The Ward Correspondence and Stationary Axisymmetric Spacetimes

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1 Introduction

Over the last half-century the study of integrability has grown into a rich and significant branch of mathematics. The main features of integrable systems are, paradoxically, nonlinearity and tractability, which allows the detailed investigation of nonlinear phenomena that would frequently be impossible using other techniques. Much of the subject is unified under a particular symmetry, called self-duality, of the Yang–Mills equations. The Yang–Mills equations arise as generalizations of Maxwell's theory of electrodynamics, and in their quantized form constitute the main building block of the Standard Model of Particle Physics, our currently most accurate and most complete model of three of the four natural forces that we know about. The fourth, gravity, is described by Einstein's General Relativity.

A Yang–Mills theory, or gauge theory, is geometrically a description of a vector bundle over a region of spacetime, and the curvature of the connection on the bundle is interpreted as the physical Yang–Mills field. The self-dual Yang–Mills equations turn out to have deep connections to Roger Penrose's twistor construction. At its heart, twistor theory is a theory based on complex geometry. The construction unifies real spactimes of signature $(+, +, +, +)$ and $(-, +, +, +)$ in a complex space with signature $(-, -, +, +)$ in which the Yang–Mills equations happen to take on a particularly nice form. Now the self-duality condition can be interpreted geometrically, which allows one to define complex manifolds that are in some sense "orthogonal" to either the self-dual or anti-self-dual solutions of the Yang–Mills equations. These are called twistor spaces. A construction called the Penrose–Ward transform will allow us to take the vector bundle defined by a gauge theory and transfer it to a *holomorphic* vector bundle over twistor space. The holomorphic bundle will *not* have a connection, but instead will encode some of the information in its holomorphic structure. It turns out that this leads to a method of generating, in principle, *all* solutions to certain symmetry reductions of the self-dual Yang–Mills equations. Einstein's equations with certain symmetries make up one class of these reductions.

We will reduce the stationary axisymmetric anti-self-dual Yang–Mills equations to four classes of solutions of Einstein's equations. The four classes are stationary axisymmetric gravitational fields, cylindrical gravitational wave solutions, the Gowdy cosmological models, and the colliding plane wave solutions. These are all exact solutions of Einstein's equations. The first two classes are self-descriptive; the Gowdy cosmological models are solutions describing a spacetime filled with a regular pattern of gravitational waves of all wavelengths, while the colliding plane wave spacetimes are another set of exact solutions describing the collision of plane waves that may produce curvature singularities.

The structure of this essay is as follows. In section 2 we establish the prerequisite mathematical background. In particular, in section 2.2 we describe Riemann–Hilbert problems, in section 2.3 we introduce the self-dual Yang–Mills equations, and define twistors in section 2.6. In section 3 we develop and prove the Ward correspondence. We then discuss symmetry reductions in section 4 and then concentrate on two particular cases of Einstein fields in sections 5 and 6. The reader unfamiliar with gauge theories may like to start at the appendix.

The material presented in this essay is classical and well-known. Much of it is taken from

Mason & Woodhouse [3], but Huggett [1] provides a nice basic introduction to the theory. Ward & Wells [7] take a slightly different approach to some aspects, but is a good companion to the more mathematically minded physicist. A good overview of the Ward correspondence in action is provided in [6], while the material on cylindrical gravitational waves is based on [9].

2 Mathematical Background

2.1 Setting

In special relativity, the spacetime is Minkowksi space, a four-dimensional affine space with the metric $\mathrm{diag}(+1,-1,-1,-1)$, and a choice of orientation. Minkowski space $\mathbb{M}$ is geometrically different from Euclidean space $\mathbb{E}$, which has the trivial metric δ_{ij}. We will be interested in studying fields on both Minkowski space and Euclidean space in a unified manner, and for this reason it will be convenient for us to allow spacetime coordinates to take complex values. That is, we will think of these real spaces as embedded in *complexified Minkowski space*. Complexified Minkowski space $\mathbb{CM}$ is the set $\mathbb{C}^4$ endowed with the metric

$$\mathrm{d}s^2 = \eta_{ab}\mathrm{d}x^a\mathrm{d}x^b = 2(\mathrm{d}z\mathrm{d}\tilde{z} - \mathrm{d}w\mathrm{d}\tilde{w})$$

and the volume form

$$\nu = \nu_{abcd}\mathrm{d}x^a \wedge \mathrm{d}x^b \wedge \mathrm{d}x^c \wedge \mathrm{d}x^d = \mathrm{d}w \wedge \mathrm{d}\tilde{w} \wedge \mathrm{d}z \wedge \mathrm{d}\tilde{z},$$

where

$$\nu_{abcd} = \frac{1}{24}\sqrt{\det(\eta)}\varepsilon_{abcd}.$$

The coordinates $(w, z, \tilde{w}, \tilde{z})$ are called *double null coordinates.* We may recover $\mathbb{E}$ and $\mathbb{M}$ by imposing reality conditions on $w, z, \tilde{w}$ and $\tilde{z}$. We retrieve standard Euclidean space, or the *Euclidean slice* $\mathbb{E}$, by imposing the conditions

$$\begin{pmatrix} \tilde{z} & w \\ \tilde{w} & z \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} x^0 + \mathrm{i}x^1 & -x^2 + \mathrm{i}x^3 \\ x^2 + \mathrm{i}x^3 & x^0 - \mathrm{i}x^1 \end{pmatrix},$$

where x^0, x^1, x^2 and x^3 are required to be real. Notice that on $\mathbb{E}$ the metric η reduces to the standard Euclidean metric. Similarly, the Minkowski slice $\mathbb{M}$ is given by

$$\begin{pmatrix} \tilde{z} & w \\ \tilde{w} & z \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} x^0 + x^1 & x^2 - \mathrm{i}x^3 \\ x^2 + \mathrm{i}x^3 & x^0 - x^1 \end{pmatrix},$$

on which the metric reduces to the standard Minkowski metric. Complexified Minkowski space $\mathbb{CM}$ contains other real slices, such as the ultrahyperbolic slice $\mathbb{U}$, which we do not study in detail. For more details the reader should consult [3].

2.2 Riemann–Hilbert Problems

Suppose we have a smooth function F defined on some smooth closed curve in the complex plane. In a general sense, Riemann–Hilbert problems are concerned with "splitting" such functions into a function that is analytic in the exterior of the curve and a function that is analytic in the interior. In the linear setting, this amounts to writing F as a difference of two functions that are boundary values of a function analytic in the exterior, and a function analytic in the interior. We will need to know about when and how a Riemann–Hilbert problem is soluble when we describe the Penrose–Ward transform in the later sections.

To study the linear Riemann–Hilbert problem first, let $F : S^1 \to \mathbb{C}$ be a smooth function on the unit circle $S^1 = \{\zeta = \mathrm{e}^{\mathrm{i}\theta}\}$ in the complex ζ-plane. We can then write F as its Fourier series on S^1, in powers of $\zeta = \mathrm{e}^{\mathrm{i}\theta}$:

$$F(\theta) = \sum_{n=-\infty}^{\infty} F_n \zeta^n = \sum_{n=0}^{\infty} f_n \zeta^n - \sum_{n=0}^{\infty} \tilde{f}_n \zeta^{-n} = f(\zeta) - \tilde{f}(\zeta),$$

where f is the boundary value of a holomorphic function in the interior $|\zeta| < 1$ of the circle (defined by analytic continuation of the corresponding power series), and $\tilde{f}$ is the boundary value of a holomorphic function in the exterior $|\zeta| > 1$, including $\zeta = \infty$ (defined similarly). This splitting of F is unique up to compensating shifts in f_0 and $\tilde{f}_0$, i.e. up to $f \mapsto f + c$, $\tilde{f} \mapsto \tilde{f} + c$ for $c \in \mathbb{C}$.

A *non-linear* Riemann–Hilbert problem is to find an analogous splitting for a function $F : S^1 \to G$ when F takes values not in the additive group $\mathbb{C}$, but rather in some more general (non-abelian) complex Lie group G. If, for example, $\mathrm{G} = \mathbb{C}^\times$, the group of non-zero complex numbers under multiplication, then the problem is as follows. Given a smooth non-vanishing function F on S^1, we wish to find non-vanishing functions $f(\zeta)$ and $\tilde{f}(\zeta)$ such that $f(\zeta)$ is holomorphic in the interior $|\zeta| < 1$, $\tilde{f}(\zeta)$ is holomorphic in the exterior $|\zeta| > 1$, including $\zeta = \infty$, and such that $F = \tilde{f}^{-1} f$ on $S^1 = \{\zeta = \mathrm{e}^{\mathrm{i}\theta}\}$. Unlike in the linear (additive) case, such a factorization need not exist. Indeed, if it does, then we must have

$$\oint_{S^1} \frac{\mathrm{d}F}{F} = \oint_{S^1} \mathrm{d}(\log(\tilde{f}^{-1} f)) = \oint_{S^1} \frac{\mathrm{d}f}{f} - \oint_{S^1} \frac{\mathrm{d}\tilde{f}}{\tilde{f}} = 0$$

by Cauchy's theorem, as $\tilde{f}$ and f are holomorphic and non-vanishing on the upper and lower half of the Riemman sphere[1] respectively. This imposes a condition on F. Thus we could only hope to factorize F if its winding number

$$k = \frac{1}{2\pi \mathrm{i}} \oint_{S^1} \frac{\mathrm{d}F}{F} \in \mathbb{Z}$$

vanishes. In the event that it does, $\log F$ is then single-valued and we may construct f and $\tilde{f}$ by expanding $\log F$ in its Fourier series, splitting it as before, and exponentiating. Note that by virtue of this construction, in particular the exponentiation, the resulting functions are non-vanishing.

Nevertheless, even in the case of a non-zero winding number, not all is lost. If F has finite winding number k, then $\zeta^{-k} F$ has zero winding number, and can therefore be factorized. Indeed,

$$\oint_{S^1} \frac{\mathrm{d}(\zeta^{-k} F)}{\zeta^{-k} F} = \oint_{S^1} -k\zeta^{-1} \mathrm{d}\zeta + \oint_{S^1} \frac{\mathrm{d}F}{F} = 0.$$

Thus a smooth non-vanishing function on the circle can always be factorized as

$$F = \tilde{f}^{-1} \zeta^k f,$$

where k is the winding number of F, f is holomorphic in the interior of the circle, and $\tilde{f}$ is holomorphic in the exterior, including infinity. Thus we have completely solved the Riemann–Hilbert problem in the case $\mathrm{G} = \mathbb{C}^\times$.

[1] Here we think of the identification of $\mathbb{C} \cup \{\infty\}$ with the Riemann sphere S^2 by stereographic projection from the North Pole. Then the exterior of the unit circle $|\zeta| > 1$, together with $\zeta = \infty$, is identified with the upper half, while the interior $|\zeta| < 1$ is identified with the lower half of the Riemann sphere.

2.2.1 Birkhoff's factorization theorem

A more general result is provided by Birkhoff's factorization theorem. Let G be a compact Lie group.

Definition 2.1. The *loop group* of G is the group of smooth maps $\mathrm{LG} = \{F : S^1 \to \mathrm{G}\}$ under the composition inherited from the composition law of G.

So $\mathrm{LGL}(n, \mathbb{C})$ is the group of smooth maps $F : S^1 \to \mathrm{GL}(n, \mathbb{C})$ under pointwise matrix multiplication. For this particular loop group we also denote by $\mathrm{LGL}_+(n, \mathbb{C})$ the subset of loops that are boundary values of holomorphic maps on $\{|\zeta| < 1\}$, and by $\mathrm{LGL}_-(n, \mathbb{C})$ the subset of loops that are boundary values of holomorphic maps on $\{|\zeta| > 1\} \cup \{\infty\}$.

Theorem 2.1 (Birkhoff's factorization theorem [3, p. 146])**.** *Any loop $F \in \mathrm{LGL}(n, \mathbb{C})$ can be factorized as*

$$F = \tilde{f}^{-1} \Delta f,$$

where $f \in \mathrm{LGL}_+(n, \mathbb{C})$, $\tilde{f} \in \mathrm{LGL}_-(n, \mathbb{C})$, and $\Delta = diag(\zeta^{k_1}, \dots, \zeta^{k_n})$ for some $k_i \in \mathbb{Z}$. Furthermore, the k_is are unique up to permutation, and the loops for which $\Delta = \mathbb{1}$ are a dense open subset of the component of $\mathrm{LGL}(n, \mathbb{C})$ connected to the identity. For loops with $\Delta = \mathbb{1}$ the factorization is unique up to $\tilde{f} \mapsto c\tilde{f}$, $f \mapsto cf$ for some constant $c \in \mathrm{GL}(n, \mathbb{C})$.

The same result holds with $\mathrm{GL}(n, \mathbb{C})$ replaced with $\mathrm{SL}(n, \mathbb{C})$, whence all matrices are required to have unit determinant (in particular, $\sum k_i = 0$). Moreover, the theorem still holds if instead of holomorphic functions of ζ we work with rational functions of ζ, or with analytic functions of ζ.

2.2.2 Jumping points

A consequence of the way Birkhoff's theorem is proved in [4] is that if a factorization of a loop $F(w, \zeta)$, depending smoothly on some parameters $w = (w_1, w_2, \dots)$, exists with $\Delta = \mathbb{1}$ at some point w, then in fact a factorization with $\Delta = \mathbb{1}$ exists in an open neighbourhood of w. Moreover, the factors f and $\tilde{f}$ may be chosen to depend smoothly and regularly on the parameters. As the following proposition explicates, the above statement also holds when 'smooth' is replaced with 'holomorphic': if $F(w, \zeta)$ depends holomorphically on ζ in a neighbourhood of the unit circle, and holomorphically on w, then a factorization with $\Delta = \mathbb{1}$ at a point extends to a factorization with $\Delta = \mathbb{1}$ in an open neighbourhood, and the factors f and $\tilde{f}$ may be chosen to be regular almost everywhere and depend holomorphically on the coordinates w. Let V, $\tilde{V}$ be a two-set open cover of the Riemann sphere, where V is a neighbourhood of $\zeta = 0$, $\tilde{V}$ is a neighbourhood of $\zeta = \infty$, such that $A = V \cap \tilde{V}$ is an annulus in the complex plane containing the unit circle.

Proposition 1 (Ward (1984), [3, pp. 148–149])**.** *Let W be an open ball in $\mathbb{C}^k$ and let*

$$F : W \times A \to \mathrm{GL}(n, \mathbb{C})$$

be holomorphic. Suppose that for some point of W there is a Birkhoff factorization of F as a function of ζ with $\Delta = \mathbb{1}$. Then there exist holomorphic maps $f : W \times V \to \mathbb{C}^{n\times n}$, $\tilde{f} : W \times \tilde{V} \to \mathbb{C}^{n\times n}$ such that

(i) $\tilde{f}F = f$ on $W \times A$, and

(ii) for almost all $w \in W$, $\det f \neq 0$ on V and $\det \tilde{f} \neq 0$ on $\tilde{V}$.

As described in [3, §9.3], attempting to extend the $\Delta = \mathbb{1}$ factorization to the whole parameter space typically fails on a submanifold of codimension 1, where Δ 'jumps' to a matrix other than the identity. Proposition 1 says that in the holomorphic case the jumping singularities are at worst poles, because if f is holomorphic, then f^{-1} cannot have an essential singularity. As we will see in section 3, in the Ward construction, the parameters w are the spacetime coordinates and the jumping points of Δ give rise to singularities in the ASDYM potential Φ. In light of this we should like to know when we can ensure that the condition $\Delta = \mathbb{1}$ is satisfied.

Proposition 2 (Gohberg and Krein (1958), [3, p. 149])**.** *Suppose that $F \in \mathrm{LGL}(n, \mathbb{C})$ and that $F + F^\dagger$ is positive definite. Then in the Birkhoff factorization of F we have $\Delta = \mathbb{1}$.*

Proof. Suppose F has the factorization $F = \tilde{f}^{-1}\Delta f$, where $\Delta = \mathrm{diag}(\zeta^{k_1}, \ldots, \zeta^{k_n})$. Put

$$Q = f\tilde{f}^\dagger \quad \text{and} \quad P = \tilde{f}(F + F^\dagger)\tilde{f}^\dagger.$$

Then for any $z \in \mathbb{C}^n$, as $F + F^\dagger$ is positive definite,

$$z^\dagger P z = z^\dagger \tilde{f}(F + F^\dagger)\tilde{f}^\dagger z = (\tilde{f}^\dagger z)^\dagger (F + F^\dagger)(\tilde{f}^\dagger z) > 0,$$

i.e. P is positive definite. Moreover,

$$\begin{aligned} \Delta Q + Q^\dagger \Delta^\dagger &= \Delta f \tilde{f}^\dagger + \tilde{f} f^\dagger \Delta^\dagger \\ &= \tilde{f}(\tilde{f}^{-1}\Delta f + f^\dagger \Delta^\dagger \tilde{f}^{-\dagger})\tilde{f}^\dagger \\ &= \tilde{f}(F + F^\dagger)\tilde{f}^\dagger = P. \end{aligned}$$

Now as f is holomorphic on $|\zeta| < 1$, and $\tilde{f}$ is holomorphic on $|\zeta| > 1$, including $\zeta = \infty$, it follows that the Fourier series of the entries of Q contain only positive powers of ζ, while the Fourier series of the entries of $Q^\dagger$ contain only negative powers of ζ. As P is positive definite, the diagonal entries of P are positive real functions of ζ. Thus we must have $k_i \leqslant 0$ for each i. An analogous argument with Q replaced with $R = f^{-\dagger}(F + F^\dagger)f^{-1}$ shows that $k_i \geqslant 0$ for each i, whence $k_i = 0$ for each i. □

Proposition 2 holds for almost all $w \in W$ whenever F, f and $\tilde{f}$ depend on additional parameters w, by virtue of proposition 1.

2.3 The Yang–Mills Equations

Consider a gauge theory with a connection Φ on a vector bundle $E \to M$ over a manifold M. The connection defines a differential operator D (which we also sometimes refer to as the connection) that maps sections s of E to E-valued 1-forms. In a local trivialization,

$$\mathrm{D}s = (\mathrm{D}_a s)\mathrm{d}x^a = \mathrm{d}s + \Phi s.$$

We define the *curvature* of D to be the matrix-valued 2-form $F = F_{ab}\mathrm{d}x^a \wedge \mathrm{d}x^b$, where

$$F_{ab} = [\mathrm{D}_a, \mathrm{D}_b] = \mathrm{D}_a\mathrm{D}_b - \mathrm{D}_b\mathrm{D}_a.$$

The *Yang–Mills equations* are $\mathrm{D}F = 0$ and $\mathrm{D} * F = 0$. In components these read

$$\mathrm{D}_{[a}F_{bc]} = 0 \quad \text{and} \quad \mathrm{D}^a F_{ab} = 0$$

respectively. These are generalizations of Maxwell's equations, as described in appendix A.3. The first of these is a consequence of the Jacobi identity for the operator D_a, while the second is the Euler–Lagrange equation of the Lagrangian density

$$\mathcal{L} = \frac{1}{4}\mathrm{Tr}(F_{ab}F^{ab}).$$

2.3.1 The anti-self-dual Yang–Mills equation

The *anti-self-dual Yang–Mills* (ASDYM) *equation* is the condition

$$*F = -F. \tag{1}$$

As the terminology suggests, a solution to (1) necessarily satisfies the Yang–Mills equations, since $\mathrm{D}F = 0$ is true for an arbitrary connection, and (1) implies that $\mathrm{D} * F = -\mathrm{D}F = 0$.

One alternative way of expressing the anti-self-duality condition is in terms of the commutativity of a so-called Lax pair. Let Φ be a connection on a complex rank-n vector bundle E over some region U in real or complex spacetime, and let F be its curvature 2-form. In a local trivialization the components of F are

$$F_{ab} = \partial_a \Phi_b - \partial_b \Phi_a + [\Phi_a, \Phi_b].$$

In double null coordinates $(w, z, \tilde{w}, \tilde{z})$ the anti-self-duality condition $*F = -F$ becomes

$$\partial_z \Phi_w - \partial_w \Phi_z + [\Phi_z, \Phi_w] = 0, \tag{2}$$
$$\partial_{\tilde{z}} \Phi_{\tilde{w}} - \partial_{\tilde{w}} \Phi_{\tilde{z}} + [\Phi_{\tilde{z}}, \Phi_{\tilde{w}}] = 0, \tag{3}$$
$$\partial_z \Phi_{\tilde{z}} - \partial_{\tilde{z}} \Phi_z - \partial_w \Phi_{\tilde{w}} + \partial_{\tilde{w}} \Phi_w + [\Phi_z, \Phi_{\tilde{z}}] - [\Phi_w, \Phi_{\tilde{w}}] = 0. \tag{4}$$

Writing

$$\mathrm{D}_w = \partial_w + \Phi_w, \quad \mathrm{D}_z = \partial_z + \Phi_z, \quad \mathrm{D}_{\tilde{w}} = \partial_{\tilde{w}} + \Phi_{\tilde{w}} \quad \text{and} \quad \mathrm{D}_{\tilde{z}} = \partial_{\tilde{z}} + \Phi_{\tilde{z}},$$

these become

$$[\mathrm{D}_z, \mathrm{D}_w] = 0, \quad [\mathrm{D}_{\tilde{z}}, \mathrm{D}_{\tilde{w}}] = 0 \quad \text{and} \quad [\mathrm{D}_z, \mathrm{D}_{\tilde{z}}] - [\mathrm{D}_w, \mathrm{D}_{\tilde{w}}] = 0,$$

or equivalently that the *Lax pair* of operators

$$L = \mathrm{D}_w - \zeta \mathrm{D}_{\tilde{z}} \quad \text{and} \quad M = \mathrm{D}_z - \zeta \mathrm{D}_{\tilde{w}} \tag{5}$$

should commute for all complex values of the *spectral parameter* ζ. The operators L and M are referred to as a *linear system* for the anti-self-dual Yang–Mills equations.

2.3.2 Yang's equation

The anti-self-duality condition $*F = -F$ is coordinate independent and manifestly gauge-invariant, as well as invariant under conformal isometries of spacetime, which we will introduce in section 2.5. However, it is possible to break one or more of these symmetries to rewrite the equation in a more tractable way. The first anti-self-duality condition in double null coordinates (2) is the statement that the operators $\partial_w + \Phi_w$ and $\partial_z + \Phi_z$ commute, which is a local Frobenius integrability condition for the existence of a matrix-valued function h such that

$$\partial_w h + \Phi_w h = 0,$$
$$\partial_z h + \Phi_z h = 0.$$

It is uniquely determined by Φ up to $h \mapsto h\tilde{P}$, where $\tilde{P}$ depends only on $\tilde{w}$ and $\tilde{z}$. Similarly, (3) is a local integrability condition for the existence of a matrix-valued function $\tilde{h}$ such that

$$\partial_{\tilde{w}} \tilde{h} + \Phi_{\tilde{w}} \tilde{h} = 0,$$
$$\partial_{\tilde{z}} \tilde{h} + \Phi_{\tilde{z}} \tilde{h} = 0,$$

which is uniquely determined by Φ up to $\tilde{h} \mapsto \tilde{h}P$, where P depends only on w and z. How do h and $\tilde{h}$ transform under a gauge transformation? Under a gauge transformation

$$\Phi \to g^{-1}\Phi g + g^{-1}\mathrm{d}g$$

so that, for example,

$$\Phi_w \to g^{-1}\Phi_w g + g^{-1}\partial_w g.$$

We then wish to find out how h transforms under a gauge transformation, say $h \to g(h)$, in a way that

$$(\partial_w h)h^{-1} = -\Phi_w$$

becomes

$$(\partial_w g(h))g(h)^{-1} = -g^{-1}\Phi_w g - g^{-1}\partial_w g.$$

Observing that if $h \to g(h) = g^{-1}h$, then

$$(\partial_w g(h))g(h)^{-1} = -g^{-1}(\partial_w g) + g^{-1}(\partial_w h)h^{-1}g,$$

we conclude that $h \to g^{-1}h$ is the correct gauge transformation for h. Similarly, of course, $\tilde{h} \to g^{-1}\tilde{h}$ under a gauge transformation. Notice that then

$$\tilde{h}^{-1}h \to (g^{-1}\tilde{h})^{-1}g^{-1}h = \tilde{h}^{-1}gg^{-1}h = \tilde{h}^{-1}h,$$

so that $J = \tilde{h}^{-1}h$ is a gauge invariant quantity. The matrix J is called *Yang's matrix*. It is determined by Φ up to $J \mapsto P^{-1}J\tilde{P}$, and conversely J determines Φ. Indeed,

$$\begin{aligned} J^{-1}\tilde{\partial}J &= J^{-1}\partial_{\tilde{w}}J\mathrm{d}\tilde{w} + J^{-1}\partial_{\tilde{z}}J\mathrm{d}\tilde{z} = h^{-1}(\Phi_{\tilde{w}}\mathrm{d}\tilde{w} + \Phi_{\tilde{z}}\mathrm{d}\tilde{z})h, \\ J\partial J^{-1} &= J\partial_w J^{-1}\mathrm{d}w + J\partial_z J^{-1}\mathrm{d}z = \tilde{h}^{-1}(\Phi_w\mathrm{d}w + \Phi_z\mathrm{d}z)\tilde{h}, \end{aligned} \tag{6}$$

where $\partial = \mathrm{d}w\partial_w + \mathrm{d}z\partial_z$ and $\tilde{\partial} = \mathrm{d}\tilde{w}\partial_{\tilde{w}} + \mathrm{d}\tilde{z}\partial_{\tilde{z}}$ are the components of the Dolbeault decomposition of the exterior derivative $\mathrm{d} = \partial + \tilde{\partial}$. Comparing these to the gauge transformations

$$\Phi \to h^{-1}\Phi h + h^{-1}\mathrm{d}h, \tag{7}$$

$$\Phi \to \tilde{h}^{-1}\Phi\tilde{h} + \tilde{h}^{-1}\mathrm{d}\tilde{h}, \tag{8}$$

we see that $J^{-1}\tilde{\partial}J$ and $J\partial J^{-1}$ are both simply Φ in the gauges (7) and (8) respectively. So it is enough to consider only one of them, say $J^{-1}\tilde{\partial}J$, to deduce Φ. It is easy to check that the first two anti-self-duality conditions (2) and (3) are satisfied identically by $J^{-1}\tilde{\partial}J$. Putting $J^{-1}\tilde{\partial}J$ and

$$\begin{aligned} (h^{-1}\mathrm{d}h)_w &= h^{-1}\partial_w h = -\Phi_w \\ (h^{-1}\mathrm{d}h)_z &= h^{-1}\partial_z h = -\Phi_z \end{aligned}$$

into the third condition (4), we find that, since $\partial_{\tilde{z}}(J^{-1}\partial_z J) = 0 = \partial_{\tilde{w}}(J^{-1}\partial_w J)$,

$$\partial_z\Phi_{\tilde{z}} - \partial_{\tilde{z}}\Phi_z - \partial_w\Phi_{\tilde{w}} + \partial_{\tilde{w}}\Phi_w + [\Phi_z, \Phi_{\tilde{z}}] - [\Phi_w, \Phi_{\tilde{w}}] = 2\partial_z(J^{-1}\partial_{\tilde{z}}J) - 2\partial_w(J^{-1}\partial_{\tilde{w}}J).$$

So the third anti-self-duality condition (4) is satisfied if and only if

$$\boxed{\partial_w(J^{-1}\partial_{\tilde{w}}J) - \partial_z(J^{-1}\partial_{\tilde{z}}J) = 0.} \tag{9}$$

Equation (9) is called *Yang's equation*, and is equivalent to the ASDYM equations. However, it is not covariant under coordinate transformations which change the tangent planes spanned by ∂_w and ∂_z, and by $\partial_{\tilde{w}}$ and $\partial_{\tilde{z}}$.

We will see that J is the zeroth Fourier coefficient of F, i.e. that the factorization $F = \tilde{f}^{-1}(\zeta)f(\zeta)$ can be reduced to give $J = \tilde{f}^{-1}(\infty)f(0)$. If we impose the gauge condition that $f(0) = 1$, then J is determined uniquely by F and given by

$$J = \tilde{f}^{-1}(\infty).$$

2.4 Spinors

The two main particle types that occur in nature, bosons and fermions, are characterized by an observationally inferrable quantity s, called *spin*, which takes values in $\{0, 1/2, 1, 3/2, 2, \dots\}$. Bosons are the particles with integer spin, while fermions are the ones with half-integer spin. A familiar example of a boson is the photon (a boson of spin 1), which is described by the Maxwell 2-form F. The correct mathematical tools for describing bosons are tensor fields, but to describe fermions it turns out that we need objects called *spinors*. Our main use of spinors will be to make covariance of certain equations explicit, but their power extends much further.

Our starting point is the isomorphism of groups

$$\mathrm{SO}(4, \mathbb{C}) \simeq \mathrm{SL}(2, \mathbb{C})_L \times \mathrm{SL}(2, \mathbb{C})_R/\mathbb{Z}_2, \tag{10}$$

under which complex rotations in four dimensions are decomposed into products of left and right rotations (corresponding to the two factors of $\mathrm{SL}(2, \mathbb{C})$, respectively), up to a sign. Tensors in complex spacetime, then, since they transform under $\mathrm{SO}(4, \mathbb{C})$, may be replaced by products of spinors, which will transform under $\mathrm{SL}(2, \mathbb{C})$.

Definition 2.2. We denote by $\mathcal{S}$ the fundamental representation of $\mathrm{SL}(2, \mathbb{C})_L$ and by $\mathcal{S}'$ the anti-fundamental (or conjugate) representation of $\mathrm{SL}(2, \mathbb{C})_R$. We call the complex vector space $\mathcal{S}$ the *spin space* and $\mathcal{S}'$ the *primed spin space*.

Definition 2.3. A *spinor of type* (m, n, m', n') is an element of the tensor product

$$\overbrace{\mathcal{S} \otimes \cdots \otimes \mathcal{S}}^{m} \otimes \overbrace{\mathcal{S}^* \otimes \cdots \otimes \mathcal{S}^*}^{n} \otimes \overbrace{\mathcal{S}' \otimes \cdots \otimes \mathcal{S}'}^{m'} \otimes \overbrace{\mathcal{S}'^* \otimes \cdots \otimes \mathcal{S}'^*}^{n'},$$

where * denotes the dual space.

The abstract spinor indices are capital Roman letters, either primed $(A', B', \dots)$ or unprimed $(A, B, \dots)$, and we denote the elements of $\mathcal{S}, \mathcal{S}^*, \mathcal{S}', \mathcal{S}'^*$ by $\alpha^A, \beta_B, \gamma^{C'}$, and $\delta_{D'}$ respectively. We use primed indices in the primed spin spaces and lower indices in the dual spaces. The four spin spaces are two-dimensional, and we make the convention that the values of the indices run over the two values 0 and 1, and denote by $0'$ and $1'$ the particular values of primed indices to be able to distinguish which spaces our spinors live in.

The $\mathrm{SL}(2, \mathbb{C})$ transformations of $\mathcal{S}$ and $\mathcal{S}'$ are symplectic: they preserve the skew-symmetric 2-spinors $\varepsilon^{AB}, \varepsilon^{A'B'}, \varepsilon_{AB}, \varepsilon_{A'B'}$, each with components

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We may interpret ε_{AB} as a bilinear form on $\mathcal{S}$, whence ε^{AB} is the dual symplectic form on $\mathcal{S}^*$ by virtue of the fact that $\varepsilon^{AB}\varepsilon_{CB} = \delta^A_C$. This provides an identity endomorphism on both $\mathcal{S}$ and $\mathcal{S}^*$, and the symplectic forms ε_{AB} and ε^{AB} provide natural isomorphisms between $\mathcal{S}$ and $\mathcal{S}^*$ which we use to raise and lower indices. Due to the skew-symmetry it is important to keep track of the order of indices, and we follow the convention of Ward & Wells [7] and Mason & Woodhouse [3] and define

$$\varepsilon^{AB}\psi_B = \psi^A$$
$$\psi^B \varepsilon_{BA} = \psi_A.$$

A useful mnemonic is 'adjacent indices, descending to the right'. Note that $\alpha^A\beta_A = -\alpha_A\beta^A$.

Of course all of this structure has a primed version, and the two are, by definition, related by complex conjugation. More precisely, complex conjugation is an anti-isomorphsim from the primed to the unprimed spin space,

$$\overline{} : \mathcal{S} \to \mathcal{S}',$$
$$\psi^A \mapsto \overline{\psi}^{A'}.$$

For mixed spinors such as $\psi^A{}_{BA'}$ the rules for abstract indices dictate that we observe the ordering of primed indices, whether subscripts or superscripts, and also the ordering of unprimed indices. But the relative ordering between a primed and unprimed index is irrelevant. Moreover, the two-dimensionality of the spin spaces implies that 'skew spinors are pure traces'. For example, if ξ_{AB} is any spinor, then

$$\xi_{AB} - \xi_{BA} = \varepsilon_{AB}\xi_C{}^C.$$

2.4.1 Tensors as spinors

Let T be the space of complex 4-vectors in $\mathbb{CM}$. We construct the identification $T = \mathcal{S} \otimes \mathcal{S}'$ by identifying the displacement vector from the origin $x^a = (w, z, \tilde{w}, \tilde{z})$ in double null coordinates with the 2-spinor $x^{AA'}$ given by

$$x^{AA'} = \begin{pmatrix} \tilde{z} & w \\ \tilde{w} & z \end{pmatrix}. \tag{11}$$

The dual of this map of course identifies $T^* = \mathcal{S}^* \otimes \mathcal{S}'^*$, so that a 1-form ω_a in $\mathbb{CM}$ is identified with a 2-spinor $\omega_{AA'}$. In particular, the coordinate derivatives ∂_a become

$$\partial_{AA'} = \begin{pmatrix} \partial_{\tilde{z}} & \partial_w \\ \partial_{\tilde{w}} & \partial_z \end{pmatrix}.$$

Under a coordinate transformation of the double null system,

$$\begin{pmatrix} \tilde{z} & w \\ \tilde{w} & z \end{pmatrix} \mapsto \Lambda \begin{pmatrix} \tilde{z} & w \\ \tilde{w} & z \end{pmatrix} \tilde{\Lambda}^t$$

for some $\Lambda, \tilde{\Lambda} \in \mathrm{SL}(2, \mathbb{C})$, which are determined up to a sign. In spinor notation,

$$x^{AA'} \mapsto \Lambda^A{}_B \tilde{\Lambda}^{A'}{}_{B'} x^{BB'}.$$

So in a similar way to tensorial expressions, which, if true in one coordinate system, then are true in all coordinate systems, spinorial expressions are "covariant up to a sign". The idea is that if we work with spinors but eventually reconvert our expressions to tensorial expressions, the sign ambiguity disappears[2].

[2] In flat spacetime. See [3, §9.9]

With this notation the spinor equivalents of the Minkowski metric η_{ab} and the alternating tensor ε_{abcd} are

$$\eta_{ab} = \eta_{ABA'B'} = \varepsilon_{AB}\varepsilon_{A'B'},$$
$$\varepsilon_{abcd} = \varepsilon_{ABCDA'B'C'D'} = \varepsilon_{AC}\varepsilon_{BD}\varepsilon_{A'D'}\varepsilon_{B'C'} - \varepsilon_{AD}\varepsilon_{BC}\varepsilon_{A'C'}\varepsilon_{B'D'}.$$

Thus

$$\mathrm{d}s^2 = \varepsilon_{AB}\varepsilon_{A'B'}\mathrm{d}x^{AA'}\mathrm{d}x^{BB'}.$$

We also extend the domain of definition of the space time connection $\nabla_a = \nabla_{AA'}$ so that it acts on spinor as well as tensor fields. This is done in such a way that the derivative of ε vanishes: $\nabla_{AA'}\varepsilon_{BC} = 0$.

2.4.2 SD and ASD 2-forms

A *spin frame* in $\mathcal{S}$ is a basis o^A, ι^A of $\mathcal{S}$ such that

$$o_A \iota^A = 1.$$

Unless otherwise stated, we will use the standard spin frames for $\mathcal{S}$ and $\mathcal{S}'$ given by $o^A = (1,0) = o^{A'}$ and $\iota^A = (0,1) = \iota^{A'}$.

Any 2-form $\gamma_{ab} = \gamma_{ABA'B'}$ can be written as

$$\gamma_{ABA'B'} = \gamma_{(AB)[A'B']} + \gamma_{[AB](A'B')},$$

since $\gamma_{ab} = -\gamma_{ba}$. As noted above, any skew-symmetric 2-spinor is necessarily a multiple of ε, and so

$$\gamma_{ABA'B'} = \phi_{AB}\varepsilon_{A'B'} + \psi_{A'B'}\varepsilon_{AB}$$

for some symmetric spinors ϕ an ψ. This is the decomposition of γ into its ASD and SD parts respectively, as can be checked by using the spinor decomposition of the alternating tensor.

Thus for the curvature 2-form F_{ab} of a connection D_a, the ASDYM equation is the statement that the SD part of F vanishes, i.e. that

$$F_{ABA'B'} = \psi_{AB}\varepsilon_{A'B'},$$

where $\psi_{AB} = \psi_{(AB)}$.

2.4.3 Spinorial version of Yang's equation

Noting (11), we may rewrite Yang's equation (9) in spinorial form:

$$\partial_{01'}(J^{-1}\partial_{10'}J) - \partial_{11'}(J^{-1}\partial_{00'}J) = 0.$$

By raising the first index on the outer partial derivatives, we get

$$\partial^1{}_{1'}(J^{-1}\partial_{10'}J) + \partial^0{}_{1'}(J^{-1}\partial_{00'}J) = 0,$$

which is

$$\iota^{A'}\partial^B{}_{A'}(J^{-1}o^{B'}\partial_{BB'}J) = 0. \tag{12}$$

This form of Yang's equation is manifestly covariant, and will be useful for our geometric description of the Penrose–Ward transform.

2.5 Compactified Complexified Minkowski Space

2.5.1 Geometry of null planes in $\mathbb{CM}$

Definition 2.4. We say that a 2-plane in $\mathbb{CM}$ is *totally null* if $\eta(A, B) = 0$ for every pair of tangent vectors A, B. With each totally null plane Π we associate a *tangent bivector* $\pi = A \wedge B$, that is $\pi^{ab} = A^{[a}B^{b]}$, for two independent tangent vectors A and B. A tangent bivector π determines the tangent space to the 2-plane, and is determined by the tangent space up to multiplication by a non-zero scalar.

Proposition 3. *If* Π *is a null 2-plane, then* $\pi_{ab}\pi^{ab} = 0$ *and* $\pi = \pi_{ab}\mathrm{d}x^a \wedge \mathrm{d}x^b$ *is either self-dual or ant-self-dual.*

Proof. A calculation shows that $\pi_{ab}\pi^{ab} = -2(\eta_{ab}A^aB^b)^2 = 0$. We also calculate

$$(*\pi)_{ab} = \sqrt{\det(\eta)}\varepsilon_{abcd}A^cB^d,$$

so as A and B span Π, for any $P \in \Pi$ we have $(*\pi)_{ab}P^a = 0$. In fact this characterizes $(*\pi)$ up to a non-zero scalar multiple. But we also have $\pi_{ab}P^a = 0$ since A and B span Π. Thus $(*\pi) = \mu\pi$ for some $\mu \neq 0$. But the eigenvalues of the Hodge star $*$ here are ± 1, so $*\pi = \pm\pi$. $\square$

Definition 2.5. We say a totally null plane Π is an α-plane whenever π is self-dual and a β-plane whenever π is anti-self-dual.

It is an exercise in applying the Hodge star operator to check that the self-dual 2-forms on $\mathbb{CM}$ are spanned by

$$\begin{aligned}\omega_1 &= \mathrm{d}w \wedge \mathrm{d}z,\\ \omega_2 &= \mathrm{d}w \wedge \mathrm{d}\tilde{w} - \mathrm{d}z \wedge \mathrm{d}\tilde{z},\\ \omega_3 &= \mathrm{d}\tilde{z} \wedge \mathrm{d}\tilde{w},\end{aligned}$$

and the anti-self-dual 2-forms on $\mathbb{CM}$ are spanned by

$$\begin{aligned}\rho_1 &= \mathrm{d}w \wedge \mathrm{d}\tilde{z},\\ \rho_2 &= \mathrm{d}w \wedge \mathrm{d}\tilde{w} + \mathrm{d}z \wedge \mathrm{d}\tilde{z},\\ \rho_3 &= \mathrm{d}\tilde{w} \wedge \mathrm{d}z.\end{aligned}$$

Furthermore, $\omega_i \wedge \rho_j = 0$ for all $i, j = 1, 2, 3$. We thus immediately see that surfaces of constant w, z are α-planes, as are surfaces of constant $\tilde{w}, \tilde{z}$. A general α-plane has a tangent bivector corresponding to the 2-form $\pi = \theta^1\omega_1 + \theta^2\omega_2 + \theta^3\omega_3$ for some complex numbers θ^i, $i = 1, 2, 3$. A mundane calculation to raise the indices of π_{ab} and split it into a wedge product of two linearly independent vectors then shows the following. Every α-plane passing through the origin, apart from the plane $w = z = 0$, has a unique (up to a constant) tangent bivector π given by

$$\pi^{ab} = l^{[a}m^{b]},$$

where

$$l = \partial_w - \zeta\partial_{\tilde{z}}, \qquad m = \partial_z - \zeta\partial_{\tilde{w}}$$

for some $\zeta \in \mathbb{C}$. Conversely, for every $\zeta \in \mathbb{C}$ the span of l and m is an α-plane through the origin. Thus by associating with $\zeta = \infty$ the span of $\partial_{\tilde{w}}$ and $\partial_{\tilde{z}}$ we obtain a one-to-one correspondence

$\Pi_\zeta \longleftrightarrow \zeta$ between α-planes through the origin and points on the Riemann sphere $\mathbb{C} \cup \{\infty\}$. The parameter ζ is called the *spectral parameter*.

The similarities between the tangent vectors l and m and the Lax pair of operators L and M for the ASDYM equation are not accidental; indeed, by the same reasoning the parameter ζ of section 2.3 is an affine coordinate on $\mathbb{P}\mathcal{S}' = \mathbb{CP}^1 = S^2 = \mathbb{C} \cup \{\infty\}$. We can even make this correspondence explicitly covariant by using spinor notation as follows. Since the components $\pi_{0'}$ and $\pi_{1'}$ of $\pi_{A'} = (\pi_{0'}, \pi_{1'})$ are homogeneous coordinates on $\mathcal{S}'$, defining

$$\zeta = \frac{\pi_{1'}}{\pi_{0'}}$$

gives a coordinate on $\mathbb{P}\mathcal{S}'$. For $\zeta \neq \infty$ we may take $\pi_{0'} = 1$ and $\pi_{1'} = \zeta$, or

$$\pi_{A'} = \zeta o_{A'} - \iota_{A'}$$

in the standard spin frame. Similarly, for $\zeta \neq 0$ we may write $\tilde{\zeta} = 1/\zeta$ and take

$$\pi_{A'} = o_{A'} - \tilde{\zeta}\iota_{A'}.$$

Now as we saw in section 2.4.2, the spinor equivalent of an SD 2-form is $\psi_{A'B'}\varepsilon_{AB}$, so any null SD 2-form has the form $\pi_{A'}\pi_{B'}\varepsilon_{AB}$. Since conversely $\pi_{A'}\pi_{B'}\varepsilon_{AB} = \pi_{ab}$ defines an α-plane, we see that every α-plane through the origin is labelled by a non-zero spinor $\pi_{A'}$, up to scale. Putting $\pi_{A'} = \zeta o_{A'} - \iota_{A'}$ in $V = \{\zeta \neq \infty\}$ and $\pi_{A'} = o_{A'} - \tilde{\zeta}\iota_{A'}$ in $\tilde{V} = \{\zeta \neq 0\}$, we obtain the same labelling of the manifold $\mathbb{CP}^1$ of null planes as above.

We shall see shortly that in the twistor construction a general α-plane, one not necessarily passing through the origin, is labelled by three complex coordinates: the parameter ζ, which determines the tangent space, together with $\zeta w + \tilde{z}$ and $\zeta z + \tilde{w}$, which are constant on the α-plane. This can be seen immediately by considering the action of the vectors l and m on $\zeta w + \tilde{w}$ and $\zeta z + \tilde{w}$. The whole space of α-planes, including those at infinity, turns out to be $\mathbb{CP}^3$ as a manifold.

2.5.2 The complex conformal group

A natural question to consider in connection with spinor calculus is how to define the Lie derivative of a spinor α^A along a vector field K. A sensible definition would have to satisfy the Leibniz rule, so the Lie derivative of the symplectic form ε_{AB}, being skew-symmetric, would have to be a multiple of itself:

$$\mathcal{L}_K \varepsilon_{AB} = \lambda \varepsilon_{AB}.$$

Recalling the decomposition of the metric $\eta_{ab} = \varepsilon_{AB}\varepsilon_{A'B'}$, we see that this would entail that

$$\mathcal{L}_K \eta_{ab} = (\lambda + \overline{\lambda})\eta_{ab} = k\eta_{ab} \tag{13}$$

for some real number k. Thus we could only hope to define the Lie derivative of a spinor along a vector field K if K satisfies eq. (13). Such vector fields are called *conformal Killing vector fields.* Equation (13) is equivalent to

$$\nabla_a K_b + \nabla_b K_a = k\eta_{ab},$$

which is in turn the same as

$$\partial_a K_b + \partial_b K_a = \frac{1}{4}\partial_c K^c \eta_{ab}. \tag{14}$$

The general solution to eq. (14) is

$$K_a = T_a + L_{ab}x^b + Rx^a + x^b x_b S_a - 2S_b x^b x_a,$$

where the coefficients are constants, and $L_{ab} = -L_{ba}$. Thus a general conformal Killing field has 15 parameters, which are four translations T_a and six rotations L_{ab} corresponding to the Poincaré group, one dilation R, and four *special conformal transformations* S_a. To investigate the latter, we consider the integral curves of S_a, which are solutions x^a to

$$\frac{\mathrm{d}x^a}{\mathrm{d}t} = S^a x^b x_b - 2S_b x^b x^a.$$

A calculation shows that the solution to this is given by

$$x^a(t) = \frac{x^a(0) + tS^a x^b(0)x_b(0)}{1 + 2tS_b x^b(0) + t^2 S_b S^b x_c(0)x^c(0)}.$$

From this we immediately see that the vector field $x^a(t)$ is incomplete on $\mathbb{CM}$: the special conformal transformations may map finite points of $\mathbb{CM}$ to infinity. Thus in order to have a proper description of the complex conformal group we adjoin to $\mathbb{CM}$ a light cone at infinity to obtain *compactified complexified Minkowski space* $\mathbb{CM}^{\#}$. This has a conformal structure (that is, it is a complex manifold with a metric defined up to the conformal equivalence $\eta_{ab} \sim \Omega^2\eta_{ab}$) and an orientation. We call transformations ρ of $\mathbb{CM}^{\#}$ such that $\rho^*\eta = \Omega^2\eta$ and $\rho^*\nu = \Omega^4\nu$ *proper conformal transformations.* These map $\mathbb{CM}^{\#} \to \mathbb{CM}^{\#}$, and form a fifteen-dimensional group called the *complex conformal group.*

Conformal Killing vector fields are natural objects when the geometry of $\mathbb{CM}^{\#}$ is considered in terms of α-planes and β-planes. It can be shown that the flow along a conformal Killing vector K moves α-planes into α-planes, and β-planes into β-planes. Moreover, if the 2-form $\mathrm{d}K^\flat$ is everywhere self-dual, then the flow along K maps β-planes to parallel β-planes, and if $\mathrm{d}K^\flat$ is everywhere anti-self-dual, the flow along K maps α-planes to parallel α-planes. The conformal Killing vector K is said to be SD (ASD) whenever the 2-form $\mathrm{d}K^\flat$ is SD (ASD) everywhere.

We were led to the conformal group by the desire to define the Lie derivative of a spinor. Although we will not be using it, for completeness we now give this definition.

Definition 2.6. For a spinor α^A and a conformal Killing vector K we define

$$\mathcal{L}_K\alpha^A = K^b\nabla_b\alpha^A - \phi^A{}_B\alpha^B - \frac{k}{4}\alpha^A,$$

where $k = \frac{1}{2}\nabla_b K^b$ and ϕ_{AB} is the ASD part of the 2-form F_{ab} given by

$$F_{ab} = \nabla_a K_b - \frac{k}{2}\eta_{ab}.$$

2.6 Twistors

2.6.1 The twistor space of $\mathbb{CM}$ and $\mathbb{CM}^{\#}$

In general, an α-plane passing through the point $x^a = (w, z, \tilde{w}, \tilde{z})$ has equations of the form

$$\zeta w + \tilde{z} = \lambda \quad \text{and} \quad \zeta z + \tilde{w} = \mu, \tag{15}$$

where λ and μ are constant. Its tangent space is spanned by the vectors

$$l = \partial_w - \zeta\partial_{\tilde{z}} \quad \text{and} \quad m = \partial_z - \zeta\partial_{\tilde{w}},$$

or by $\partial_{\tilde{z}}$ and $\partial_{\tilde{w}}$ in the limiting case $\zeta = \infty$. Thus the α-planes in complex spacetime, other than those corresponding to infinite ζ, are labelled by the three complex coordinates λ, μ and ζ, and the set of α-planes through a given point has the structure of a Riemann sphere with affine coordinate ζ, as discussed in section 2.5.1. We see then that the set of all α-planes in complex spacetime is a three-dimensional complex manifold, which we denote by $\mathcal{P}_{\mathbb{CM}}$.

We can rewrite (15) in the homogeneous form

$$\tilde{z}Z^2 + wZ^3 = Z^0 \quad \text{and} \quad \tilde{w}Z^2 + zZ^3 = Z^1 \tag{16}$$

for some complex Z^α, $\alpha = 0, 1, 2, 3$, where $Z^2 \neq 0$. Equations (15) and (16) are equivalent with

$$\lambda = \frac{Z^0}{Z^2}, \quad \mu = \frac{Z^1}{Z^2}, \quad \text{and} \quad \zeta = \frac{Z^3}{Z^2}.$$

Moreover, we may identify the case $Z^2 = 0$, $Z^3 \neq 0$ with $\zeta = \infty$, when the tangent space is spanned by $\partial_{\tilde{w}}$ and $\partial_{\tilde{z}}$ (this corresponds to the α-planes of constant w and z). Thus interpreting $\{Z^\alpha\}$ as homogeneous coordinates, we may identify the twistor space of $\mathbb{CM}$,

$$\mathcal{P}_{\mathbb{CM}} = \{(w, \tilde{w}, z, \tilde{z}) \in \mathbb{CM} : \zeta w + \tilde{z} = \lambda,\ \zeta z + \tilde{w} = \mu \text{ for some } \lambda, \mu \in \mathbb{C}\},$$

with an open subset of $\mathbb{CP}^3$. The points of $\mathbb{CP}^3$ that are excluded lie on the line $I = \{Z^2 = 0 = Z^3\}$, i.e. the set $\mathbb{CP}^3 \supset \{Z^0, Z^1, 0, 0\} = \mathbb{CP}^1$. Thus as a complex manifold, the twistor space of $\mathbb{CM}$ is $\mathcal{P}_{\mathbb{CM}} = \mathbb{CP}^3 - \mathbb{CP}^1$. It may be covered by two charts, V and $\tilde{V}$, given by

$$\begin{aligned} V &= \{Z^2 \neq 0\} = \{\zeta \neq \infty\}, \\ \tilde{V} &= \{Z^3 \neq 0\} = \{\zeta \neq 0\}. \end{aligned}$$

On the coordinate patch V we simply use λ, μ, and ζ as coordinates, whereas on $\tilde{V}$ we use

$$\tilde{\lambda} = \frac{Z^0}{Z^3}, \quad \tilde{\mu} = \frac{Z^1}{Z^3}, \quad \text{and} \quad \tilde{\zeta} = \frac{Z^2}{Z^3}.$$

On the intersection $V \cap \tilde{V} = \{\zeta \neq 0\} \cap \{\zeta \neq \infty\}$ we then have

$$\tilde{\lambda} = \frac{\lambda}{\zeta}, \quad \tilde{\mu} = \frac{\mu}{\zeta}, \quad \text{and} \quad \tilde{\zeta} = \frac{1}{\zeta}.$$

Moreover, we denote by $\mathbb{T}$ the copy of $\mathbb{C}^4$ on which (Z^0, Z^1, Z^2, Z^3) are linear coordinates, and by $\mathbb{PT}$ the corresponding projective space, which is of course a copy of $\mathbb{CP}^3$. To construct the twistor space of $\mathbb{CM}$ in the previous paragraph we excised the line I, a copy of $\mathbb{CP}^1$, out of $\mathbb{CP}^3$. It turns out that the excluded points of I can be interpreted as α-planes at infinity (see [3, §10.3]). We may then adjoin I to $\mathcal{P}_{\mathbb{CM}}$ to recover $\mathbb{PT} = \mathbb{CP}^3$ in a manner similar to the conformal compactification of $\mathbb{CM}$, and it turns out that the *entire* twistor space $\mathbb{PT}$ is the twistor space of the conformal compactification of $\mathbb{CM}$, that is $\mathbb{PT} = \mathcal{P}_{\mathbb{CM}^\#}$ in the notation of the following subsections. Moreover, the action of the conformal group on the α-planes is given by the natural action of $\mathrm{GL}(4, \mathbb{C})$ on $\mathbb{CP}^3$. For the precise details of these constructions we refer the reader to [3, §§9,10].

It should be noted that the space $\mathbb{T}$ is what is sometimes called the twistor space in general relativity, while $\mathbb{PT}$ is referred to as the projective twistor space. We will stick to the differential geometry nomenclature and call $\mathbb{PT}$, and its subsets, twistor spaces.

2.6.2 The twistor space of an elementary set

In the previous paragraph we defined the twistor space of $\mathbb{CM}$ and the twistor space of all of $\mathbb{CM}^{\#}$. We can also define the twistor space of $U \subset \mathbb{CM}$ as follows. Suppose $U \subset \mathbb{CM}$ and that the intersection of U with each α-plane is connected and simply connected (such a U is called *elementary*). Then the *twistor space of* U is

$$\mathcal{P}_U = \{Z \in \mathbb{PT} : Z \cap U \neq \emptyset\}.$$

If U is open in $\mathbb{CM}$, then $\mathcal{P}_U$ is open in $\mathbb{PT}$. When $U = \mathbb{CM}$, $\mathcal{P}_U$ is the complement of the line $I = \{Z^2 = 0 = Z^3\}$, as discussed in the previous paragraph. As long as $U \subset \mathbb{CM}$, $\mathcal{P}_U$ may be covered by the coordinate patches V and $\tilde{V}$.

Occasionally we do not wish to specify the subset U of complex spacetime and simply work with a twistor space $\mathcal{P}$.

2.6.3 The correspondence space

It is useful when passing between $U \subset \mathbb{CM}$ and the corresponding twistor space $\mathcal{P}_U$ to make use of the *correspondence space* $\mathcal{F}_U$.

Definition 2.7. The *correspondence space* $\mathcal{F}_U$ is the set of ordered pairs (x, Z), where $x \in U$ and Z is an α-plane through x.

The correspondence space $\mathcal{F}_U$ fibres over U and $\mathcal{P}_U$ by the surjective projections

$$\begin{aligned} q : \mathcal{F}_U &\to U, \\ (x, Z) &\mapsto x, \end{aligned}$$

and

$$\begin{aligned} p : \mathcal{F}_U &\to \mathcal{P}_U, \\ (x, Z) &\mapsto Z; \end{aligned}$$

$$\begin{tikzcd} & \mathcal{F}_U \arrow[dl, two heads, "q"'] \arrow[dr, two heads, "p"] & \\ U & & \mathcal{P}_U. \end{tikzcd}$$

We may label the points of $\mathcal{F}_U$ by $(w, z, \tilde{w}, \tilde{z}, \zeta)$, including $\zeta = \infty$, whence the two projections are given by

$$\begin{aligned} p : (w, z, \tilde{w}, \tilde{z}, \zeta) &\mapsto (\lambda, \mu, \zeta) = (\zeta w + \tilde{z}, \zeta z + \tilde{w}, \zeta) \\ q : (w, z, \tilde{w}, \tilde{z}, \zeta) &\mapsto (w, z, \tilde{w}, \tilde{z}). \end{aligned}$$

Then, as before, the tangent spaces to the "leaves" of the fibration p are spanned at each point by the vector fields

$$l = \partial_w - \zeta \partial_{\tilde{z}} \quad \text{and} \quad m = \partial_z - \zeta \partial_{\tilde{w}}$$

(or $\partial_{\tilde{z}}$ and $\partial_{\tilde{w}}$ when $\zeta = \infty$) on $\mathcal{F}_U$. A function on $\mathcal{P}_U$ is a function of the three *twistor coordinates* (λ, μ, ζ). By pulling it back by p, we may represent it as a function on the correspondence space, so that it will be constant along l and m.

2.6.4 The twistor space of a point: lines in $\mathbb{PT}$

There is a different way of reading equations (16), which is to hold the spacetime coordinates $(w, z, \tilde{w}, \tilde{z})$ fixed and allow Z^α to vary. These are two equations in four variables, so they determine a two-dimensional subspace of $\mathbb{T}$. One of these dimensions collapses under the projection $\mathbb{T} \to \mathbb{PT}$, so they also determine a *projective line* in $\mathbb{PT}$. As we saw in section 2.5.1, geometrically this is the Riemann sphere of α-planes through the spacetime point $x^a = (w, z, \tilde{w}, \tilde{z})$. We denote the projective line corresponding to $x \in \mathbb{CM}$ by $\hat{x}$. In fact, $\hat{x}$ is the twistor space of the set $U = \{x\}$, that is $\hat{x} = \mathcal{P}_x$.

The geometry of null vectors in $\mathbb{CM}$ is neatly encoded in the geometry of $\mathbb{PT}$. Suppose we have two points $x, y \in \mathbb{CM}$, and consider the lifts $q^*(x)$ and $q^*(y)$ to $\mathcal{F}$, i.e. the sets of ordered pairs $\{(x, Z)\}$ and $\{(y, W)\}$, where Z is any α-plane passing through x, and W is any α-plane passing through y. The points x and y are null-separated exactly when there exists a common α-plane that they lie on, that is there exists a Z such that $(x, Z) \in q^*(x)$ and $(y, Z) \in q^*(y)$. Under the projection $p : \mathcal{F} \to \mathcal{P}$ the lines $\hat{x}$ and $\hat{y}$ will intersect at an α-plane Z exactly when such a Z as above exists, thus two lines in $\mathbb{PT}$ intersect if and only if the corresponding spacetime points are null-separated.

3 The Twistor Correspondence

3.1 The Penrose–Ward Transform

The Penrose–Ward transform is a way of relating solutions to the ASDYM equation on a domain $U \subset \mathbb{CM}$ and holomorphic vector bundles on the twistor space $\mathcal{P} = \mathcal{P}_U$ of U. For a general *analytic* solution, the vector bundle can be represented by a patching matrix F, which eats three complex variables. The matrix F patches the two open sets covering $\mathcal{P}$ (see section 2.6), and the three complex variables are coordinates on $\mathcal{P}$. It turns out that F can be constructed from a linear system (a Lax pair), and that the solution to the ASDYM equation can be recovered from F by solving a Riemann–Hilbert problem. The key observation is that the ASDYM equation is equivalent to the vanishing of the curvature $F_{ab} = [D_a, D_b]$ on every α-plane. We first describe a constructive, coordinate-dependent picture of this correspondence, and later give a purely geometric description.

Let $U \subset \mathbb{CM}$ and let $\mathrm{D} = \mathrm{d} + \Phi$ be an anti-self-dual connection (that is, the curvature 2-form of Φ be ASD) on a vector bundle $E \to U$ with fibre $\mathbb{C}^n$. Suppose that U is an *elementary* open subset of $\mathbb{CM}$, i.e. each α-plane Z intersects U in a connected and simply connected set. We denote by $\mathcal{P}_U$ the twistor space of U, and by V, $\tilde{V}$ the two-set open cover of $\mathcal{P}_U$, as defined in section 2.6.

Proposition 4. *The curvature F is ASD if and only if for every α-plane Z that intersects U, the restriction of D to $Z \cap U$ is integrable, or flat.*

Sketch proof. Self-dual 2-forms are orthogonal to anti-self-dual 2-forms, so the restriction of an ASD curvature to a SD 2-plane vanishes, and therefore the restricted connection is flat. □

3.1.1 The fundamental solutions

By theorem A.3, the compatibility condition $[L, M] = 0$ for the Lax pair

$$L = \mathrm{D}_w - \zeta \mathrm{D}_{\tilde{z}}, \quad M = \mathrm{D}_z - \zeta \mathrm{D}_{\tilde{w}}$$

implies that the linear system

$$Ls = 0, \quad Ms = 0,$$

where s is a section of E, represented by a column vector of length n, is integrable for each fixed value of ζ. We can put together n independent solutions to form an $n \times n$ matrix-valued fundamental solution f, the columns of which form a frame field for E. This frame field is made up of sections that are covariantly constant on the α-planes tangent to $\partial_w - \zeta\partial_{\tilde{z}}$ and $\partial_z - \zeta\partial_{\tilde{w}}$ (recall that the curvature of Φ is assumed to be ASD, so that $Ls = 0 = Ms$ does indeed mean that s is covariantly constant on α-planes with these tangent vectors). The sections are single-valued because the α-planes Z intersect U in simply-connected sets. So f satisfies

$$\mathrm{D}_w f - \zeta \mathrm{D}_{\tilde{z}} f = 0, \tag{17}$$

$$\mathrm{D}_z f - \zeta \mathrm{D}_{\tilde{w}} f = 0, \tag{18}$$

and we can make f holomorphic in $\zeta \in \mathbb{C}$, as well as holomorphic in the spacetime coordinates $w, z, \tilde{w}, \tilde{z}$, but we cannot make f regular (holomorphic with non-vanishing determinant) at $\zeta = \infty$ as well, because then f would be a holomorphic function on the whole Riemann sphere and by Liouville's theorem would thus be constant in ζ. Then we would have $\mathrm{D}_a f = 0$ for each coordinate a, which would make the connection Φ flat everywhere.

In a given gauge f is unique up to $f \mapsto fH$ for some non-singular matrix-valued function $H = H(\zeta, w, z, \tilde{w}, \tilde{z})$ that is holomorphic on $V = \{\zeta \neq \infty\}$ and satisfies

$$\partial_w H - \zeta \partial_{\tilde{z}} H = 0, \quad \partial_z H - \zeta \partial_{\tilde{w}} H = 0. \tag{19}$$

That is, f is unique up to a function H that is constant along tangents to a given α-plane, i.e. up to a function only of $\lambda = \zeta w + \tilde{z}$, $\mu = \zeta z + \tilde{w}$ and ζ.

As we have seen, when D is not flat it is impossible to choose f so that it is regular on the whole Riemann sphere. However, we can find another fundamental solution, $\tilde{f}$, which is holomorphic on $\zeta \neq 0$, including $\zeta = \infty$, by setting $\tilde{\zeta} = 1/\zeta$ and solving the system

$$\tilde{\zeta} \mathrm{D}_w \tilde{f} - \mathrm{D}_{\tilde{z}} \tilde{f} = 0, \quad \tilde{\zeta} \mathrm{D}_z \tilde{f} - \mathrm{D}_{\tilde{w}} \tilde{f} = 0. \tag{20}$$

Analogously to f, $\tilde{f}$ is unique up to $\tilde{f} \mapsto \tilde{f}\tilde{H}$, where is $\tilde{H}$ is a non-singular matrix-valued function of λ, μ and ζ that is holomorphic on $\tilde{V} = \{\zeta \neq 0\}$.

3.1.2 The patching matrix

On $V \cap \tilde{V}$, where the domains of f and $\tilde{f}$ overlap, they are related by

$$f = \tilde{f} F,$$

where F satisfies (19). The matrix F is called the *patching matrix* associated with D. Due to the non-uniqueness of f and $\tilde{f}$ discussed above, F is determined by D up to $F \mapsto \tilde{H}^{-1} F H$, the matrix $\tilde{H}$ being regular on $\tilde{V}$, and H regular on V. We set up an equivalence relation on patching matrices, $F \sim \tilde{H}^{-1} G H$, and call the equivalence class of F the *patching data* of D. Now if F lies in the equivalence class of the identity, it can, of course, be factorized as

$$F = \tilde{H}^{-1} H.$$

Then the fundamental solution $fH = \tilde{f}\tilde{H}$ is regular on the Riemann sphere, and so by Liouville's theorem is independent of ζ. In this case (17) and (18) imply that the columns of f are covariantly constant, so the curvature vanishes. Conversely, when such a factorization does not exist, the curvature is non-zero. In fact, as we will see shortly, the clash of notation is deliberate: it turns out that the patching matrix F encodes the ASDYM field, since D can be recovered from F. The map that assigns the patching data to an ASDYM field is called the *forward Penrose–Ward transform.*

3.1.3 The reverse transform

Conversely, suppose we start with a given holomorphic matrix-valued function $F(\lambda, \mu, \zeta)$ with non-vanishing determinant, defined on $V \cap \tilde{V}$. By applying Birkhoff's theorem at each spacetime point, we obtain a factorization of F in the form

$$F(\zeta w + \tilde{z}, \zeta z + \tilde{w}, \zeta) = \tilde{f}^{-1} \Delta f,$$

where $f(w, z, \tilde{w}, \tilde{z}, \zeta)$ is regular for $|\zeta| \leqslant 1$, $\tilde{f}(w, z, \tilde{w}, \tilde{z}, \zeta)$ is regular for $|\zeta| \geqslant 1$, including $\zeta = \infty$, and $\Delta = \operatorname{diag}(\zeta^k, \ldots, \zeta^m)$ for some integer-valued functions $k, \ldots, m$. If F happens to factorize with $\Delta = \mathbb{1}$ at some point in spacetime, then $\Delta = \mathbb{1}$ in an open neighbourhood U of that point, as discussed in section 2.2.2. We show that in this case F is the patching matrix associated with some solution to the ASDYM equation on U. Because F is constant along $\partial_w - \zeta \partial_{\tilde{z}}$, differentiating $\tilde{f} F = f$ along this vector field shows that

$$(\partial_w f - \zeta \partial_{\tilde{z}} f) = (\partial \tilde{f} - \zeta \partial_{\tilde{z}} \tilde{f}) F = (\partial_w \tilde{f} - \zeta \partial_{\tilde{z}} \tilde{f}) \tilde{f}^{-1} f,$$

and so

$$(\partial_w f - \zeta \partial_{\tilde{z}} f) f^{-1} = (\partial_w \tilde{f} - \zeta \partial_{\tilde{z}} \tilde{f}) \tilde{f}^{-1} \tag{21}$$

in U, for all ζ in some neighbourhood of the unit circle. Similarly,

$$(\partial_z f - \zeta \partial_{\tilde{w}} f) f^{-1} = (\partial_z \tilde{f} - \zeta \partial_{\tilde{w}} \tilde{f}) \tilde{f}^{-1}. \tag{22}$$

Now since f is regular (and so invertible) on $|\zeta| \leqslant 1$, the left-hand side of (21) is holomorphic for $|\zeta| < 1$, and similarly the right-hand side of (21) is holomorphic for $|\zeta| > 1$, except for a simple pole at $\zeta = \infty$. So both sides of (21) must be an entire function of ζ with a simple pole at $\zeta = \infty$, and by the generalized Liouville's theorem (theorem A.2) must thus be a polynomial of order 1 in ζ. We write

$$(\partial_w f - \zeta \partial_{\tilde{z}} f) f^{-1} = (\partial_w \tilde{f} - \zeta \partial_{\tilde{z}} \tilde{f}) \tilde{f}^{-1} = -\Phi_w + \zeta \Phi_{\tilde{z}}, \tag{23}$$

and interpret the constants $\Phi_w(w, z, \tilde{w}, \tilde{z})$ and $\Phi_{\tilde{z}}(w, z, \tilde{w}, \tilde{z})$ as two of the components of the connection Φ that we aim to construct. We perform the same construction starting from (22) to construct the other two components of Φ, and rearranging arrive at

$$\begin{aligned} \mathrm{D}_w f - \zeta \mathrm{D}_{\tilde{z}} f = 0, \quad \mathrm{D}_z f - \zeta \mathrm{D}_{\tilde{w}} f = 0, \\ \mathrm{D}_w \tilde{f} - \zeta \mathrm{D}_{\tilde{z}} \tilde{f} = 0, \quad \mathrm{D}_z \tilde{f} - \zeta \mathrm{D}_{\tilde{w}} \tilde{f} = 0, \end{aligned}$$

where $\mathrm{D} = \mathrm{d} + \Phi$ acts on the columns of f and $\tilde{f}$. By Frobenius' theorem, the linear system associated with Φ is integrable, so $[L, M] = 0$, i.e. Φ is ASD.

One might reasonably wonder about the uniqueness of this construction. Indeed, by the uniqueness statement in Birkhoff's theorem, if we choose to factorize F as

$$F(\zeta w + \tilde{z}, \zeta z + \tilde{w}, \zeta) = \tilde{f}^{-1}(\cdot, \zeta) f(\cdot, \zeta),$$

(where the dots represent dependence on other variables) then any other factorization must be given by

$$f' = g f, \quad \tilde{f}' = g \tilde{f},$$

where g is an element of the gauge group $\mathrm{GL}(n, \mathbb{C})$ and is independent of ζ. Putting these into our definition of the potential (23) (and an analogous equation for the other two components Φ_z and $\Phi_{\tilde{w}}$), we find that the new potential Φ' satisfies

$$\Phi = g^{-1} \Phi' g + g^{-1} \mathrm{d} g.$$

That is, F determines Φ up to a choice of gauge.

Thus given a holomorphic matrix of three complex coordinates such that it has a Birkhoff factorization with $\Delta = \mathbb{1}$ at at least one point, we can recover the connection Φ, up to a gauge transformation, in an open subset of spacetime. Moreover, this connection is a solution to the ASDYM equation in said open set.

3.1.4 Recovering the gauge potential

Once we have factorized the patching matrix $F(\cdot,\zeta) = \tilde{f}^{-1}(\cdot,\zeta)f(\cdot,\zeta)$, we may write down the solution to the ASDYM equation explicitly. Recall that Φ is ASD if and only if eq. (9) is satisfied.

Proposition 5. *The gauge potential Φ is given in terms of f and $\tilde{f}$ by*

$$\Phi = h\partial h^{-1} + \tilde{h}\tilde{\partial}\tilde{h}^{-1},$$

where $h = f(0)$ and $\tilde{h} = \tilde{f}(\infty)$.

Proof. Setting $\zeta = 0$ in eqs. (17) and (18), we obtain

$$\Phi_w = -(\partial_w h)h^{-1} = h\partial_w h^{-1} \quad \text{and} \quad \Phi_z = -(\partial_z h)h^{-1} = h\partial_z h^{-1},$$

and similar expressions for $\Phi_{\tilde{z}}$ and $\Phi_{\tilde{w}}$ by setting $\tilde{\zeta} = 0$ in eq. (20). □

This also shows that the functions of spacetime coordinates $h = f(0)$ and $\tilde{h} = \tilde{f}(\infty)$ (we omit the variables $(w, z, \tilde{w}, \tilde{z})$ for simplicity) actually satisfy the same defining relations as the functions h and $\tilde{h}$ of section 2.3.2, so consistency of notation is salvaged. We then have that Yang's matrix is given by

$$J = \tilde{h}^{-1}h = \tilde{f}^{-1}(\infty)f(0),$$

that is J is given by the zeroth Fourier coefficient of $F = \tilde{f}^{-1}(\zeta)f(\zeta)$. Recalling proposition 1, we note that if the Birkhoff factorization of F does not have $\Delta = \mathbb{1}$, the functions $f(w, z, \tilde{w}, \tilde{z}, \zeta)$ and $\tilde{f}(w, z, \tilde{w}, \tilde{z}, \zeta)$ may possess singularities. These would then carry over to h and $\tilde{h}$, and by extension to Φ. Thus the jumping points of Δ give rise to singularities in the ASDYM potential. Conversely, if $\Delta = \mathbb{1}$ then everything is regular and we may impose the gauge condition $f(0) = \mathbb{1}$ (this amounts to choosing a gauge in which $\Phi_w = 0 = \Phi_z$) to set

$$J = \tilde{f}^{-1}(\infty).$$

Example 1 (The scalar wave equation [3, p. 175])**.** Consider the patching matrix[3]

$$F = \begin{pmatrix} \zeta & \gamma \\ 0 & \zeta^{-1} \end{pmatrix},$$

defining a vector bundle over $\mathcal{P}$, where $\gamma = \gamma(\zeta)$ is any holomorphic function on $V \cap \tilde{V}$. As usual, let $\lambda = \zeta w + \tilde{z}$ and $\mu = \zeta z + \tilde{w}$. We may expand γ in a Laurent series valid on the whole Riemann sphere,

$$\gamma(\zeta) = \sum_{n=-\infty}^{\infty} \gamma_{-n}\zeta^n = \gamma_+ + \phi + \gamma_-,$$

[3]This is known as the Atiyah–Ward ansatz.

where the γ_n's are functions of λ, μ and ζ, γ_+ contains only positive powers of ζ, γ_- contains only negative powers of ζ, and $\phi = \gamma_0$. As, by assumption, $l\gamma = 0 = m\gamma$, we obtain the recursion relations

$$\partial_w \gamma_n = \partial_{\tilde{z}} \gamma_{n+1} \quad \text{and} \quad \partial_z \gamma_n = \partial_{\tilde{w}} \gamma_{n+1}. \tag{24}$$

The Birkhoff factorization $F = \tilde{f}^{-1} f$ is given by[4]

$$f = \frac{1}{\sqrt{\phi}} \begin{pmatrix} \zeta & \phi + \gamma_+ \\ -1 & -\zeta^{-1}\gamma_+ \end{pmatrix} \quad \text{and} \quad \tilde{f} = \frac{1}{\sqrt{\phi}} \begin{pmatrix} 1 & -\zeta\gamma_- \\ -\zeta^{-1} & \phi + \gamma_- \end{pmatrix}$$

whenever $\phi \neq 0$. When $\phi = 0$, F is diagonal so we must take

$$\Delta = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}.$$

These are examples of jumping points: the gauge potential Φ is singular at the points $(w, z, \tilde{w}, \tilde{z})$ where $\phi = 0$. Now for $\phi \neq 0$ we find that

$$h = \frac{1}{\sqrt{\phi}} \begin{pmatrix} 0 & \phi \\ -1 & -\gamma_{-1} \end{pmatrix} \quad \text{and} \quad \tilde{h} = \frac{1}{\sqrt{\phi}} \begin{pmatrix} 1 & -\gamma_1 \\ 0 & \phi \end{pmatrix},$$

so

$$J = \tilde{h}^{-1} h = -\frac{1}{\phi} \begin{pmatrix} \gamma_1 & \gamma_1\gamma_{-1} - \phi^2 \\ 1 & \gamma_{-1} \end{pmatrix}.$$

A somewhat tedious calculation shows that Yang's equation reduces to the wave equation on $\log \phi$,

$$\Box_\eta \log \phi = (\partial_w \partial_{\tilde{w}} - \partial_z \partial_{\tilde{z}}) \log \phi = 0.$$

We may also explicitly recover the gauge potential. Using proposition 5 and the recursion relations (24), we calculate

$$\Phi = \frac{1}{2\phi} \begin{pmatrix} \tilde{\partial}\phi - \partial\phi & 2(\partial_z \phi \, \mathrm{d}\tilde{w} + \partial_w \phi \, \mathrm{d}\tilde{z}) \\ 2(\partial_{\tilde{z}} \phi \, \mathrm{d}w + \partial_{\tilde{w}} \phi \, \mathrm{d}z) & \partial\phi - \tilde{\partial}\phi \end{pmatrix}.$$

3.2 The Ward Correspondence

We have now given a concrete description of the Penrose–Ward transform, which is useful for explicit constructions. However, from a mathematical point of view, this approach gives a false impression that the choice of the two-set cover V and $\tilde{V}$ and the coordinates plays a special role. There exists a purely geometric description of the correspondence between ASDYM fields on U and holomorphic vector bundles over $\mathcal{P}_U$. In this form, the correspondence between ASD connections and holomorphic bundles is manifestly covariant with respect to coordinate transformations of complex spacetime. The problem is that of transferring the gauge potential Φ, a connection on a vector bundle E over a subset of complex spacetime U, to a vector bundle over the twistor space. It is worth emphasizing that in the $\mathbb{CM}$-picture one has a vector bundle with a connection, while in the $\mathcal{P}$-picture one has only a vector bundle and *no* connection. All of the information about the connection D on $E \to U$ is encoded in the holomorphic structure of the corresponding vector bundle over $\mathcal{P}$.

[4] In general there is no systematic way of performing the Birkhoff factorization, and for this reason this is the most problematic part of the construction. Nevertheless, there are many special constructions that deal with particular forms of the matrix F. For more details see [7, §8.2].

Theorem 3.1 (Ward (1977), [3, p. 177]). *Let $U \subset \mathbb{CM}$ be an open set such that for every α-plane Z, the intersection $Z \cap U$ is connected and simply connected whenever $Z \cap U$ is non-empty. Then there is a one-to-one correspondence between solutions to the ASDYM equation on U with gauge group* $\mathrm{GL}(n, \mathbb{C})$ *and holomorphic vector bundles $E' \to \mathcal{P}_U$ such that $E'|_{\hat{x}}$ is trivial for every $x \in U$.*

Proof. To go in the forward direction, let D be an ASD connection on a rank-n vector bundle $E \to U$ and define a vector bundle $E' \to \mathcal{P}$ by setting the fibre of E' at $Z \in \mathcal{P}$ to be

$$E'_Z = \{s \in \Gamma(Z \cap U, E) : \mathrm{D}s|_{Z\cap U} = 0\},$$

where $\Gamma(Z \cap U, E)$ is the space of sections of E over $Z \cap U$. As we saw in proposition 4, because D is ASD, its curvature vanishes on the restriction to Z, and because $Z \cap U$ is connected and simply connected, the covariantly constant sections on $Z \cap U$ are single-valued and uniquely determined by their values at any one point. Hence E'_Z is an n-dimensional complex vector space. Furthermore, it varies holomorphically with Z.

Conversely, suppose that we are given a holomorphic bundle $E' \to \mathcal{P}$ such that $E'|_{\hat{x}}$ is trivial for every $x \in U$. Define a bundle $E \to U$ by setting

$$E_x = \Gamma(\hat{x}, E),$$

where Γ is the space of holomorphic sections of E' over $\hat{x}$. Now $E'|_{\hat{x}} = \hat{x} \times \mathbb{C}^n$, so in this trivialization the global sections of $E'_{\hat{x}}$ are holomorphic maps $\hat{x} \to \mathbb{C}^n$. Since these are globally holomorphic, by Liouville's theorem they are constant, and so E_x is the space of constant maps with values in $\mathbb{C}^n$, i.e. an n-dimensional complex vector space.

Our goal is to construct a connection D on E such that for each $Z \in \mathcal{P}$, E'_Z is the space of covariantly constant sections of E over $Z \cap U$. Now an element of E_x is, by definition, a section of $E'|_{\hat{x}}$, so for each α-plane $Z \in \mathcal{P}$ we identify the fibres E_x, for $x \in Z \cap U$, with E'_Z by evaluation at Z. But this is a characterization of parallel transport over α-planes! If D exists, then the covariantly constant sections over $Z \cap U$ are the constant sections in E'_Z. Since null vectors tangent to α-planes span the tangent space at each point of spacetime, the connection is unique if it exists.

Now to show existence of the connection D, we will have to work on the correspondence space $\mathcal{F}$. Consider the pullbacks p^*E' and q^*E, which are bundles over $\mathcal{F}$. By construction, $p^*E' = q^*E$.

$$\begin{tikzcd}
q^*E \arrow[rr, equal] \arrow[dr] & & p^*E' \arrow[dl] \\
& \mathcal{F} \arrow[dl, "q"'] \arrow[dr, "p"] & \\
U & & \mathcal{P} \\
E \arrow[u] & & E' \arrow[u]
\end{tikzcd}$$

Figure 1: Constructing the connection D

Let $Z \in \mathcal{P}$. By the definition of the pullback, the fibre of p^*E' at $p^{-1}(Z) \in \mathcal{F}$ is E'_Z, so $p^*E'\big|_{p^{-1}(Z)}$ is the product bundle $p^{-1}(Z) \times E'_Z$. The leaves of the foliation $p : \mathcal{F} \to \mathcal{P}$ are spanned by the vector fields

$$l = \partial_w - \zeta\partial_{\tilde{z}} \quad \text{and} \quad m = \partial_z - \zeta\partial_{\tilde{w}}.$$

We can define a 'partial' connection D that allows us to differentiate the sections of p^*E' along the twistor fibration by requiring that on each leaf $p^{-1}(Z)$ we must have

$$\mathrm{D}_l s = l(s) \quad \text{and} \quad \mathrm{D}_m s = m(s)$$

in the trivialization $p^*E' = p^{-1}(Z) \times E'_Z$. The sections for which $\mathrm{D}_l s$ and $\mathrm{D}_m s$ vanish are then the pullbacks of $\mathcal{F}$ to local sections of E'.

Now pick a local trivialization of E over some open subset of U. This determines a local trivialization of p^*E' in which

$$\mathrm{D}_l = l + \Phi_l, \qquad \mathrm{D}_m = m + \Phi_m,$$

where Φ_l and Φ_m are matrix-valued functions of ζ and the spacetime coordinates $(w, z, \tilde{w}, \tilde{z})$. Then $\zeta^{-1}l$ and $\zeta^{-1}m$ are regular at $\zeta = \infty$. By considering the partial connection along these rescaled vector fields, we see that $\zeta^{-1}\Phi_l$ and $\zeta^{-1}\Phi_m$ must also be regular at $\zeta = \infty$. Hence by the generalized Liouville's theorem A.2, Φ_l and Φ_m are polynomials in ζ of degree at most 1, that is

$$\Phi_l = \Phi_w - \zeta\Phi_{\tilde{w}}, \qquad \Phi_m = \Phi_z - \zeta\Phi_{\tilde{w}}.$$

Then

$$\Phi = \Phi_w \mathrm{d}w + \Phi_z \mathrm{d}z + \Phi_{\tilde{w}} \mathrm{d}\tilde{w} + \Phi_{\tilde{z}} \mathrm{d}\tilde{z}$$

is independent of ζ and gives the desired connection via $\mathrm{D} = \mathrm{d} + \Phi$.

□

Finally, we note that an analogous theorem holds when we include the α-planes at infinity, that is the correspondence extends to a correspondence between solutions of the ASDYM equation on $U \subset \mathbb{CM}^{\#}$ and holomorphic vector bundles over subset of $\mathbb{PT}$ of α-planes that intersect U.

4 Symmetry Reductions

We have now seen that the Penrose–Ward transform gives a correspondence between solutions to the ASDYM equation and holomorphic vector bundles over twistor space. The central aim of this essay is to study not the full ASDYM equation, however, but systems obtained under the reduction by a certain subroup of the conformal group. We hope to study Einstein's equations in particular, and it turns out that the ASDYM equation may be reduced to many special cases of the Einstein equations possessing certain symmetries. We impose these symmetries by defining Killing vectors on complex spacetime $\mathbb{CM}$, but eventually we will need a way of seeing how the symmetries behave in twistor space. We discuss broadly how this works in section 4.3.

4.1 The Ernst Equation

One physically relevant reduction of the ASDYM equation by a two-dimensional subgroup of the conformal group is generated by the vector fields

$$X = w\partial_w - \tilde{w}\partial_{\tilde{w}}, \quad Y = \partial_{\tilde{z}} + \partial_z. \tag{25}$$

If we restrict ourselves to the Minkowski slice $\mathbb{M}$ of complex spacetime $\mathbb{CM}$, we may change coordinates by

$$w = \frac{r}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}\theta}, \quad \tilde{w} = \frac{r}{\sqrt{2}}\mathrm{e}^{\mathrm{i}\theta}, \quad z = \frac{1}{\sqrt{2}}(t-x), \quad \tilde{z} = \frac{1}{\sqrt{2}}(t+x) \tag{26}$$

for (t, x, r, θ) real. Then the metric becomes

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}r^2 - r^2\mathrm{d}\theta^2, \tag{27}$$

and the symmetries generated by X and Y become rotations $\theta \mapsto \theta + \theta_0$ and time translations $t \mapsto t + t_0$ respectively. Having imposed these symmetries, we are looking at stationary axisymmetric solutions to the ASDYM equations, *and their continuations to* $\mathbb{CM}$. As originally noticed by Witten in [8], these reduced ASDYM equations turn out to be equivalent to the Ernst equation for stationary axisymmetric gravitational fields in general relativity. It will be useful to know the reduction of Yang's equation (9) under the action of the group generated by X and Y. If we start with eq. (9), make the change of variables (26), and perform the calculations assuming J is independent of θ and t, then Yang's equation becomes

$$\partial_x(J^{-1}\partial_x J) + \frac{1}{r}\partial_r(rJ^{-1}\partial_r J) = 0.$$

Following Witten's observation, we may study stationary axisymmetric solutions of Einstein's equations using the twistor methods that we have developed. In fact, while one set of reality conditions corresponds to actual stationary axisymmetric solutions to Einstein's equations, others correspond to cylindrical gravitational waves, or a pair of colliding plane waves, or to the Gowdy cosmological models. These four sets of solutions are essentially characterized by whether the reality conditions make the Killing fields X and Y both timelike, both spacelike, or one timelike and one spacelike. In particular, if $x = \mathrm{i}t$ is purely imaginary, we get cylindrical gravitational waves and Yang's matrix becomes

$$\partial_t(J^{-1}\partial_t J) - \frac{1}{r}\partial_r(rJ^{-1}\partial_r J) = 0.$$

4.2 Reductions of Einstein's Equations

Let (M, g) be a four-dimensional manifold, either real or complex, and let X_1, X_2 be two commuting Killing vectors. By commuting, of course, here we mean that their Lie bracket vanishes, $[X_1, X_2] = 0$.

Definition 4.1. We say X_1 and X_2 generate an *orthogonally transitive* isometry group if whenever vector fields U and V are orthogonal to the orbits, then so is $[U, V]$.

Suppose X_1 and X_2 generate an orthogonally transitive isometry group with non-null 2-dimensional orbits. Put $J = (J_{ij}) = (g_{ab}X_i^a X_j^b)$, and let S be any of the 2-surfaces orthogonal to the orbits (i.e. the quotient space). Let h be the metric on S and D the corresponding Levi–Civita connection. We will use the roman indices $a, b, \ldots$ to denote components of tensors on the full four-dimensional manifold M, and the greek indices $\mu, \nu, \ldots$ to denote components of tensors on the two-dimensional quotient space S.

Proposition 6. *Let g_{ab} be a solution to the vacuum Einstein equations in four dimensions. Suppose that it admits two commuting Killing vectors with orbits orthogonal to a family of non-null surfaces, and suppose that the gradient of r is non-null. Then the metric on S can be written in the form*

$$h = \pm\Omega^2(\mathrm{d}r^2 + \mathrm{d}x^2),$$

and $J(x, r)$ is Yang's matrix corresponding to a stationary axisymmetric solution to the ASDYM equation with gauge group $\mathrm{GL}(n, \mathbb{C})$.

Sketch proof. Let ∇ denote the Levi–Civita connection of g. Since the Killing vectors commute, J is constant along the orbits: $\nabla_{X_k} J_{ij} = 0$. Furthermore, by the definition of a Killing vector,

$$\nabla_a X_{ib} + \nabla_b X_{ia} = 0$$

for all i. So because $[X_i, X_j] = 0$ for all i and j, we have

$$X_j^a \nabla_a X_{ib} = X_i^a \nabla_a X_{jb} = -X_j^a \nabla_b X_{ia} = -X_i^a \nabla_b X_{ja} = -\frac{1}{2}\partial_b(J_{ij}).$$

Also, for any vector fields U and V orthogonal to the orbits, that is $U^b X_{ib} = 0 = V^b X_{ib}$ for all i,

$$\begin{aligned} U^a V^b \nabla_a X_{ib} - V^a U^b \nabla_a X_{ib} &= -X_{ib} U^a \nabla_a V^b + X_{ib} V^a \nabla_a U^b \\ &= -X_{ib}(U^a \nabla_a V^b - V^a \nabla_a U^b) \\ &= -X_{ib}[U, V]^b = 0 \end{aligned}$$

by orthogonal transitivity. A calculation then shows that

$$\nabla_a X_{ib} = \frac{1}{2} J^{jk} \left((\partial_a J_{ki}) X_{jb} - (\partial_b J_{ki}) X_{ja} \right), \tag{28}$$

where J^{jk} is the inverse of J_{jk}. Now by virtue of the first Bianchi identity, any Killing vector X satisfies the differential equation

$$\nabla_b \nabla_c X_d = R_{abcd} X^a,$$

where R_{abcd} is the Riemann tensor. Thus by taking the covariant derivative of eq. (28) and contracting over a pair of indices we obtain

$$R_{ab} X_i^a X_j^b = -\frac{1}{2} J_{ik} \frac{1}{\sqrt{\det g}} \partial_a \left(\sqrt{\det g} g^{ab} J^{kl} \partial_b J_{lj} \right),$$

where $R_{ab} = R^c{}_{acb}$ is the Ricci tensor. If the vacuum Einstein equations

$$R_{ab} = 0$$

are satisfied, then

$$\frac{1}{\sqrt{\det g}} \partial_a (\sqrt{\det g} g^{ab} J^{-1} \partial_b J) = 0. \tag{29}$$

This equation reduces to an equation on S: we have that $\det g = -r^2 \det h$, where $r^2 = -\det J$, so eq. (29) becomes

$$D_\mu (r J^{-1} h^{\mu\nu} D_\nu J) = 0. \tag{30}$$

Since J is two-dimensional, we easily calculate that

$$\mathrm{Tr}(J^{-1} \mathrm{D}_\mu J) = \frac{1}{r^2} \mathrm{D}_\mu r^2 = \frac{2}{r} \mathrm{D}_\mu r.$$

Thus by taking the trace of eq. (30) we find that $\mathrm{D}_\mu \mathrm{D}^\mu r = 0$, so that r is a harmonic function on S. Since we assumed that the gradient of r is non-null, a standard result in geometry tells us that the metric on S is conformal to $\mathrm{d}r^2 + \mathrm{d}x^2$, where x is the harmonic conjugate to r. Recall that the harmonic conjugate x to r is the function such that $\xi = x + \mathrm{i}r$ is holomorphic, and is defined up to the addition of a constant. So

$$h = \pm \Omega^2 (\mathrm{d}r^2 + \mathrm{d}x^2),$$

the sign being chosen depending on whether h is timelike or spacelike, and in these coordinates eq. (30) reduces to

$$\partial_x(J^{-1}\partial_x J) + \frac{1}{r}\partial_r(rJ^{-1}\partial_r J) = 0,$$

which is Yang's equation for a stationary axisymmetric solution to the ASDYM equations with gauge group $\mathrm{GL}(n, \mathbb{C})$, as we saw in section 4.1. □

Proposition 6 has a partial converse, which makes the ASDYM equations a useful tool for studying Einstein's equations.

Proposition 7. *Every real symmetric solution to*

$$\partial_x(J^{-1}\partial_x J) + \frac{1}{r}\partial_r(rJ^{-1}\partial_r J) = 0 \tag{31}$$

such that $\det J = -r^2$ *determines a solution to the vacuum Einstein equations.*

Sketch proof. If we reconstruct a metric from a given solution J to eq. (31) and an Ω in a vein similar to that in the proof of proposition 6, then, as we saw, eq. (31) is equivalent to the vanishing of the Ricci tensor R_{ab} along the Killing vectors, while the remaining components of $R_{ab} = 0$ reduce to

$$2\mathrm{i}\partial_\xi \log(r\Omega^2) = r\mathrm{Tr}(\partial_\xi J^{-1}\partial_\xi J),$$

where $\xi = x + \mathrm{i}r$. This latter equation is of course equivalent to two real equations (when x and r are real), which are overdetermined but compatible when Yang's equation (31) is satisfied. Conversely, if J is known, they determine Ω up to a multiplicative constant. □

Proposition 6 and proposition 7 in fact hold in any dimension $n+s$ and for spacetimes admitting an arbitrary number s of Killing vector fields, but in our case, when $n = s = 2$, the condition $\det J = -r^2$ is enough to determine the components of J in terms of two functions of x and r. We write

$$J = \begin{pmatrix} f\alpha^2 - r^2 f^{-1} & -f\alpha \\ -f\alpha & f \end{pmatrix},$$

where f and α are *complex* functions of x and r. As locally the metric is

$$\mathrm{d}s^2 = J_{\mu\nu}\mathrm{d}y^\mu \mathrm{d}y^\nu - \Omega^2\left(\mathrm{d}r^2 + \mathrm{d}x^2\right), \tag{32}$$

where $\partial/\partial y^1$ and $\partial/\partial y^2$ are the Killing vectors, it becomes

$$\mathrm{d}s^2 = f(\mathrm{d}t - \alpha\mathrm{d}\theta)^2 - f^{-1}r^2\mathrm{d}\theta^2 - \Omega^2(\mathrm{d}r^2 + \mathrm{d}x^2).$$

When $f = f(x,r)$ and $\alpha = \alpha(x,r)$ are real for real x and r, this is a stationary axisymmetric gravitational field in Weyl canonical coordinates. These reality conditions correspond to the existence of one timelike and one spacelike Killing vector. Other reality conditions correspond to different types of solutions to Einstein's equations. For example, when α is purely imaginary for purely imaginary x, the metric becomes a cylindrical gravitational wave. Indeed, writing

$$t = \mathrm{i}\tilde{x}, \quad x = \mathrm{i}\tilde{t}, \quad \alpha\left(r, \mathrm{i}\tilde{t}\right) = -\mathrm{i}\tilde{\alpha}\left(r, \tilde{t}\right)$$

for real $\tilde{x}, \tilde{t}$ and $\tilde{\alpha}\left(r, \tilde{t}\right)$ corresponds to the metric[5] (dropping the tildes)

$$\mathrm{d}\tau^2 = -f(\mathrm{d}x + \alpha\mathrm{d}\theta)^2 - f^{-1}r^2\mathrm{d}\theta^2 - \Omega^2(\mathrm{d}r^2 - \mathrm{d}t^2),$$

[5]We must also change the sign of f and Ω^2 so that the line element $\mathrm{d}s^2$ becomes timelike.

which is a gravitational wave in Weyl's canonical coordinates. The reduction to a cylindrical gravitational wave corresponds to the existence of two spacelike Killing vectors. Indeed, since the Killing vectors in the stationary axisymmetric case were $\partial/\partial\theta$ and $\partial/\partial t$, spacelike and timelike respectively, these became $\partial/\partial\theta$ and $-\mathrm{i}\partial/\partial\tilde{x}$, the latter turning timelike due to the factor of i. Yet other reality conditions correspond to colliding plane waves, or to the Gowdy cosmological models. We study the former two in sections 5 and 6.

4.3 Reductions of the Penrose–Ward Transform

In order to use twistor methods to study the reductions of Einstein's equations to stationary axisymmetric solutions, or cylindrical gravitational waves, or any other system that is not the full ASDYM equation, we require a way of transferring the action of the conformal group on $\mathbb{CM}^{\#}$, or a subset U (perhaps $\mathbb{CM}$), to the corresponding twistor space $\mathcal{P}$. The correspondence space $\mathcal{F}$ introduced in section 2.6.3 comes in handy in passing from complex spacetime to the twistor space. We do not aim to give a full and precise account of the reductions of the Penrose–Ward transform, since for our purposes it will be sufficient to only know broadly how the reductions are constructed, and in particular to be aware of the existence of an *invariant spectral parameter*, a complex variable which will effectively replace some of the complex variables that the patching matrix F depends on.

Recall that for $U \subset \mathbb{CM}$, the twistor space $\mathcal{P}_U$ is the quotient of $\mathcal{F}_U$ by the flows of the vector fields

$$l = \partial_w - \zeta\partial_{\tilde{z}} \quad \text{and} \quad m = \partial_z - \zeta\partial_{\tilde{w}},$$

and we have the projections

$$\begin{tikzcd} & \mathcal{F}_U \arrow[dl, two heads, "q"'] \arrow[dr, two heads, "p"] & \\ U & & \mathcal{P}_U. \end{tikzcd}$$

One transfers the action of the conformal group from spacetime to twistor space by lifting the conformal Killing vectors from U to $\mathcal{F}_U$ and then projecting them onto $\mathcal{P}_U$. The proper conformal transformations of complex spacetime map α-planes to α-planes, and hence induce holomorphic motions of twistor space. These coincide with the natural action of $\mathrm{GL}(4,\mathbb{C})$ on $\mathbb{CP}^3$. If a given ASDYM field is invariant under a subgroup of the conformal group, then its transform, a holomorphic bundle over twistor space, is invariant under the corresponding subgroup of $\mathrm{GL}(4,\mathbb{C})$. In simple cases we can transfer the symmetry group to the twistor space essentially by quotienting out by our symmetries to construct a *reduced twistor space* $\mathcal{R}$. These simpler cases correspond to situations where we can simply ignore some of the coordinates. It should be noted that it is not always possible to perform this procedure, when non-trivial information is encoded in the action of the symmetry group on the fibres over the *singular set* of the symmetry group, the set of α-planes that are fixed by a non-trivial subgroup of the symmetry group. In this case one has to work with invariant bundles over a larger space.

4.3.1 Stationary axisymmetric solutions

Reduction by the commuting Killing vectors

$$X = w\partial_w - \tilde{w}\partial_{\tilde{w}} \quad \text{and} \quad Y = \partial_{\tilde{z}} + \partial_z$$

gives the stationary axisymmetric case, as we saw in section 4.1. We would expect that a reduction by two Killing vectors should reduce the dimension of the twistor space from three to one, and indeed the reduced twistor space $\mathcal{R}$ turns out to be a compact, one-dimensional, but *non-Hausdorff* complex manifold, [10]. It is, in a sense, two Riemann spheres glued together at a face, and requires four coordinate patches to cover. The invariant spectral parameter on $\mathcal{R}$ may be taken to be

$$\gamma = x + \frac{1}{2} r \left(\zeta^{-1} - \zeta\right),$$

(see [3, §11.3], [10] for details), but due to the way the four coordinate patches cover $\mathcal{R}$, we must keep the coordinate ζ to describe the most general holomorphic bundle over $\mathcal{R}$. The patching matrix describing such a holomorphic vector bundle has the form

$$F(\gamma, \zeta) = \begin{pmatrix} \phi & (-\zeta)^k \psi \\ \zeta^{-k} \psi & \chi \end{pmatrix}, \tag{33}$$

where $\det F = -1$, k is an integer, and $\phi = \phi(\gamma)$, $\psi = \psi(\gamma)$ and $\chi = \chi(\gamma)$ are meromorphic functions of the complex parameter γ which satisfy the reality conditions $\overline{\Upsilon(\gamma)} = \Upsilon(\overline{\gamma})$ for $\Upsilon = \phi, \chi, \psi$. The integer k is in some sense a "winding number" describing how the symmetry acts on the axis $r = 0$. If $k = 0$, for example, the symmetry acts trivially on the axis. Moreover, in the case of stationary axisymmetric *gravitational fields*, the solution J is regular on the axis $r = 0$ if and only if $k = 1$, since, in the notation of the next section, the norm squared of the timelike Killing vector Y on the axis is $-r^{1-k}\phi(\gamma)^{-1}$. In the case $k \neq 0$ the construction of J, which we describe in section 5, is slightly different from the construction we described in the previous chapters. The details can be found in [6] and [10].

5 Stationary Axisymmetric Gravitational Fields

Suppose we have a vacuum spacetime admitting two commuting Killing vector fields $X = \partial/\partial\theta$ and $Y = \partial/\partial t$, X being spacelike and Y timelike. If the spacetime symmetry group generated by X and Y is orthogonally transitive, and the determinant $(X^a X_a)(Y^b Y_b) - (X^a Y_a)^2$ is non-constant, then locally the spacetime metric can be written as [6, 10]

$$\mathrm{d}s^2 = r J_{\mu\nu} \mathrm{d}y^\mu \mathrm{d}y^\nu - \Omega^2 \left(\mathrm{d}r^2 + \mathrm{d}x^2\right), \tag{34}$$

where $\partial/\partial y^1 = \partial/\partial\theta$ and $\partial/\partial y^2 = \partial/\partial t$ are the Killing vectors, $\Omega = \Omega(r, x)$, and J is a symmetric 2×2 matrix of real-valued functions of r and x with $\det J = -1$. This is the same as the metric (32), except we have taken a factor of r out of J. The only coordinate freedom remaining in writing (34) is that of making constant $\mathrm{SL}(2, \mathbb{C})$ transformations on y^1 and y^2.

As we saw in section 4.2, the vacuum Einstein equations $R_{ab} = 0$ of (34) reduce to

$$\partial_x \left(J^{-1} \partial_x J\right) + \frac{1}{r} \partial_r \left(r J^{-1} \partial_r J\right) = 0 \tag{35}$$

together with $4\mathrm{i}\partial_\xi \log\Omega = r\mathrm{Tr}(\partial_\xi J^{-1} \partial_\xi J) - \frac{1}{r}$, or the two equivalent real equations

$$\partial_x \log\Omega^2 = -\frac{1}{2} r \mathrm{Tr}\left(\partial_r J^{-1} \partial_x J\right), \tag{36}$$

$$\partial_r \log\Omega^2 = \frac{1}{4} r \mathrm{Tr}\left(\partial_x J^{-1} \partial_x J\right) - \frac{1}{4} r \mathrm{Tr}\left(\partial_r J^{-1} \partial_r J\right) - \frac{1}{r}. \tag{37}$$

Equation (35) is Yang's equation (9) under the reduction by the symmetry group generated by the Killing vectors $\partial/\partial\theta$ and $\partial/\partial t$, as we saw in section 4.1, and ensures the compatibility of the overdetermined system (36)–(37), as can be quickly shown by cross-differentiating. Moreover, eqs. (36) and (37), are easily integrable once J is known because their right-hand sides do not depend on Ω. Thus it is evident that solving eq. (35) is the core of the problem of finding (all) stationary axisymmetric solutions of Einstein's equations.

5.1 The Twistor Solution to Yang's Equation

As we saw, Yang's equation can be written in spinorial form, where it is an equation in background Minkowski space. So eq. (35), even though originally arising as an equation from curved spacetime, can effectively be rewritten as an equation in flat space,

$$\iota^{A'}\partial^{B}{}_{A'}(J^{-1}o^{B'}\partial_{BB'}J) = 0, \tag{12}$$

which is manifestly covariant. Here we work on the Minkowski slice $\mathbb{M}$, that is we impose the reality conditions

$$x^{AA'} = \begin{pmatrix} \tilde{z} & w \\ \tilde{w} & z \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} x^0 + x^1 & x^2 - \mathrm{i}x^3 \\ x^2 + \mathrm{i}x^3 & x^0 - x^1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} t+x & r\mathrm{e}^{-\mathrm{i}\theta} \\ r\mathrm{e}^{\mathrm{i}\theta} & t-x \end{pmatrix},$$

where in the second equality we make use of Ernst's coordinates (26). Notice that the $\mathrm{GL}(n,\mathbb{C})$ gauge freedom here reduces to the freedom

$$J \mapsto J' = WJV,$$

where $W = W(x^{AA'}o_{A'}) = W(w,z)$ and $V = V(x^{AA'}\iota_{A'}) = V(\tilde{w},\tilde{z})$ are $\mathrm{GL}(n,\mathbb{C})$ matrices. That is, if J is a solution to eq. (12), then so is J'.

To construct solutions of eq. (35), we thus wish to solve eq. (12). An abstract characterization of solutions to eq. (12) is of course provided by theorem 3.1, which here states the following.

There is a natural one-to-one correspondence between analytic solutions J to eq. (12) *(equivalently, eq.* (35)*) on $U \subset \mathbb{CM}$, modulo the freedom $J \mapsto J'$, and holomorphic rank-*2 *vector bundles E over $\mathcal{P}_U$, such that $E|_{\hat{x}}$ is trivial for all $x \in U$.*

So solutions to eq. (35) correspond to holomorphic vector bundles, which can be described by a patching matrix F. We next construct the solution J from such a patching matrix, following a generalized version of the construction introduced in section 3.1.

5.2 Constructing Solutions

The reduced twistor space $\mathcal{R}$ in the stationary axisymmetric case is one-dimensional, and the patching matrix has the form (33). We apply Birkhoff's theorem to split the patching matrix

$$F\left(x + \frac{1}{2}r\zeta - \frac{1}{2}r\zeta^{-1},\, \zeta\right) = \tilde{f}^{-1}(r,x,\zeta)f(r,x,\zeta),$$

where f is analytic on $|\zeta| \leqslant 1$, $\tilde{f}$ is analytic on $|\zeta| \geqslant 1$, including $\zeta = \infty$, and both f and $\tilde{f}$ are non-singular almost everywhere in $\mathbb{M}$. We then let

$$J(r,x) = P\tilde{f}^{-1}(r,x,\infty)f(r,x,0)P,$$

where $P = \mathrm{diag}(r^{k/2}, r^{-k/2})$. Evidently $\det J = -1$, and of course the construction ensures that J satisfies the reduced Yang's equation. We saw that it is equivalent to the reduced ASDYM equation, which in turn is (essentially) equivalent to the reduced Einstein equation. Furthermore, the case $k = 0$, which corresponds to F not depending on ζ, clearly reduces to our usual construction of J.

6 Cylindrical Gravitational Waves

Section 5 explored the reality conditions that reduce stationary axisymmetric solutions to the ASDYM equation to stationary axisymmetric gravitational fields. We next study the reduction to cylindrical gravitational waves. We saw that the invariant spectral parameter on $\mathcal{R}$ is

$$\gamma = x + \frac{1}{2}r\left(\zeta - \zeta^{-1}\right),$$

which on the unit circle $\zeta = \mathrm{e}^{\mathrm{i}\theta}$ becomes

$$\gamma = x + \mathrm{i}r\sin\theta = \mathrm{i}(t + r\sin\theta)$$

after we impose the reality condition $x = \mathrm{i}t$. So in the cylindrical gravitational wave case we may use, abusing the notation slightly, $x = t + r\sin\theta$. We study the hyperbolic version of Ward's $k = 0$ ansatz that we introduced in the section 4.3.1, with a quick digression into the $k = 1$ case.

Let us step back for a moment and consider the scalar wave equation on $\mathbb{M}$ in cylindrical polar coordinates. This has cylindrically symmetric solutions of the form

$$\varphi(r,t) = \frac{1}{2\pi}\int_0^{2\pi} P(t + r\sin\theta)\mathrm{d}\theta, \tag{38}$$

where $P(x)$ is an arbitrary generating function. The generating function is equal to the solution on the symmetry axis $r = 0$ and is determined by the Cauchy data on $t = 0$ by

$$P(x) = u(0) + \frac{x}{2}\int_0^{\pi}\left(u'(x\sin\theta) + v(x\sin\theta)\sin\theta\right)\mathrm{d}\theta, \tag{39}$$

where u and v are the even functions such that $u(r) = J(r,0)$ and $v(r) = J_t(r,0)$ for $r \geqslant 0$. Some details on where these come from are given in [9]. Notice that eq. (38) equates $\varphi(r,t)$ to the zeroth Fourier coefficient of P. This is a pattern we have already seen in the Ward construction, and indeed our purpose here is to describe the nonlinear, or *exponentiated*, versions of eqs. (38) and (39). In the nonlinear theory we use the notation $\varphi \rightsquigarrow J$ and $P \rightsquigarrow F$. We have said a lot about the construction of solutions J, but nothing about how the generating function F is propagated from Cauchy data. Woodhouse, [9, §4], describes precisely this in the nonlinear setting. For proofs the reader should consult the Appendix of [9].

As we saw, the metric of a cylindrical gravitational wave can be put in the form

$$\mathrm{d}\tau^2 = \Omega^2(\mathrm{d}t^2 - \mathrm{d}r^2) - f(\mathrm{d}x + \alpha\mathrm{d}\theta)^2 - f^{-1}r^2\mathrm{d}\theta^2,$$

and the hyperbolic version of Yang's equation is

$$\partial_t(J^{-1}\partial_t J) - \frac{1}{r}\partial_r(rJ^{-1}\partial_r J) = 0. \tag{40}$$

If we perform the Bäcklund transformation by defining, up to the addition of a constant, $\psi(r,t)$ by

$$r\partial_r\psi = \phi^2\partial_t\alpha \quad \text{and} \quad r\partial_t\psi = \phi^2\partial_r\alpha, \tag{41}$$

then Yang's equation (40) still holds with J replaced with

$$J' = \phi^{-1} \begin{pmatrix} \phi^2 + \psi^2 & -\psi \\ -\psi & 1 \end{pmatrix}.$$

The motivation for this transformation is as follows. Whereas previously the determinant of the matrix J, defined in terms of the Killing vectors via $J_{ij} = g_{ab} X_i^a X_j^b$, was $-r^2$, the determinant of J' is manifestly 1. Thus J' is regular on the axis $r = 0$, unlike J. This is essentially because the above Bäcklund transformation shifts k from 0 to 1.

The remaining field equations are

$$\partial_r \log f\Omega^2 = -r\mathrm{Tr}(J_t(J^{-1})_t + J_r(J^{-1})_r), \tag{42}$$

$$\partial_t \log f\Omega^2 = -2r\mathrm{Tr}(J_r(J^{-1})_t). \tag{43}$$

As before, eqs. (42) and (43) are overdetermined, and cross differentiation shows that they are compatible if eq. (40) is satisfied.

Equation (40) is a nonlinear generalization of the scalar wave equation for cylindrical waves. One class of its solutions, given by $J = \mathrm{diag}(\mathrm{e}^{\varphi}, \mathrm{e}^{-\varphi})$, are called the Einstein–Rosen waves, and reduce to eq. (38) in the sense that such a J is a solution of eq. (40) if and only if φ satisfies eq. (38).

6.1 The Nonlinear Generating Function

Given a generating function $F : \mathbb{R} \to \mathrm{GL}(2, \mathbb{R})$, we construct J in the usual way as follows. We let $x = t + r\sin\theta$, and using Birkhoff's theorem we factorize

$$F(t + r\sin\theta) = \tilde{f}^{-1}(r, t, \zeta) f(r, t, \zeta),$$

where $\tilde{f} \in \mathrm{LGL}_+(2, \mathbb{C})$ and $f \in \mathrm{LGL}_-(2, \mathbb{C})$. The matrix J is then given by

$$J(r, t) = \tilde{f}^{-1}(r, t, \infty) f(r, t, 0).$$

This is the hyperbolic version of the Ward $k = 0$ ansatz.. If F is symmetric and has determinant 1, the so does J and the splitting can be made unique by requiring that $f(r, t, 0) = \mathbb{1}$. Note that the cylindrical gravitational wave reality conditions reverse the sign of the determinant of J.

6.2 Propagation from Cauchy Data

Equation (39) exponentiates similarly. Define, for $0 \leqslant \theta \leqslant \pi$ and fixed x, $S(\theta)$ by

$$\frac{\mathrm{d}S}{\mathrm{d}\theta} = \frac{1}{2} x S(\theta) \left((1 + \cos\theta) u'(x\sin\theta) + v(x\sin\theta)\sin\theta\right) \quad \text{and} \quad S(0) = 1. \tag{44}$$

Integrating shows that $\mathrm{e}^{u(0)} S(\pi) = \mathrm{e}^{P(x)}$. Notice that the right-hand side is not exactly the integrand in the definition of $P(x)$, but that the term $\cos\theta u'(x\sin\theta)$ does not contribute to $S(\pi)$. Equation (44) may thus be used to generalize eq. (39) to the nonlinear setting. There, S, u' and v are matrix-valued, with u' and v given by

$$u'(r) = J^{-1}(r, 0) J_r(r, 0) \quad \text{and} \quad v(r) = J^{-1}(r, 0) J_t(r, 0)$$

for $r \geqslant 0$. The generating function for $J(r, t)$ is then given by $F(x) = J(0, 0) S(\pi)$. For a proof we refer the reader to [9].

It is manifest from eq. (44) that if J is real, then so are u' and v, and thus so is S, and hence F. If J has unit determinant, then so does F, since $\det F = \det J(0,0) \det S(\pi)$, and

$$\begin{aligned}
\det S(\pi) &= \exp\left(-\mathrm{Tr}(u(0)) + \mathrm{Tr}(P(x))\right) \\
&= \exp\left(\left(\frac{1}{2}x\int_0^\pi \mathrm{Tr}(u'(x\sin\theta)) + \mathrm{Tr}(v(x\sin\theta))\sin\theta \mathrm{d}\theta\right)\right) \\
&= \exp\left(\frac{1}{2}x\int_0^\pi \mathrm{Tr}(J^{-1}J_r + J^{-1}J_t\sin\theta)(x\sin\theta, 0)\mathrm{d}\theta\right) \\
&= 1.
\end{aligned}$$

To see that the symmetry of J implies that of F, notice that if $S(\theta)$ satisfies eq. (44), the so does $\hat{S}(\theta)$, defined by

$$\hat{S}(\pi - \theta)^t = J(x\sin\theta, 0)S(\theta)^{-1}.$$

It then follows that F is given by

$$F(x) = J(0,0)L(x)J(x,0)^{-1}L(x)^t J(0,0)^t, \tag{45}$$

where $L(x) = S(\pi/2)$. This form of F is manifestly symmetric whenever J is symmetric, and also positive definite whenever J is positive definite.

We stated earlier that the Einstein–Rosen solutions of eq. (40) reduce to the solutions of the scalar wave equation. The scalar wave equation of course possesses the superposition principle, and this linearity carries over to a superposition principle for the Einstein–Rosen solutions of the exponentiated equation (40) in the following way. If J_1 and J_2 are two Einstein–Rosen solutions of eq. (40) with generating functions F_1 and F_2 respectively, then J_1J_2 is also an Einstein–Rosen solution, and has generating function F_1F_2. Interestingly, it turns out that eq. (45) implies that a restricted version of this superposition principle holds for general cylindrical gravitational waves. Suppose that J_1 and J_2 are solutions of eq. (40) whose Cauchy data at $t = 0$ is supported in the non-intersecting intervals on the r-axis $[a_1, b_1]$ and $[a_2, b_2]$, respectively. Let $L_1(x)$ and $L_2(x)$ be their corresponding L functions as in the above construction, and let J be the solution obtained by combining the two sets of Cauchy data. Then this solution J is generated by

$$F(x) = J(0,0)L(x)J(x,0)^{-1}L(x)^t J(0,0)^t,$$

where $L(x) = L_1(x)L_2(x)$.

A Appendix

A.1 Liouville's Theorems

The following Liouville's theorems are stated, in a somewhat more general form including the infinite-dimensional case, in [5]. In the case $X = \mathbb{C}$, theorem A.1 reduces to the familiar statement that *a bounded entire function must be constant.* In the article we frequently make use of this, and also the generalized version of Liouville's theorem, theorem A.2, with $X = \mathrm{GL}(n, \mathbb{C})$, or another gauge group G, as appropriate.

Theorem A.1 (Liouville's Theorem)**.** *Let X be a finite-dimensional complex topological vector space and let $f : \mathbb{C} \to X$ be holomorphic. If $f(\mathbb{C})$ is a bounded subset of X, then f is constant.*

Theorem A.2 (Generalized Liouville's Theorem)**.** *Let X be a finite-dimensional complex Banach space, $f : \mathbb{C} \to X$ be holomorphic, and suppose that there exist positive numbers K and γ such that*

$$\|f(\zeta)\| \leqslant K\,(1 + \|\zeta\|^{\gamma})$$

for all $\zeta \in \mathbb{C}$. Then $f(\zeta)$ is a polynomial in ζ of degree at most γ.

A.2 Frobenius Theorem

The Frobenius theorem is a local existence theorem for a maximal set of independent solutions to an underdetermined system of first-order homogeneous linear PDEs. Its most general formulation is in terms of differential geometric concepts, but in simple cases it reduces to a more elementary and tractable form.

Definition A.1. A *subbundle* E of a vector bundle $F \to M$ over a manifold M is a collection of linear subspaces E_x of the fibres F_x of the bundle F such that $E \to M$ is itself a vector bundle.

Definition A.2. Let E be a tangent subbundle over M, i.e. a subbundle of the tangent bundle $\mathrm{T}M$. We say E is *integrable* at $x \in M$ if there exists a submanifold N of M such that at each point $y \in N$ the differential map of the inclusion $\iota : N \hookrightarrow M$,

$$\mathrm{d}_y\iota : \mathrm{T}_yN \hookrightarrow \mathrm{T}_yM,$$

induces a toplinear isomorphism of T_yN on E_y. We say E is integrable if it is integrable at every point.

Theorem A.3 (Frobenius Theorem, [2, §6.1])**.** *Let M be a manifold of class C^k, $k \geqslant 2$ and let E be a tangent subbundle over M. Then E is integrable if and only if for all points $x \in M$ and all vector fields X, Y (defined in a neighbourhood of x) such that at x they lie in E, the Lie bracket of the vector fields $[X, Y]$ at x also lies in E.*

Thus theorem A.3 states that, e.g., on an n-dimensional manifold a set of r first-order linear differential operators (vector fields) $L_i = l_i^k(x)\partial_k(x)$ are *involutive*, i.e.

$$[L_i, L_j](u)(x) = \alpha_{ij}^k(x)L_k(u)(x)$$

for some functions $\alpha_{ij}^k(x)$, if and only if locally there exist $n-r$ solutions $u_1, \ldots, u_{n-r}$ to the system $L_iu(x) = 0$ for all $1 \leqslant i \leqslant r$ such that their gradients $\nabla u_1, \ldots, \nabla u_{n-r}$ are linearly independent.

A.3 Gauge Theories

Gauge theories arise as generalizations of Maxwell's theory of electrodynamics, which is a set of partial differential equations for the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. Maxwell's equations can be recast in terms of a 4-vector potential Φ_μ, which determines $\mathbf{E}$ and $\mathbf{B}$, but is itself only defined up to the *gauge transformation* $\Phi_\mu \to \mathrm{e}^{\mathrm{i}\theta}\Phi_\mu$. Since the gauge transformations $\{\mathrm{e}^{\mathrm{i}\theta}\}$ form the group $\mathrm{U}(1)$, Maxwell's theory is called a $\mathrm{U}(1)$ gauge theory. The step up to a general gauge theory requires a more geometric approach. A first change is to regard the potential Φ as the fundamental variable of the theory in place of $\mathbf{E}$ and $\mathbf{B}$. One then interprets Φ as a *connection 1-form*, that is encodes it in a differential operator $\mathrm{D} = \mathrm{d} + \mathrm{i}\Phi$. This allows one to change the *gauge group* $\mathrm{U}(1)$ to a more general Lie group G. The theory produces a system of PDEs, called the *Yang–Mills equations*, for each choice of the gauge group G, and different choices of G can produce systems with quite different properties. For example, the corresponding system of PDEs is linear exactly when G is abelian. Of course, in the case G = U(1) these equations reduce to Maxwell's equations.

In what follows we give several rather informal geometric definitions which should provide us with sufficient background for a quick description of gauge theories.

Definition A.3. A *vector bundle* of rank n over a manfold M is a manifold E together with the projection map $\pi : E \to M$ such that each *fibre* $\pi^{-1}(x)$ (for $x \in M$) has the structure of an n-dimensional vector space. The projection is required to be *locally trivial*, so that for each $x \in M$ there exists a neighbourhood $U \subset M$ of x such that $\pi^{-1}(U) = U \times \mathbb{R}^n$ or $U \times \mathbb{C}^n$, depending on whether the vector bundle is real or complex. We frequently write $E \to M$ to denote a vector bundle E over a manifold M.

Definition A.4. A *local section* of a vector bundle E is a map $s : U \subset M \to E$ such that $\pi(s(x)) = x$ for all $x \in U$. The map s is a *global* section if it is defined on all of M, that is $U = M$. If M is a real manifold, then the fibres can be either real or complex vector spaces, and the sections are required to be smooth. If M is a complex manifold, then the fibres must be complex vector spaces and the sections are required to be holomorphic. In this case the vector bundle E is said to be a *holomorphic vector bundle*. We say more about these below.

Definition A.5. A *local frame field* of a vector bundle $E \to M$ is a family of local sections $e_1, \ldots, e_n$ such that $\{e_i(x)\}$ is a basis for E_x at each x. Given a local frame field we represent a local section s by a column vector with components $s_1, \ldots, s_n$ and write $s = s_j e_j$.

Two local frame fields are related by $\tilde{e}_j = e_i g_{ij}$, and the corresponding sections are related by $s_i = g_{ij}\tilde{s}_j$. The maps $g = (g_{ij})$ relating the local sections are called *transition functions* or *patching matrices*, and form a group. The map g takes values in $\mathbb{C}^{n\times n}$, and is defined wherever the domains of definitions of the local frame fields overlap. The group of matrices $\{g\}$ is called the *structure group*, and in the absence of any special structure on the bundle is $\mathrm{GL}(n, \mathbb{C})$. If the fibres of the vector bundle have extra structure, e.g. a Hermitian metric, then the structure group reduces to a subgroup of $\mathrm{GL}(n, \mathbb{C})$. In the case of the existence of a Hermitian metric it is $\mathrm{U}(n)$.

Definition A.6. A *connection* on E is a first-order differential operator D that maps sections s of E to E-valued 1-forms. In a local trivialization this is given by

$$\mathrm{D}s = \mathrm{D}_a s \; \mathrm{d}x^a = \mathrm{d}s + \Phi s,$$

where $\Phi = \Phi_a \mathrm{d}x^a$ is a matrix-valued 1-form which we also call the connection, or sometimes the *gauge potential*.

A connection defines a *covariant exterior derivative* on E-valued forms via

$$\mathrm{D}\alpha = \mathrm{d}\alpha + \Phi \wedge \alpha.$$

In a gauge theory, a choice of local trivialization is called a *gauge*, and the structure group is referred to as the *gauge group*. A *gauge transformation* is a change of local trivialization, $\tilde{e}_j = e_i g_{ij}$. Under this change the local sections transform as $s \to \tilde{s} = g^{-1}s$, and the gauge potential transforms as

$$\Phi \to \tilde{\Phi} = g^{-1}\Phi g + g^{-1}\mathrm{d}g.$$

Definition A.7. The *curvature* of D is the matrix-valued 2-form $F = F_{ab}\mathrm{d}x^a \wedge \mathrm{d}x^b$ with components

$$F_{ab} = \partial_a \Phi_b - \partial_b \Phi_a + [\Phi_a, \Phi_b].$$

It measures the extent to which the operators $\mathrm{D}_a, \mathrm{D}_b$ fail to commute, since

$$F_{ab} = [\mathrm{D}_a, \mathrm{D}_b] = \mathrm{D}_a\mathrm{D}_b - \mathrm{D}_b\mathrm{D}_a.$$

The curvature F takes values in the Lie algebra of the structure group.

Under a gauge transformation the curvature transforms by

$$F \to g^{-1}Fg,$$

so it is an obstruction to finding a gauge in which $\Phi = 0$. Indeed, if there exists a frame in which $\Phi = 0$, then F must be zero in all frames. Conversely, if $F = 0$, then there exists a local gauge such that $\Phi = 0$, since then $F = 0$ is the local Frobenius integrability condition for the system of linear equations

$$D_a e_i = 0, \quad i = 1, \ldots, n.$$

As F takes values in the Lie algebra, we have

$$\mathrm{D}_a F_{bc} = \partial_a F_{bc} + [\Phi_a, F_{bc}].$$

Then $\mathrm{D}F = \mathrm{D}_{[a}F_{bc]}\mathrm{d}x^a \wedge \mathrm{d}x^b \wedge \mathrm{d}x^c = \mathrm{d}F + \Phi \wedge F - F \wedge \Phi$. A calculation yields the *Jacobi identity*,

$$[\mathrm{D}_a, [\mathrm{D}_b, \mathrm{D}_c]] + [\mathrm{D}_b, [\mathrm{D}_c, \mathrm{D}_a]] + [\mathrm{D}_c, [\mathrm{D}_a, \mathrm{D}_b]] = 0,$$

which implies the *Bianchi identity*

$$\mathrm{D}F = 0.$$

This is true for an arbitrary connection.

A.4 Holomorphic Vector Bundles

Let a complex manifold M be covered by open sets $\{V_\sigma\}$, and suppose that we have a vector bundle $E \to M$ of rank n such that on each chart V_σ, the bundle E has a given holomorphic frame field $e_{\sigma i}$, where $i = 1, \ldots, n$. On each non-empty intersection $V_\sigma \cap V_\tau$,

$$(e_{\tau 1}, \ldots, e_{\tau n}) = (e_{\sigma 1}, \ldots, e_{\sigma n})F_{\sigma\tau}$$

for some holomorphic maps

$$F_{\sigma\tau} : V_\sigma \cap V_\tau \to \mathrm{GL}(n, \mathbb{C}).$$

For a fixed pair of (σ, τ) we call $F_{\sigma\tau}$ the *patching matrix* from V_σ to V_τ, and the collection of all patching matrices the *patching data* of the vector bundle E. By the definition of a complex vector bundle, the patching data satisfy the following conditions.

(i) Each patching matrix $F_{\sigma\tau}$ is holomorphic and non-singular,

(ii) $F_{\sigma\tau} = F_{\tau\sigma}^{-1}$ whenever $V_\sigma \cap V_\tau$ is non-empty,

(iii) $F_{\sigma\tau} \circ F_{\tau\nu} \circ F_{\nu\sigma} = \mathbb{1}$ whenever $V_\sigma \cap V_\tau \cap V_\nu$ is non-empty.

Any collection of matrices satisfying these conditions will define a holomorphic vector bundle.

We say two holomoprhic vector bundles E and E' are *equivalent* if around every point on M there exist local trivializations for E and E', covered by the same open sets V_σ, such that their patching matrices are related by

$$F_{\sigma\tau} = h_\sigma^{-1} F'_{\sigma\tau} h_\tau$$

for some family of holomorphic maps $h_\sigma : V_\sigma \to \mathrm{GL}(n, \mathbb{C})$. In particular, E is in the equivalence class of the trivial bundle if and only if its patching matrices can be factorized as $F_{\sigma\tau} = h_\sigma^{-1} h_\tau$. Note that this definition of equivalence of vector bundles amounts to saying that there exists a biholomorphic map $E \to E'$ that maps the fibres of E linearly onto the corresponding fibres of E'.

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