phi \to G_\Sigma$, and D_Σ is precisely the set of homomorphisms $G_\Phi \to \mathbb{F}$ that factor through $G_\Phi \to G_\Sigma$.

In contrast to the case of varieties, there is in general no descending chain condition for Diophantine sets.

Example 5.3. Consider an infinite chain of strict inclusions of the free group of rank 2 into itself:

$$F_2 \overset{\iota_1}{\hookrightarrow} F_2 \overset{\iota_2}{\hookrightarrow} F_2 \overset{\iota_3}{\hookrightarrow} \cdots \overset{\iota_{n-1}}{\hookrightarrow} F_2 \overset{\iota_n}{\hookrightarrow} \cdots$$

The composition of inclusions $\iota_n \circ \ldots \circ \iota_1 : F_2 \hookrightarrow F_2$ defines a strictly decreasing sequence of Diophantine sets. This sequence clearly does not terminate after finitely many steps.

Moreover, the intersection of a sequence of Diophantine sets may not be itself a Diophantine set. The problem seems to be that there is no systematic measure of complexity for a Diophantine set.

Is there an example?

Of course, we are not interested in general sequences of Diophantine sets—we are interested in sequences of the form

$$\text{Tower} \to \text{Quotient} \to \text{Tower} \to \text{Quotient} \to \cdots$$

Again, there is in general no descending chain condition. It follows that we cannot naively apply the Generalized Merzlyakov Theorem iteratively, and hope to verify the truth of a sentence in finitely many steps.

Example?

5.4 The anvil

The anvil is the tool that one uses to overcome the problem that there is no descending chain condition for Diophantine sets. In general one has the set-up

$$L \overset{\nu}{\hookrightarrow} M \to N$$

where L is a limit group, M is a tower and N is the quotient $M/\langle\langle\Psi\rangle\rangle$, where the quotient is taken in the category of limit groups. We would like to know that the image of L in N is strictly simpler than L. However, in general the relation Ψ can involve all the generators of M.

Is this right?

Are there examples where L always embeds in N?

The anvil provides a way of keeping the relations in an 'envelope' of the image L. The idea is to resolve the top level of the completion M before resolving the lower levels.

To this end, consider the JSJ decomposition of L, with rigid vertices $V_1, \ldots, V_r$. Treat the images $\nu(V_1), \ldots \nu(V_r)$ as parameter subgroups in M and consider a resolution

$$M \to M_1 \to M_2 \to \ldots \to M_t$$

in the Makanin–Razborov diagram for M relative to the subgroups $\nu(V_1), \ldots \nu(V_r)$. The following lemma is the crucial technical point that makes the anvil work.

Lemma 5.1. *The map $L \to M_t$ is not injective.*

Proof. Suppose the map is injective. As L is not free, M_t must be solid (or rigid). Every splitting of M_t induces a splitting of L, so the rigid vertices $V_1, \ldots, V_n$ embed into the rigid vertices of the JSJ decomposition of M_t. Therefore the *relative* JSJ decomposition of M_t is equal to the *full* JSJ decomposition:

$$JSJ(M_t) = JSJ(M_t; V_1, \ldots, V_n).$$

But then the relative modular group of M_t is equal to the full modular group of M_t. So M_t cannot be solid (or rigid). □

This enables us to define the anvil.

Definition 5.2. Let L_t be the image of L in M_t. Let P be the completion of the (relative) resolution of M chosen above. Let Q be the completion of a resolution of L_t. Then L_t embeds in P and in Q. The *anvil* is defined as the amalgamation

How wrong is this?

$$\mathscr{A} = P *_{L_t} Q.$$

One then defines the *induced resolution* of L. Roughly speaking, the resolution begins as

$$L \to L_1 \to L_2 \to \ldots \to L_t$$

where L_i is the image of L in M_i. The resolution then continues with the chosen resolution of L_t:

How wrong is this?

$$L \to L_1 \to L_2 \to \ldots \to L_t \to L_{t+1} \to \ldots \to F_k.$$

Let R be the completion of the induced resolution of L.

Lemma 5.2. *The natural map $L \to \mathscr{A}$ extends to an embedding*

Is this right?

$$R \to \mathscr{A}.$$

The image of R is the 'envelope' of L in $\mathscr{A}$, within which the relation Ψ is confined. In this context, there *is* a descending chain condition.

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