The 3-Dimensional Geometrisation Conjecture Of W. P. Thurston

May 14, 2002

Contents

1 Introduction **2**

2 $\mathcal{G}$-manifolds **3**
2.1 Definitions and basic examples 3
2.2 Foliations 4
2.3 Contact Structures 5
2.4 The Developing Map and Completeness 6

3 Model Geometries **7**
3.1 Definition and Basic Examples 7
3.2 More Examples 8
3.3 Hyperbolic Geometry 11
3.4 Thurston's Theorem 12

4 Seifert bundles **15**
4.1 Orbifolds 15
4.2 Seifert Fibre Spaces 18

5 Geometrisation **19**
5.1 Manifold decompositions 19
5.2 Conjectures 20

1 Introduction

Let M be a compact surface with a geometric structure. The idea of geometric structure will be defined more precisely later, but for the moment it suffices to think of this as meaning that M is the quotient of some homogeneous simply-connected Riemannian manifold $\tilde{M}$ by a discrete group of isometries. It follows from a theorem of differential geometry that $\tilde{M}$ is one of $\mathbf{S}^2$, $\mathbf{E}^2$ or $\mathbf{H}^2$ depending on whether the curvature, K, is positive, zero or negative, respectively, and it can be easily seen which structure M admits. Triangulate M. By definition, for each triangle, K times its area is equal to the sum of its interior angles less π. Summing over the whole surface,

$$K\text{vol}(M) = 2\pi\chi(M)$$

so the sign of the curvature is also the sign of the Euler characteristic. In particular, the only orientable surface with spherical geometry is the sphere, the only compact orientable surface with Euclidean geometry is the torus, and all other compact orientable surfaces can only admit hyperbolic geometry.

The natural question is, is something similar possible in 3 dimensions? And the initial answer is no. However, every 3-manifold does have a canonical decomposition into primes, and every irreducible 3-manifold has a canonical decomposition, due to Johannson and Jaco-Shalen, into Seifert fibred pieces and atoroidal pieces. So the natural modification of this idea of geometrisation becomes, can we give every Seifert-fibred or atoroidal manifold a geometric structure? This is Thurston's Geometrisation Conjecture.

While the conjecture has not been completely solved, geometrisation has nonetheless proved a fruitful source of topological information. For example, unlike in the 2-dimensional case where any hyperbolic manifold admits infinitely many such structures, in higher dimensions hyperbolic manifolds exhibit behaviour known as *Mostow Rigidity.*

Theorem 1.1 *Let M, N be complete hyperbolic manifolds of finite volume and dimension greater than 2, with isomorphic fundamental groups. Then they are isomorphic as hyperbolic manifolds.*

This means that any hyperbolic invariant, like the volume, is a topological invariant and, indeed, a homotopy invariant.

This essay describes the results that lead to Thurston's conjecture, and the progress that has been made in its proof. It concentrates specifically on the 3-dimensional model geometries, and Thurston's proof that there are only 8. A compact manifold with hyperbolic structure is also explicitly constructed. Throughout, all manifolds and maps are assumed to be smooth.

2 $\mathcal{G}$-manifolds

2.1 Definitions and basic examples

A *pseudogroup* $\mathcal{G}$ on a topological space X is a set of homeomorphisms between open sets of X such that:

1. the domains of the elements of $\mathcal{G}$ cover X;
2. the restriction of an element of $\mathcal{G}$ to an open set contained in its domain is also in $\mathcal{G}$;
3. when the composition of two elements of $\mathcal{G}$ exists it is in $\mathcal{G}$;
4. the inverse of an element of $\mathcal{G}$ lies in $\mathcal{G}$;
5. if $g : U \to V$ is such that each $x \in U$ has an open neighbourhood $U_x \subseteq U$ with $g|_{U_x} \in \mathcal{G}$ then $g \in \mathcal{G}$.

A pseudogroup on a topological vector space can be thought of as the natural object containing the transition functions of a manifold. Here are some examples of pseudogroups:

1. $\mathcal{C}^0$, the pseudogroup of homeomorphisms between open subsets of $\mathbb{R}^n$;
2. $\mathcal{C}^r$, the pseudogroup of r-times differentiable maps between open subsets of $\mathbb{R}^n$ with r-times differentiable inverses;
3. $\mathcal{C}^\infty$, the pseudogroup of diffeomorphisms between open subsets of $\mathbb{R}^n$;
4. $\mathcal{H}$, the pseudogroup of holomorphic bijections between open subsets of $\mathbb{C}^n$;
5. the restrictions of the elements of any Lie Group G of homeomorphisms of X to all the open subsets generate a pseudogroup, denoted (G, X).

These make it possible to unify many different concepts of geometric structure on a manifold. An *atlas*, $\mathcal{A}$, for a topological space M is a collection of homeomorphisms from open sets in M to open sets in X, called *charts*, whose domains cover M. Two such charts, φ and ψ, are $\mathcal{G}$-*compatible* if their *transition function* $\vartheta = \psi \circ \varphi^{-1}$ lies in $\mathcal{G}$. A $\mathcal{G}$-*atlas* is an atlas for which all pairs of charts are $\mathcal{G}$-compatible, and a $\mathcal{G}$-*manifold* is a Hausdorff second countable topological space with a maximal $\mathcal{G}$-atlas $\mathcal{A}$.

For example, a topological manifold is a $\mathcal{C}^0$-manifold, a differential manifold is a $\mathcal{C}^\infty$-manifold, and a complex manifold is a $\mathcal{H}$-manifold. However, we will be most interested in (G, X)-manifolds, whose geometry is locally like that of X with isometry group G. Let $\mathbf{E}^n$ be n-dimensional Euclidean space, $\mathbf{S}^n$ the n-sphere and $\mathbf{H}^n$ n-dimensional hyperbolic space, all with the usual groups of isometries. Then a *Euclidean manifold* is an $(\mathrm{Isom}(\mathbf{E}^n), \mathbf{E}^n)$-manifold, a *spherical manifold* is an $(\mathrm{Isom}(\mathbf{S}^n), \mathbf{S}^n)$-manifold and a *hyperbolic manifold* is an $(\mathrm{Isom}(\mathbf{H}^n), \mathbf{H}^n)$-manifold. Henceforth all manifolds will be assumed to be smooth.

Usually we shall think of G as a group of isometries. But when can we do this?

Proposition 2.1 *Let G be a Lie Group acting transitively on a smooth manifold X. Then there exists a Riemannian metric on X invariant under the action of G if and only if for some (and hence any) $x \in X$ the image of the stabilizer G_x in $GL(T_xX)$ has compact closure.*

Proof: If such a metric exists then the image is conjugate to a subgroup of $O(T_xX)$ which is compact.

Conversely, if the image has compact closure H_x let $< \bullet, \bullet >$ be any positive-definite symmetric bilinear form on T_xX, and define

$$(u, v) = \int_{H_x} < dgu, dgv > dg$$

for $u, v \in T_xX$, where $\int_{H_x}$ is Haar measure on H_x. This gives an inner product on T_xX which is invariant under the action of G_x, and is finite because H_x is compact. This can now be propagated to the whole of X using the action of G. *QED*

2.2 Foliations

Consider $\mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k$, and let $\mathcal{G}$ be the pseudogroup of diffeomorphisms on open sets which preserve horizontal factors; that is, which take the form

$$\vartheta(x, y) = (\vartheta_1(x, y), \vartheta_2(y))$$

for $x \in \mathbb{R}^{n-k}, y \in \mathbb{R}^k$. Then a *foliation* $\mathcal{F}$ for a manifold M is a $\mathcal{G}$-structure. In this situation, the horizontal planes given by the charts piece together to form the *leaves* of the foliation, which are smooth k-manifolds (though not necessarily submanifolds of M). Each point of M lies in a unique leaf. The set of leaves will often, abusively, also be denoted $\mathcal{F}$.

For example, the foliation of $\mathbb{R}^2$ by horizontal lines is preserved by

$$f : (x, y) \mapsto (2x, 2y)$$

so the quotient, a torus, inherits a foliation. See figure 1.

The only leaf of the foliation preserved by f is $\{x = 0\}$, which has image two circles under the quotient map. This is represented by a bold line in the figure. All the other leaves spiral around the torus, accumulating near the two circles.

Let τ be a plane field on M. Then τ is called *integrable* if τ is the tangent field of a foliation. In particular, suppose τ is a connection on the smooth fibre bundle $p : M \to B$. Then τ is required to be locally the tangent field of the horizontal foliation induced by a local trivialization. But if τ is flat then all the transition functions are locally constant, so the local foliations piece together. Therefore τ is flat if and only if it is integrable.

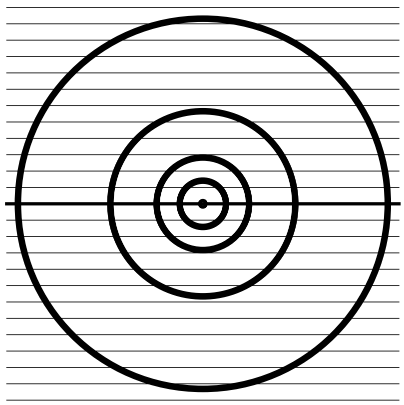

Figure 1: The induced foliation of the torus

2.3 Contact Structures

Let τ be the plane field in $\mathbb{R}^3$ spanned at each (x, y, z) by the vectors $(1, 0, 0)$ and $(0, 1, x)$ as illustrated in figure 2. Alternatively, it can be seen as the kernel of the *contact form* $\omega = -xdy + dz$. It is in fact a connection for the trivial bundle given by $p : (x, y, z) \mapsto (x, y)$. It is certainly not flat: in fact, any two points in $\mathbb{R}^3$ can be joined by a curve always tangent to τ, so it cannot be integrable.

Note that $d\omega = -dx \wedge dy$. Let γ be a closed curve in the (x, y)-plane. Let $\tilde{\gamma}$, its lift tangent to τ, be completed to a closed path by a vertical segment, δ. Let S' be a surface bounded by $\tilde{\gamma}$ and δ, projecting to S bounded by γ. Then

$$\int_S dx \wedge dy = \int_{S'} dx \wedge dy = \int_{\tilde{\gamma}} \omega + \int_\delta \omega$$

but $\int_{\tilde{\gamma}} \omega = 0$ and $\int_\delta \omega = \tilde{\gamma}(0) - \tilde{\gamma}(1)$, so it follows that τ has curvature -1.

Let $\mathcal{G}$ be the pseudogroup of diffeomorphisms between open sets in $\mathbb{R}^3$ which preserve τ. Then a *contact structure* on a 3-manifold is simply a $\mathcal{G}$-structure.

Example 2.2 $\mathbb{R}P^1 \times \mathbb{R}^2$ *can be given a natural contact structure, extending the standard one on* $\mathbb{R}^3$ *by setting* $\tau(\infty, y, z) = \ker(dy)$.

There is a convenient way of generating automorphisms of this contact structure.

Lemma 2.3 *Let* φ *be any automorphism of the* (y, z)*-plane. Then there exists a contact automorphism* $\tilde{\varphi}$ *of* $\mathbb{R}^3$ *which projects to* φ.

Proof: Set $\tilde{\varphi}_1(x, y, z)$ to be the gradient of the vector $d\varphi(1, x)$. This gives a contact automorphism by definition. *QED*

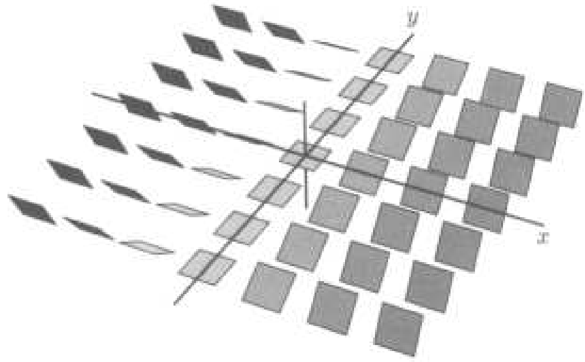

Figure 2: The contact field on the (x, y)-plane. This illustration is taken from page 169 of [1].

Proposition 2.4 *The tangent circle bundle of a smooth surface has a contact structure, preserved by the derivative of any diffeomorphism.*

Proof: $\mathbb{R}P^1 \times \mathbb{R}^2$ is double-covered by the tangent circle bundle to the plane, $\mathbb{R}^2 \times S^1$, and so induces a contact structure on it which is also preserved by diffeomorphisms of $\mathbb{R}^2$. Therefore for any surface, the local contact structures can be pieced together to give a global contact structure. *QED*

It can be shown that the Levi-Civita connection on the unit tangent bundle of any Riemannian surface S is a contact structure if and only if the Gaussian curvature of S is strictly non-zero. For details see [1].

2.4 The Developing Map and Completeness

Let $\varphi_0 : U_0 \to X$ and $\varphi_1 : U_1 \to X$ be (G, X)-compatible charts on a (G, X)-manifold M having domains with non-empty simply connected intersection. Then their transition function is some $g \in G$, so

$$x \mapsto \begin{cases} \varphi_0(x) & x \in U_0 \\ g \circ \varphi_1(x) & x \in U_1 \end{cases}$$

is a well-defined map $U_0 \cup U_1 \to X$. This process could be continued to cover the entire manifold if it were not for the fact that if M is not simply connected then the resulting map need not be well-defined. However $\tilde{M}$, the universal cover of M, inherits a (G, X)-structure from M and is simply connected. So we have a map

$$D : \tilde{M} \to X$$

called the *developing map*, defined up to a choice of chart about x_0 or, equivalently, multiplication by an element of G. Explicitly, consider a simple curve $\gamma : I \to M$, with $\gamma(0) = x_0$, corresponding to $\sigma = [\gamma] \in \tilde{M}$. Let γ have image covered by the domains $U_0, U_1, \dots, U_k$ of charts $\varphi_0, \varphi_1, \dots, \varphi_k$ with $U_i \cap U_j \neq \emptyset$ if and only if i and j are consecutive. Write $g_i = \varphi_{i-1} \circ \varphi_i^{-1}$ for $i = 1, \dots, k$, and $\varphi_0^\sigma = g_1 g_2 \dots g_k \circ \varphi_k$. Then in a neighbourhood of σ, $D = \varphi_0^\sigma \circ \pi$, where π is the projection $\tilde{M} \to M$.

In particular, if σ is a class of loops in M (that is $\sigma \in \pi_1(M)$) then the domain of φ_0^σ intersects the domain of φ_0 so by (G, X)-compatibility, the transition function, denoted $H(\sigma)$, lies in G. Let T_σ be the covering transformation of $\tilde{M}$ associated with σ. Then we now have

$$D \circ T_\sigma = \varphi_0^\sigma \circ \pi \circ T_\sigma = H(\sigma) \circ \varphi_0 \circ \pi = H(\sigma) \circ D$$

so $H : \pi_1(M) \to G$ is a group homomorphism, called the *holonomy* of M, and its image is called the *holonomy group.*

A (G, X)-manifold M is called *complete* if the developing map is a covering map. In this case, the holonomy group Γ determines the manifold, since $M \cong X/\Gamma$. Completeness in this sense is in fact precisely equivalent to metric-space completeness for M. There follows a convenient sufficient condition.

Proposition 2.5 *Let G be a Lie Group acting transitively and analytically on a smooth manifold X with compact point stabilizers. Then any closed (G, X)-manifold is complete.*

Proof: By 2.1 there exists a Riemannian metric on X preserved by G. The charts on G, combined with partitions of unity on M, allow this invariant metric to be pulled back to M. Thence it also extends to $\tilde{M}$.

Since M is compact there exists positive ϵ such that for all $p \in M$, $B_\epsilon(p)$ is ball-like (homeomorphic to a ball in $T_p(M)$ under the exponential map) and convex. Since G acts transitively on X, it can also be assumed that all ϵ-balls in X are ball-like and convex.

Now for $y \in \tilde{M}$, D maps $B_\epsilon(y) \subseteq \tilde{M}$ isometrically onto $B_\epsilon(D(y)) \subseteq X$. It is a local isometry by definition of the metric, and it is injective because if $D(y_1) = D(y_2)$ then the geodesic from y_1 to y_2 maps to a geodesic through $D(y_1)$ and $D(y_2)$ which must be a point by convexity.

Now for any point in $\tilde{M}$, the $\epsilon/2$ ball about it must be the disjoint union of isometric copies. So D is a covering map. *QED*

3 Model Geometries

3.1 Definition and Basic Examples

The intuitive idea of different manifolds having the same geometry can now be formalized.

Definition 3.1 *A model geometry (G, X) is a manifold X with a Lie Group G of diffeomorphisms of X such that:*

1. *X is connected and simply connected;*
2. *G acts transitively on X with compact point stabilizers;*
3. *G is not contained in any larger groups of diffeomorphisms of X with compact point stabilizers;*
4. *there exists at least one compact (G, X)-manifold.*

A manifold M is modelled on (G, X) if it has a (G, X)-structure.

Note that criterion 2 ensures by proposition 2.1 that all model geometries have an invariant Riemannian metric and by 2.5 that all modelled manifolds are complete.

For a Riemannian manifold X we will often abusively write X for $(\text{Isom}(X), X)$. In this case there is a nice sufficient condition for criterion 3.

Lemma 3.2 *Let (X^n, g) be a Riemannian manifold, and let $G = \text{Isom}(X, g)$ act transitively on X with point stabilizers isomorphic to the orthogonal group O_n. Then G is maximal among all groups of diffeomorphisms acting transitively on X with compact stabilizers.*

Proof: Suppose H is a group of diffeomorphisms acting transitively on X with compact stabilizers and containing G. Then by 2.1 there exists a Riemannian metric h preserved by H. Fix $x \in X$. Then since h is preserved by G which is transitive, h is determined by h_x. So it suffices to consider inner products on T_xX which are preserved by G_x. Take g_x to be the standard inner product, so G_x acts as the orthogonal group O_n. Recall that any other inner product has matrix of the form $U\Lambda U^t$ for orthogonal U and positive diagonal Λ. Then it is preserved precisely by the matrix group $(U\sqrt{\Lambda})O_n(U\sqrt{\Lambda})^{-1}$. This contains O_n if and only if $\sqrt{\Lambda}$ is in $N(O_n) = \mathbb{R}^*O_n$, in which case Λ is just a multiple of the identity and h_x is a multiple of g_x, so h is a multiple of g and $H = G$. *QED*

Example 3.3 *Consider $\mathbf{E}^3$. It clearly satisfies criteria 1 and 2, and satisfies criterion 3 by lemma 3.2. The 3-torus $\mathbf{T}^3 = \mathbf{E}^3/\mathbb{Z}^3$ is an example of a compact manifold modelled on $\mathbf{E}^3$.*

Example 3.4 *Consider $\mathbf{S}^3$, which has isometry group O_4. Again, it clearly satisfies criteria 1 and 2, and satisfies criterion 3 by lemma 3.2. It is itself compact.*

3.2 More Examples

There are also examples with smaller point stabilizers. Henceforth we consider only dimension 3.

Example 3.5 *Consider* $\mathbf{S}^2 \times \mathbb{R}$*, which has isometry group* $G = O_3 \times \mathbb{R}$*. The point stabilizers are isomorphic to* O_2*, and act as rotations of the spherical factor. There follows a result which ensures criterion 3 is satisfied.* $\mathbf{S}^2 \times \mathbf{S}^1 = \mathbf{S}^2 \times \mathbb{R}/\mathbb{Z}$ *is a compact manifold modelled on* $\mathbf{S}^2 \times \mathbb{R}$*.*

Lemma 3.6 *Let* X *be a 3-dimensional Riemannian manifold, and let* $G = \mathrm{Isom}(X)$ *have point stabilizer isomorphic to* O_2*. Then* G *is maximal.*

Proof: Just as in lemma 3.2 it suffices to consider inner products on the tangent space T_xX which are preserved by the stabilizer G_x. It can easily be seen that, in an appropriate basis, such an inner product must be of the form

$$\begin{pmatrix} a & 0 \\ 0 & \mathbf{R} \end{pmatrix}$$

where $a \in \mathbb{R}$ and $\mathbf{R}$ is some 2-dimensional inner product preserved by O_2. But a simple calculation shows that any such $\mathbf{R}$ is proportional to the identity. So any such inner product has matrix of the form

$$\mathbf{S} = \begin{pmatrix} a & 0 \\ 0 & b\mathbf{I} \end{pmatrix}$$

which is preserved by the group $\sqrt{\mathbf{S}}O_3\sqrt{\mathbf{S^{-1}}}$. Another simple calculation shows that this group only contains G_x if $a = b$, in which case the inner product is merely a multiple of the standard inner product. *QED*

Example 3.7 *Consider* $\mathbf{H}^2 \times \mathbb{R}$*, which has isometry group* $\mathrm{Isom}H^2 \times \mathbb{R}$*. This also has 1-dimensional point stabilizers, and so satisfies criterion 3 by lemma 3.6. Any compact hyperbolic 2-manifold (for example an orientable surface of genus greater than 1) crossed with the circle is an example of a compact manifold modelled on this geometry.*

So far it is clear that all the geometries we have seen are distinct. Any isomorphism would have to be an isometry, but it is clear that these all have different curvature.

Example 3.8 *Consider* $\mathbf{H}^2$*. We extend the Riemannian metric to* $T\mathbf{H}^2$ *as follows.*

Let $x \in \mathbf{H}^2$ *and let* $v \in T\mathbf{H}^2$ *lie above* x*. Then there are horizontal and vertical subspaces,* H *and* V *respectively, of* $T_vT\mathbf{H}^2$*.* V *is naturally identified with* $T_x\mathbf{H}^2$ *by the exponential map, so inherits an inner product. The natural projection* $p : T\mathbf{H}^2 \to \mathbf{H}^2$ *identifies* H *with* T_x *isomorphically, so induces an inner product. Finally it is stipulated that* H *and* V *should be orthogonal.*

For this Riemannian metric, if $f : \mathbf{H}^2 \to \mathbf{H}^2$ *is an isometry then so is* $df : T\mathbf{H}^2 \to T\mathbf{H}^2$*. This property descends with the restriction of the Riemannian metric to the unit sphere bundle* $U\mathbf{H}^2$*. Note that* $PSL_2\mathbb{R}$*, the group of orientation preserving isometries of* $\mathbf{H}^2$*, acts transitively on* $U\mathbf{H}^2$ *with trivial stabilizer, so there is a natural identification of* $U\mathbf{H}^2$ *with* PSL_2*. It follows that* *there is an induced metric on $\widetilde{SL_2}$ invariant under the action of PSL_2, and hence under the action of $\widetilde{SL_2}$.*

Since $\pi_1(U\mathbf{H}^2) \cong \mathbb{Z}$ there is an exact sequence

$$0 \to \mathbb{Z} \to \widetilde{SL_2} \to PSL_2 \to 0$$

with $\mathbb{Z}$ in the centre of $\widetilde{SL_2}$. But PSL_2 is centreless, so $\mathbb{Z}$ is precisely the centre of $\widetilde{SL_2}$. The action of $\mathbb{Z}$ is translation of the fibres of $\widetilde{SL_2}$ by a fixed amount.

There is also an isometric action of $\mathbb{R}$ on $\widetilde{SL_2}$ induced from rotations of the fibres in $T\mathbf{H}^2$. This is also translation of the fibres and commutes with the action of $\widetilde{SL_2}$, so intersects that action precisely in its centre.

From these facts it follows that the isometry group generated by $\widetilde{SL_2}$ and $\mathbb{R}$ is a connected 4-dimensional Lie Group which preserves the bundle structure of $\widetilde{SL_2}$. Later it will be shown that the identity component of the point stabilizer subgroup of the isometry group of a 3-dimensional model geometry must be either trivial, SO_2 or SO_3, so $\mathrm{Isom}(\widetilde{SL_2})$ is 3-, 4- or 6-dimensional. It cannot be 3-dimensional, so the stabilizer must be SO_2 or SO_3. It therefore follows either from lemma 3.2 or from lemma 3.6 that the isometry group is maximal. In fact $\widetilde{SL_2}$ is distinct from $\mathbf{E}^3$, $\mathbf{S}^3$ and $\mathbf{H}^3$, so does not have constant curvature and, in particular, has point stabilizer with identity component SO_2.

It is clear that the unit circle bundle of any compact hyperbolic 2-manifold (for example the surface of genus 2) has geometry modelled on $\widetilde{SL_2}$.

Note that the non-integrable horizontal plane field on $T\mathbf{H}^2$ restricts to a non-integrable plane field in $U\mathbf{H}^2$, which induces a non-integrable horizontal plane field in $\widetilde{SL_2}$. Therefore $\widetilde{SL_2}$ must be distinct from $\mathbf{H}^2 \times \mathbb{R}$ which has an integrable horizontal plane field.

Example 3.9 *Let* **Nil** *be the Heisenberg group, namely the space of all matrices of the form*

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

for $x, y, z \in \mathbb{R}$. It is easy to write down an invariant metric, since it is determined by its value at a point. Identifying **Nil** *with $\mathbb{R}^3$ and setting $ds^2 = dx^2+dy^2+dz^2$ at the origin we arrive at the metric $ds^2 = dx^2+dy^2+(dz-xdy)^2$. Note that* **Nil***'s action on itself preserves its structure as a bundle over $\mathbb{R}^2$ given by the projection to the (x, y)-plane.*

There is also an isometric action of the circle on **Nil** *given by, if ρ_θ is the rotation through angle θ,*

$$\theta.((x, y), z) = (\rho_\theta(x, y), z + \frac{1}{2}\sin\theta(y^2\cos\theta - x^2\cos\theta - 2xy\sin\theta))$$

for any $\theta \in S^1$. The isometry group generated by this action and **Nil** *is 4-dimensional, so just as for $\widetilde{SL_2}$, it follows that this isometry group is maximal.*

An example of a compact manifold with **Nil***-geometry is given by the quotient by the Integer Heisenberg Group, namely those elements with* $x, y, z \in \mathbb{Z}$*. This is the quotient of* $\mathbb{R}^3$ *by the group generated by integer translation in the* y*- and* z*-directions and* $(x, y, z) \mapsto (x + 1, y, z + y)$*, which is clearly compact.*

Example 3.10 *Another geometry, denoted* **Sol** *and called solve-geometry, can be obtained by giving* $\mathbb{R}^3$ *yet another group structure. Define the group multiplication by*

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + e^{-z_1}x_2, y_1 + e^{z_1}y_2, z_1 + z_2)$$

for $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3$*. Then a left-invariant metric on* $\mathbb{R}^3$ *is given by* $ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$*. The maximality of the isometry group of this metric will be shown later.*

As for a compact manifold modelled on **Sol***, let* $\phi : \mathbf{T}^2 \to \mathbf{T}^2$ *be induced by the linear map of* $\mathbb{R}^2$ *given by the matrix*

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

and take the eigenvalues of $\mathbf{A}$ *as a basis. Then there exists* $t_0 \in \mathbb{R}$ *such that*

$$\psi_{t_0} : (x, y, t) \mapsto (e^{t_0}x, e^{-t_0}y, t + t_0)$$

precisely corresponds to ϕ*. The quotient of* $\mathbb{R}^3$ *by unit translations in the* x*- and* y*-directions and by* ψ_{t_0}*, which is the mapping torus of* ϕ*, is then a compact solve-manifold.*

3.3 Hyperbolic Geometry

Recall that $\mathbf{H}^n$ can be modelled as the half-space $\{\mathbf{x} \in \mathbb{R}^n | x_n > 0\}$, taking

$$ds^2 = \frac{1}{x_n^2}(dx_1^2 + \ldots + dx_n^2)$$

as a Riemannian metric. Just as in the 2-dimensional case, the geodesics turn out to be semicircles and lines intersecting the (x, y)-plane orthogonally. Analogously to the 2-dimensional case, in 3-dimensions it is convenient to identify $(x, y, z) \in \mathbb{R}^3$ with the quaternion $x + yi + zj$. Then the orientation-preserving isometry group of $\mathbf{H}^n$ is $PSL_2\mathbb{C}$, acting by Möbius transformations. In particular, this includes orthogonal maps preserving the z coordinate, translations in the x- and y- directions and inversions in hemispheres based on the (x, y)-plane.

Since the stabilizer of a point is isomorphic to O_3 it is clear that $\mathbf{H}^3$ satisfies criteria 1 to 3 of definition 3.1. However, constructing a compact hyperbolic manifold is slightly harder. The strategy is to identify the edges of a hyperbolic polyhedron. But when is such a glueing a manifold?

Theorem 3.11 *Let P be a collection of disjoint hyperbolic polygons $P_1, \dots, P_k$, let $\sim$ be an identification of the facets and let $q : P \to P/\sim = M$ be the quotient. If every point $x \in M$ has an open neighbourhood U_x and a homeomorphism $\varphi_x : U_x \to B_{\epsilon(x)}(0) \subseteq \mathbf{H}^3$ which maps x to 0 (in the Poincaré ball model) and restricts to an isometry on each component of $U_x \cap q(P - \partial P)$ then M is a hyperbolic manifold.*

Proof: It is clear that M is Hausdorff. $P \subseteq \mathbf{H}^n$ which is second-countable, and every open set in M corresponds to a distinct open set in P.

Shrinking $\epsilon(x)$ if necessary, x lies in the closure of each component of $U_x \cap q(P-\partial P)$. Note that, for each $x' \in q^{-1}(x)$ there is an isometry $h_{x'} : B_{\epsilon(x)}(x') \to B_{\epsilon(x)}(0)$ with $h_{x'}|_{P-\partial P} = \varphi_x \circ q$. The φ_x form an atlas for M. It remains to check that these charts are hyperbolically compatible; that is, that if $x, y \in M$ and X is a component of $U_x \cap U_y$ then $\varphi_x \circ \varphi_y^{-1} : \varphi_y(X) \to \varphi_x(X)$ is the restriction of a hyperbolic isometry. By assumption this is true on any component of $\varphi_y(X \cap q(P - \partial P))$. It remains to check that these are compatible. Any two points of M can be joined by a path whose interior does not intersect any faces of dimension less than $n - 1$, so it suffices to check that if $z \in X$ lies in a face of dimension $n - 1$ that $\varphi_x \circ \varphi_y^{-1}$ is a hyperbolic isometry in a neighbourhood of z.

Let $z_1, z_2 \in q^{-1}(z)$. Then there exists a unique $x_i \in q^{-1}(x)$ in the component of $q^{-1}(U_x)$ containing z_i for each i, and similarly for y. Let k be the isometry identifying the facets of z_1 and z_2. Without loss of generality, x_i and y_i lie in the same facets as the z_i, since otherwise reducing $\epsilon(x)$ eliminates any intersection with a facet. On this facet, $h_{x_1} = h_{x_2} \circ k$. But also $h_{y_1} = h_{y_2} \circ k$, and since $q^{-1}(U_x) \cap q^{-1}(U_y)$ have non-trivial intersection, k can be extended to $\mathbf{H}^n$ in such a way as to make both these relations true everywhere they make sense. But this is precisely the required compatibility condition. *QED*

The proof was taken from [5]. It enables the construction of a compact hyperbolic manifold.

Example 3.12 (Seifert-Weber Dodecahedral Space) *Let D be a hyperbolic dodecahedron. Identify opposite faces with a clockwise rotation by $3\pi/5$. Then diagram-chasing shows that all the vertices are identified, and the edges are identified in 6 groups of 5. Since all 20 vertices are identified, in order for the gluing to create a manifold by theorem 3.11 each vertex must have solid angle $4\pi/20 = \pi/5$ which is achieved by a dodecahedron with dihedral angle $2\pi/5$ (using the area of a spherical triangle). A small hyperbolic dodecahedron is similar to a Euclidean one which has dihedral angles of over $\pi/2$, and an ideal hyperbolic dodecahedron has dihedral angles of $2\pi/6$. So by the intermediate value theorem there exists an appropriate hyperbolic dodecahedron, as illustrated in figure 3.*

3.4 Thurston's Theorem

This subsection is devoted to the proof of a theorem of Thurston, which states that the only 3-dimensional model geometries are those we have discussed above.

Figure 3: A hyperbolic dodecahedron with dihedral angle $2\pi/5$. This illustration is taken from page 37 of [1].

However, first it is useful to look at the 2-dimensional case.

Theorem 3.13 *Any 2-dimensional model geometry is either* $\mathbf{E}^2$,$\mathbf{S}^2$ *or* $\mathbf{H}^2$.

The proof is a result of the following important theorem from differential geometry.

Theorem 3.14 *If M^n is a complete Riemannian manifold with constant sectional curvature K then its universal cover is one of:*

1. $\mathbf{S}^n$ *if* $K = +1$;
2. $\mathbf{E}^n$ *if* $K = 0$;
3. $\mathbf{H}^n$ *if* $K = -1$.

Proof omitted.

Proof of theorem 3.13: Let (G, X) be a 2-dimensional model geometry. G acts transitively on X, so X has constant sectional curvature. Rescaling the metric, this curvature is either +1, 0 or -1. The result now follows from theorem 3.14, since X is simply connected. *QED*

The 3-dimensional case can now be addressed. The proof of the following theorem is taken from [1].

Theorem 3.15 *Any 3-dimensional model geometry is one of* $\mathbf{E}^3$, $\mathbf{S}^3$, $\mathbf{H}^3$, $\mathbf{S}^2 \times \mathbb{R}$, $\mathbf{H}^2 \times \mathbb{R}$, $\widetilde{SL_2}$, **Nil** *or* **Sol**.

Proof: Let G' be the identity component of G, which also acts transitively on X. For $x \in X$ consider the stabilizer G'_x. It preserves the inner product on T_xX and so must be isomorphic to a subgroup of O_3. Now $\bigcup_{x\in X} G'_x/(G'_x)'$ naturally forms a covering space for X. But X is simply connected, so it follows that, for all x, G'_x is its own identity component, and so is connected. Therefore it must be isomorphic to one of SO_3, SO_2 and the trivial group. Each of these cases will be addressed in turn.

If $G'_x \cong SO_3$ then X has constant sectional curvature, and so by theorem 3.14 it must be one of $\mathbf{E}^3$, $\mathbf{S}^3$ and $\mathbf{H}^3$.

Now suppose $G'_x \cong SO_2$. Then for each x there exists an axis, varying smoothly with x, preserved by the action of G'_x. Let V be a unit left-invariant vector field on X such that, for each x, $V(x)$ is this axis. Let its flow be φ_t. This defines a foliation $\mathcal{F}$ of X. The aim is to show that the leaves are in fact the fibres of a bundle.

Since V is G'-invariant, φ_t commutes with G' for all $t \in \mathbb{R}$. It follow that two points on the same leaf of $\mathcal{F}$ have the same stabilizer. Moreover, if $g \in G'$ preserves a leaf $F \in \mathcal{F}$ then it commutes with the stabilizer of the points on that leaf. For, let $x \in F$, $h \in G'_x$ and $k \in G'$ with $y = kx \in F' \in \mathcal{F}$. Then $ghg^{-1}h^{-1}|_F = e$ and similarly $kghg^{-1}h^{-1}k^{-1}|_{F'} = e$, so since G' acts transitively on X, $gh = hg$ on every leaf of the foliation.

Fix $x \in F \in \mathcal{F}$ and $t \in \mathbb{R}$. Let g_t be such that $\varphi_t \circ g_t(x) = x$. Since g_t and φ_t both commute with G'_x, $d(\varphi_t \circ g_t)$ is an automorphism of T_xX which commutes with all rotations about $V(x)$. It follows that it must itself be a rotation, possibly combined with an expansion by a factor of λ_t.

But the expansion is in fact just the identity, because φ_t is volume-preserving. For, let ω be the volume-form on X. On the above assumptions $(\varphi_t^*\omega)_x = \lambda_t\omega_x$ and since φ_t commutes with G', λ_t doesn't depend on x. But X has a compact manifold M modelled on it, which inherits a metric, foliation and flow with the same properties. Now

$$\int_M \omega = \int_M \varphi_t^*\omega = \lambda_t \int_M \omega$$

so $\lambda_t = 1$.

Therefore, φ_t is an isometry. Because of this, disjoint leaves have disjoint neighbourhoods and $X/\mathcal{F}$ is a simply-connected surface, Y, which inherits a G'-action and invariant metric from X. $X \to Y$ is a principle bundle with fibre $\mathbb{R}$ or $\mathbf{S}^1$, and a connection given by τ the plane field orthogonal to V. If the curvature of this bundle is non-zero then, rescaling, it is without loss of generality, +1.

If τ has curvature 0 then it is a product bundle and there are the following cases:

1. if Y has curvature 0 then $X = \mathbf{E}^3$, so the isometry group is not maximal;
2. if Y has curvature +1 then $X = \mathbf{S}^2 \times \mathbb{R}$;
3. if Y has curvature -1 then $X = \mathbf{H}^2 \times \mathbb{R}$.

If τ has curvature $+1$ then it is a contact structure, and X can be taken to be the universal cover of the unit sphere bundle on Y.

1. If Y has curvature 0 then we have $X = \mathbf{Nil}$.
2. If Y has curvature $+1$ then its unit sphere bundle is $U\mathbf{S}^2 \cong SO_3$ which has universal cover $\mathbf{S}^3$. The isometry group is precisely those isometries which preserve the Hopf fibration, and so is not maximal.
3. If Y has curvature -1 then its unit sphere bundle has universal cover $\widetilde{SL_2}$.

Finally, suppose G'_x is trivial. Then $X \cong G'/G'_x \cong G'$ so, X is a Lie Group. By Lie's Theorem, it suffices to consider the Lie Algebra, $\mathfrak{g}$. It follows from the existence of a compact quotient that G is unimodular, and it follows from this that, if $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is the adjoint representation of $\mathfrak{g}$, for any $v \in \mathfrak{g}$, $\mathrm{tr}\ \mathrm{ad} v = 0$.

The Lie bracket on $\mathfrak{g}$ can be regarded as a linear map $\mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$. Fixing an inner product $< \bullet, \bullet >$ on $\mathfrak{g}$ and an orientation gives an identification of $\mathfrak{g} \wedge \mathfrak{g}$ with $\mathfrak{g}$ via $v \wedge w \mapsto v \times w$, so the Lie bracket corresponds to an endomorphism $L : \mathfrak{g} \to \mathfrak{g}$. It follows from unimodularity that L is symmetric.

Choosing a basis and a quadratic form appropriately, it is possible to ensure that L has a diagonal matrix with decreasing diagonal entries $\lambda_1, \lambda_2, \lambda_3$ all equal to ± 1 or 0,and with at least as many positive entries as negative ones. Since the Lie algebra determines the universal cover of the Lie Group up to isomorphism, this gives the following options.

1. $\lambda_1 = \lambda_2 = \lambda_3 = 1$ gives $G' = \mathbf{S}^3$.
2. $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = -1$ gives $G' = \widetilde{SL_2}$.
3. $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 0$ gives $G' = \mathbf{Sol}$.
4. $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = 0$ gives $G' = \mathbf{Nil}$.
5. $\lambda_1 = \lambda_2 = \lambda_3 = 0$ gives $G' = \mathbb{R}^3$.
6. $\lambda_1 = 1, \lambda_2 = 0$ and $\lambda_3 = -1$ gives $\widetilde{\mathrm{Isom}(\mathbf{E}^2)}$ which is a subgroup of $\mathrm{Isom}(\mathbf{E}^3)$.

So the only geometry of this form is **Sol**. *QED*

4 Seifert bundles

4.1 Orbifolds

Definition 4.1 *Let X be a Hausdorff second countable topological space. X is an n-dimensional orbifold if it is locally isomorphic to the quotient of $\mathbb{R}^n$ by a finite group action. More precisely, X comes equipped with a covering $\mathfrak{U}$ of open sets closed under finite intersection which is maximal among all such coverings* *with the following conditions: to each* $U \in \mathfrak{U}$ *there is associated a finite group* Γ_U *with an action of* Γ_U *on an open subset* $\tilde{U}$ *of* $\mathbb{R}^n$*, and a homeomorphism* $\varphi : U \to \tilde{U}/\Gamma_U$*; and if* $U \subseteq V \in \mathfrak{U}$*, then there is inclusion* $f : \Gamma_U \to \Gamma_V$ *and an embedding* $\theta_{UV} : \tilde{U} \to \tilde{V}$ *such that everything commutes.*

Of course, any orbifold is homeomorphic to a manifold, possibly with boundary. This is called the *underlying space* of the orbifold.

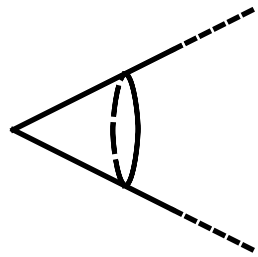

Figure 4: A cone point

Example 4.2 *Consider the action of* C_r*, the cyclic group of order* r*, on* $\mathbb{R}^2$ *by rotations. The quotient is an infinite cone formed by identifying the two edges of a wedge with angle* $2\pi/r$*. This is trivially an orbifold. The orbit of 0 is called a cone point.*

A *cone point* on a 2-dimensional orbifold X is a point x with a neighbourhood U and an isomorphism $\varphi : U \to V \subseteq \mathbb{R}^2/C_r$ such that φ maps x to the orbit of 0.

Example 4.3 *Let* C_2 *act on* $\mathbb{R}^2$ *by reflection in a line. Then the quotient is isometric to a half-plane bordered by that line. This is known as a reflector line.*

A *reflector line* on a 2-dimensional orbifold is a boundary line with a neighbourhood which is isomorphic to an open set in example 4.3.

Example 4.4 *Let* D_r*, the dihedral group of order* $2n$*, act on* $\mathbb{R}^2$ *as the symmetry group of an* r*-gon. Then the* 0 *under the quotient map is called a corner reflector.*

A *corner reflector* on a 2-dimensional orbifold is a point with a neighbourhood locally isomorphic to a neighbourhood of the image of 0 under the quotient map in example 4.4.

In general, any 2-dimensional orbifold can be thought of as a surface with cone points. The underlying surface may have a boundary, consisting of reflector lines meeting at corner reflectors.

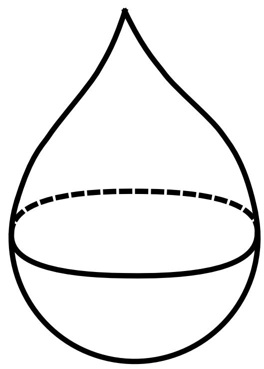

Figure 5: The teardrop orbifold $S^2(p)$

Example 4.5 *The teardrop orbifold has underlying space* $\mathbf{S}^2$ *and a single cone point of angle* $2\pi/p$*. It is denoted* $S^2(p)$*. See figure 5. Similarly, the orbifold with two cone points, of angles* $2\pi/p$ *and* $2/pi/q$*, is denoted* $S^2(p,q)$*.*

Example 4.6 *The orbifold with underlying space a disc, with one corner reflector of angle* π/p*, is denoted* $D^2(p)$*. Note that this is double-covered by* $S^2(p)$*. With another corner reflector of angle* π/q *it is denoted* $D^2(p,q)$*, which is double-covered by* $S^2(p,q)$*. .*

An *orbifold covering* is a map $f : M \to N$ of orbifolds such that every point of N has a neighbourhood U such that the restriction of f to any component of $f^{-1}(U)$ is the natural quotient map between two quotients of $\mathbb{R}^n$ by finite groups, one of which is a subgroup of the other. It can be shown that every orbifold has a universal cover, in the sense of an orbifold which has only trivial coverings. An orbifold is *good* if it is orbifold covered by a manifold, and *bad* if its universal covering is a proper orbifold. Fortunately, bad orbifolds are easily characterized.

Theorem 4.7 *The only bad orbifolds are:*

1. $S^2(p)$ *for* $p \neq 1$*;*
2. $S^2(p,q)$ *for* $p \neq q$ *and neither equal to 1;*
3. $D^2(p)$ *for* $p \neq 1$*;*
4. *and* $D^2(p,q)$ *for* $p \neq q$ *and neither equal to 1.*

Proof omitted. For details see [2].

In fact, every good compact 2-dimensional orbifold is finitely covered by a manifold.

It is now possible to generalize the concept of Euler number to orbifolds. It is clear from the definition of Euler number that if K is a finite simplicial complex and $\tilde{K}$ is an n-fold cover of K that $\chi(\tilde{K}) = n\chi(K)$. Therefore, for good 2-dimensional orbifolds it makes sense to define the Euler characteristic by this formula. This is not in general an integer, but is always rational.

In fact, the Euler characteristic can be computed from the orbifold itself. In particular, suppose the orbifold X has d-fold manifold cover $\tilde{X}$ and underlying surface Y, n cone points with angles $2\pi/q_i$ $(1 \leq i \leq n)$ and no reflector lines. Then removing small discs about the cone points to create the surface W, it is clear that $\chi(Y) = \chi(W) + n$. Let $\tilde{W}$ be the pre-image of W in $\tilde{X}$. Then $\chi(\tilde{W}) = \chi(W)$. The pre-image of each disc in $\tilde{X}$ is just d/q_i discs. So $\chi(\tilde{X}) = d\chi(W) + \sum_i d/q_i$, from which follows

$$\chi(X) = \chi(Y) - \sum_i \left(1 - \frac{1}{q_i}\right)$$

which is commonly know as the Riemann-Hurwitz formula.

This can also be used to define the Euler characteristic of bad orbifolds, since we know what they are by theorem 4.7. For example, $\chi(S^2(p)) = 1 + 1/p$. Since all bad 2-dimensional orbifolds are finitely covered by $S^2(p)$ or $S^2(p,q)$ this defines the Euler characteristic for all 2-dimensional orbifolds.

4.2 Seifert Fibre Spaces

A Seifert fibre space is, at least in the compact case, precisely a 3-manifold foliated by circles. However, this is not the definition that Seifert first wrote down, and it requires some work to prove its equivalence.

Definition 4.8 *A trivial fibred solid torus is $S^1 \times D^2$ with the product foliation by circles. That is, each leaf (usually called a fibre) is $S^1 \times y$ for $y \in D^2$. In general, a fibred solid torus is a solid torus, equipped with a foliation by circles, which is finitely covered by a trivial fibred solid torus. A fibred solid Klein bottle is a solid Klein bottle with a foliation by circles which is finitely covered by a trivial fibred solid torus.*

A fibred solid torus can be constructed by taking a solid cylinder with the product foliation by lines and identifying the ends after a (conventionally) anti-clockwise twist of $2\pi q/p$. Such a torus is denoted $T(p,q)$. Any non-central fibre winds q times round the central fibre and represents p times the generator of the fundamental group, so $T(p,q) \cong T(p',q')$ if and only if $p = p'$ and $q' \equiv \pm q \bmod p$. Choosing an orientation for $T(p,q)$, p and $q \bmod p$ are invariants. It is the quotient of the trivial fibred solid torus by the group action of C_p in which the generator rotates the S^1 factor by $2\pi/p$ and the D^2 factor by $2\pi q/p$.

A fibred solid Klein bottle can be constructed from a solid cylinder with the product foliation by lines by identifying the ends after a reflection. Since all reflections are conjugate, the fibred solid Klein bottle is unique.

Definition 4.9 *A Seifert fibre space is a 3-manifold with a decomposition into disjoint circles, such that each circle has a neighbourhood of circles which is isomorphic to fibred solid tori or the fibred solid Klein bottle.*

A fibre is *regular* if its neighbourhood is isomorphic to the trivial solid torus, and otherwise it is called *critical*. In a solid fibred torus only the central fibre is critical, in which case (p, q) are called the *orbit invariants*. In the solid Klein bottle they form a one-sided annulus. Therefore the solid fibres are either isolated or form surfaces.

Let M be a Seifert fibre space. Then the space of fibres, X, has a natural orbifold structure. Isolated critical fibres with orbit invariants (p, q) correspond to cone points of angle $2\pi p/q$, while surfaces of critical fibres correspond to reflector lines. Note there are no corner reflectors. Moreover, if M is a manifold without boundary then X is an orbifold without boundary, although that does not preclude reflector lines which do not form part of the boundary of the orbifold (though they may correspond to the boundary of the underlying space). The map $M \to X$ is, in a generalized sense, a sort of bundle, called a *Seifert bundle*.

It can be shown by detailed analysis of their isometry groups that any closed manifold modelled on $\mathbf{E}^3$, $\mathbf{S}^3$, $\mathbf{S}^2 \times \mathbb{R}$, $\mathbf{H}^2 \times \mathbb{R}$, $\widetilde{SL_2}$ or **Nil** admits a Seifert fibre structure and vice versa . Moreover, the geometry can be determined by the Euler characteristic of the base space, χ, and an invariant of the Seifert structure called the Euler number and denoted e, according to the following table.

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e = 0$	$\mathbf{S}^2 \times \mathbb{R}$	$\mathbf{E}^3$	$\mathbf{H}^2 \times \mathbb{R}$
$e \neq 0$	$\mathbf{S}^3$	**Nil**	$\widetilde{SL_2}$

It can also be shown that, if a manifold is modelled on a particular geometry, then that geometry is unique. From this it follows that **Sol** does indeed satisfy criterion 3 of definition 3.1, since otherwise it would be a subgeometry of one of the others, and so not satisfy uniqueness. For more details see [2].

5 Geometrisation

5.1 Manifold decompositions

A 3-manifold M is *prime* if, whenever it is expressed as a connected sum $X \# Y$, one of X or Y is homeomorphic to the 3-sphere. Milnor showed that any closed orientable 3-manifold has a unique decomposition as the connected sum of primes. M is irreducible if each 2-sphere in M bounds a 3-ball in M.

Lemma 5.1 *Let M be irreducible with $M = X \# Y$. Then one of X and Y is homeomorphic to the 3-sphere.*

Proof: By the definition of the connected sum, there exists an embedded 2-sphere F in M such that $M - F$ is homeomorphic to the disjoint union of $X - B$ and $Y - B$, where B is a 3-ball. By irreducibility it follows that, without loss of generality, $X - B$ is homeomorphic to B. So X is homeomorphic to the result of gluing two 3-balls along their boundaries; but this is a 3-sphere. *QED*

There is a partial converse to this result, which states that if M is prime and not irreducible then it is a 2-sphere bundle over S^1.

The aim of this section is to investigate when 3-manifolds can be given geometric structures. Unfortunately, prime decomposition is insufficient for these purposes. Having exhausted the possibilities for cutting along spheres, we turn to the next most simple surface.

An embedded torus $T \hookrightarrow M$ is *compressible* if there exists a disc $D \subseteq M$ with $D \cap T = \partial D$ and ∂D not contractible in T. Otherwise T is *incompressible.* An irreducible 3-manifold M is *atoroidal* if any $\mathbb{Z}\oplus\mathbb{Z}$ subgroup of $\pi_1(M)$ is conjugate to the inclusion of the fundamental group of a toric boundary component of M. This is equivalent to requiring that any incompressible torus is boundary-parallel, in the sense that it separates M into a copy of M and $\partial M \times I$.

Definition 5.2 *A torus decomposition of an irreducible manifold M is a collection of embedded incompressible tori $T_1, \ldots, T_n$ such that the components of $M - \cup_i T_i$ are all either atoroidal or Seifert fibred, and no proper subset of $T_1, \ldots, T_n$ has this property.*

Any compact, orientable, irreducible 3-manifold has a torus decomposition. For any compact orientable manifold M there exists a number $n(M)$ such that, if $S_1, \ldots, S_k$ is any collection of pairwise disjoint, closed, 2-sided, incompressible surfaces in M and $k \geq n(M)$ then some pair S_i, S_j bounds an I-product in M (see [3] or [4]). So there exists a maximal collection of embedded incompressible tori, $T_1, \ldots T_n$ in M which separate M into atoroidal pieces. Now, some of these pieces may admit Seifert fibred structures, and sometimes two neighbouring pieces, separated by T_i, may admit compatible such structures. In this case, remove T_i and repeat.

It can be shown that any such collection of toruses is unique up to isotopy. For details see [4].

The orientability condition merely simplifies the situation; if cutting along Klein bottles is allowed and the definition of atoroidality is modified accordingly then a similar result holds for non-orientable manifolds.

5.2 Conjectures

In the 2-dimensional case, every closed orientable manifold admits a geometric structure, depending on its Euler characteristic. Recall from theorem 3.13 that the only 2-dimensional geometries are spherical, Euclidean and hyperbolic.

Theorem 5.3 *Let M be a 2-dimensional closed orientable manifold.*

1. *If $\chi(M) > 0$ then M is a sphere and admits a spherical structure.*

2. *If $\chi(M) = 0$ then M is a torus and admits a Euclidean structure.*
3. *If $\chi(M) < 0$ then M is a surface of genus greater than 1 and admits a hyperbolic structure.*

Proof: Statement 1 is trivial and statement 2 is clear, since the 2-torus is $S^1 \times S^1 \cong \mathbf{E}^2/\mathbb{Z}^2$

Statement 3 follows from theorem 3.11. For the surface of genus $g > 1$ can be thought of as a $4g$-gon with identified edges. All the vertices are identified, so for the theorem to apply a hyperbolic polyhedron with interior angles $\pi/2g$ is required. A small hyperbolic $4g$-gon is almost Euclidean, so has angles close to $(1 - 1/2g)\pi$, while a large hyperbolic $4g$-gon is almost ideal, so has angles close to 0. Therefore, by the intermediate value theorem, an appropriate polygon can be found as long as $(1 - 1/2g) > 1/2g$, or equivalently $g > 1$. *QED*

The 3-dimensional situation is not nearly as nice. Not all 3-manifolds even admit geometric structures. But as stated in subsection 5.1, any closed orientable 3-manifold can be canonically decomposed into primes. These are all either sphere bundles over the circle or are irreducible, and so have a canonical torus decomposition. It is natural to conjecture that all the resulting pieces can be given geometric structures.

Conjecture 5.4 (Thurston's Geometrisation Conjecture) *Let M be a closed irreducible orientable 3-manifold. Then all the components of its torus decomposition admit a geometric structure.*

It has been shown that any Seifert fibred space admits a geometrical structure. So it only remains to consider the atoroidal case. Let M be a closed atoroidal manifold. Then there are essentially two cases.

If $\pi_1(M)$ is finite, then any geometry on which M is modelled must be compact, so there is only one candidate.

Conjecture 5.5 (Thurston's Elliptisation Conjecture) *Any closed orientable manifold with finite fundamental group admits a geometric structure modelled on $\mathbf{S}^3$.*

A trivial case of this is the Poincaré conjecture, which corresponds to $\pi_1(M) = 0$.

Conjecture 5.6 (Poincaré) *Any closed simply connected 3-manifold is homeomorphic to $\mathbf{S}^3$.*

In fact, the Elliptisation Conjecture follows from the Poincaré Conjecture and the following speculation.

Conjecture 5.7 *Every free action of a finite group on $\mathbf{S}^3$ is conjugate to an orthogonal action.*

For let M be a closed 3-manifold with finite fundamental group. Then by the Poincaré Conjecture its universal cover is $\mathbf{S}^3$ and by conjecture 5.7 the action of the fundamental group on $\mathbf{S}^3$ is by isometries.

The second case is if $\pi_1(M)$ is infinite. It is necessary to show that any atoroidal such M is hyperbolic. The case where M is *Haken* (M is prime and contains a connected orientable incompressible properly embedded surface other than S^2), which includes all compact manifolds with non-empty boundary, was proved by Thurston. This leaves the following.

Conjecture 5.8 *If M is closed, orientable, irreducible and atoroidal with infinite fundamental group but not Haken then M admits a hyperbolic structure.*

Very little progress has been made on this conjecture. More details are given in [2].

References

[1] W. P. Thurston, *Three-Dimensional Geometry and Topology, Volume 1 (Silvio Levy Ed)*, Princeton University Press, 1997.

[2] G. P. Scott, *The Geometries of 3-Manifolds*, Bull. London Math. Soc. 15 pp. 401-487, 1983.

[3] J. Hempel, *3-Manifolds*, Princeton University Press, 1976.

[4] A. Hatcher, *Basic Topology of 3-Manifolds*, http://www.math.cornell.edu/~hatcher/, 2000.

[5] M. Lackenby, *Hyperbolic Manifolds*, http://www.maths.ox.ac.uk/~lackenby/, 2000.