Kähler Manifolds and the Calabi Conjecture

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1 Introduction

Some motivation to study Calabi–Yau manifolds comes from theoretical physics, where, for example, in superstring theory, the extra dimensions of spacetime are sometimes conjectured to take the shape of a 6-dimensional Calabi–Yau manifold. The reason they are of interest is because, by construction, they are solutions to the vacuum Einstein equations. In full geometric jargon, Calabi–Yau manifolds are Ricci flat Kähler manifolds with vanishing first Chern class. So to begin to understand the theory of Calabi–Yau manifolds, one must first understand the theory of Kähler manifolds. Kähler manifolds are interesting objects in their own right. In fact, with two exceptions (the flat metric on the circle and Joyce metrics in dimensions 7 and 8), the only known *compact* examples of irreducible Riemannian metrics satisfying the vacuum Einstein equations are constructed on Kähler manifolds.

Kähler manifolds may be understood from the point of view of real Riemannian geometry. They are defined to be smooth manifolds with three mutually compatible structures, namely a complex structure, a Riemannian structure, and a symplectic structure. They are *complex* manifolds, and the theory of complex geometry will be of some use to us from time to time, but we will mostly follow the picture of thinking of these objects as real manifolds with extra structure.

In some sense, Kähler manifolds are like the complex objects corresponding to real Riemannian manifolds. Indeed, Kähler manifolds possess the remarkable property that around every point on the manifold there exist local *holomorphic* coordinates in which the Kähler metric osculates the standard Hermitian metric to the order 2 at a point. This is analogous to the existence of normal coordinates on real Riemannian manifolds – see Theorem C.1. Perhaps even more strikingly, it turns out that the Kähler metric around every point on a Kähler manifold is determined by a single smooth function, regardless of the dimension of the manifold. As in classical complex analysis, the *rigidity* of natural complex objects is a recurring feature.

The material presented in this essay is classical and well-known. We broadly follow the lecture notes of Moroianu, [13], padded with material from Joyce, [10]. Further references may be found at the end. We present some proofs, but skip ones that are either trivial or too involved for the scope of this essay. We refer the concerned reader to Griffiths and Harris [6], which contains almost every proof imaginable.

We assume familiarity with standard differential geometry, including the definitions of smooth manifolds, vector bundles, exterior and Lie derivatives, exterior forms and so on, although we recap the important definitions either as we go along or in the appendices. For ease of exposition, we will usually use the Einstein summation convention. The letter n will usually denote the real dimension, while m will usually denote the complex dimension of a manifold.

2 Preliminaries

2.1 Vector Bundles, Principal Bundles and G-structures

In this section we briefly review the theory of bundles, G-structures, and connections. For more details we refer the reader to Joyce [10] or Griffiths and Harris [6]. Let us first briefly recall the definition of a smooth real vector bundle.

Definition 2.1. Let M be a smooth manifold. A *smooth real rank k vector bundle E* over M, written $E \to M$, is a smooth manifold E equipped with a smooth projection $\pi : E \to M$ such that for each $p \in M$ the *fibre* $E_p := \pi^{-1}(p)$ has the structure of a vector space, there is an open neighbourhood $U(p)$ of p such that there exists a diffeomorphism $\psi : \pi^{-1}(U(p)) \to U(p) \times \mathbb{R}^k$, the vector space structures on $E_q = \pi^{-1}(q)$ and $\{q\} \times \mathbb{R}^k$ agree for all $q \in U(p)$, and the following diagram commutes,

$$\begin{tikzcd}
\pi^{-1}(U(p)) \arrow[rr, "\psi", "\cong"'] \arrow[dr, "\pi"'] & & U(p) \times \mathbb{R}^k \arrow[dl, "\mathrm{pr}_{U(p)}"] \\
& U(p) &
\end{tikzcd}$$

Definition 2.2. We say two vector bundles $E, F \to M$ are *isomorphic* if there exists a homeomorphism of total spaces $\varphi : E \to F$ with the property that for each $p \in M$

$$\varphi|_{\pi_E^{-1}(p)} : \pi_E^{-1}(p) \to \pi_F^{-1}(p)$$

is a linear isomorphism.

Definition 2.3. Let M be a smooth manifold and $E \to M$ a smooth vector bundle. A *connection* ∇^E on E is a linear map $\nabla^E : \Gamma(E) \to \Gamma(E \otimes T^*M)$ satisfying

$$\nabla^E(f\sigma) = f\nabla^E\sigma + \mathrm{d}f \otimes \sigma$$

for all smooth functions $f : M \to \mathbb{R}$ and all sections $\sigma \in \Gamma(E)$. If ∇^E is such a connection and $X \in \Gamma(TM)$ is a vector field, we write

$$\nabla^E_X\sigma := X \lrcorner \nabla^E\sigma \in \Gamma(E),$$

where $\lrcorner$ contracts X with the T^*M factor in $\nabla^E\sigma$.

Definition 2.4. Let ∇ be a connection on the tangent bundle $TM \to M$ of a smooth manifold M. We call the tensor $T^a{}_{bc}$, defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

for all vector fields $X, Y \in TM$, the *torsion* of ∇. We say ∇ is *torsion-free* if $T^a{}_{bc} = 0$.

Vector bundles are a fundamental concept in differential geometry. A slightly more general idea is the concept of a *fibre bundle*. Intuitively, fibre bundles are the same sort of structure as vector bundles, but now one is allowed to attach something other than a vector space to each point on a manifold. For example, one may consider Lie groups parametrised by points on a manifold.

Definition 2.5. A (right) group action of a Lie group G on a manifold M is a smooth map $\rho : M \times \mathrm{G} \to M$, denoted by $mg := \rho(m, g)$, such that

(i) $me = m \quad \forall m \in M$,

(ii) $m(gh) = (mg)h \quad \forall m \in M, \forall g, h \in \mathrm{G}$.

The action is called *free* if for all $m \in M$ $mg = m$ implies $g = e$, and *transitive* if for all $m, m' \in M$ there exists $g \in \mathrm{G}$ such that $m' = mg$.

Definition 2.6. Let M be a manifold, real or complex, and G a Lie group. A *principal* G-*bundle* $P \to M$ consists of the following.

(i) A manifold P,

(ii) a smooth map $\pi : P \to M$, and

(iii) a smooth (right) action $\rho : P \times \mathrm{G} \to P$ of G on P,

such that each $p \in M$ has an open neighbourhood $U(p) \subset M$ with the commutative diagram

$$\begin{tikzcd} \pi^{-1}(U(p)) \arrow[rr, "\psi", "\cong"'] \arrow[dr, "\pi"'] & & U(p) \times \mathrm{G} \arrow[dl, "\mathrm{pr}_{U(p)}"] \\ & U(p) & \end{tikzcd}$$

compatible with the G-action $h : (u, g) \mapsto (u, gh)$ on $U(p) \times \mathrm{G}$. By compatible here we mean that

$$\rho\left(\psi^{-1}(u, g), h\right) = \psi^{-1}(u, gh) \quad \forall u \in U(p), \forall g, h \in \mathrm{G}.$$

An equivalent definition is that a principal G-bundle is a fibre bundle together with a smooth right action $\rho : P \times \mathrm{G} \to P$ which restricts to a free transitive action on each fibre $\pi^{-1}(p)$. Note that this immediately implies that each fibre $\pi^{-1}(p)$ must be diffeomorphic to G: choose any $u \in \pi^{-1}(p)$ and define $L_u : \mathrm{G} \to \pi^{-1}(p)$ by $g \mapsto \rho(u, g)$ Since ρ acts smoothly, freely and transitively on $\pi^{-1}(p)$, L_u is smooth, injective and surjective, so a diffeomorphism.

Of particular importance is one specific example of a principal bundle called the *frame bundle*.

Definition 2.7. Let M be a manifold of dimension n. The *frame bundle* $F \to M$ of M is the principal $\mathrm{GL}(n, \mathbb{R})$-bundle on M defined by

$$F = \{(p, f_1, \dots, f_n) \,:\, p \in M, (f_1, \dots, f_n) \text{ is a basis of } T_pM\}$$

with projection $\pi : (p, f_1, \dots, f_n) \mapsto p$ and $\mathrm{GL}(n, \mathbb{R})$-action

$$(A_{ij}) : (p, f_1, \dots, f_n) \longmapsto \left(p, \sum_j A_{1j} f_j, \dots, \sum_j A_{nj} f_j\right).$$

Definition 2.8. Let G be a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$. A G-*structure* on an n-dimensional manifold M is a submanifold $P \subseteq F$ of the frame bundle $F \to M$ such that P is invariant under the G-action of F induced by the $\mathrm{GL}(n, \mathbb{R})$-action, and $\pi|_P : P \to M$ is a principal G-bundle. In other words, P is a principal G-subbundle of $F \to M$.

Example 2.9. Let g be a Riemannian metric on an n-dimensional manifold M and let $F \to M$ be the frame bundle. Define

$$P^g = \{(p, f_1, \dots, f_n) \in F \, : \, g|_p(f_i, f_j) = \delta_{ij}\} \, .$$

In other words, P^g is a subbundle of the frame bundle such that at each point $p \in M$ the fibre consists of frames orthonormal with respect to $g|_p$. To see what P^g is as a G-structure, consider the action of $A \in \mathrm{GL}(n, \mathbb{R})$ on the fibre $(P^g)_p$. We require that

$$\begin{aligned} \delta_{ij} &= g|_p \left(\sum_k A_{ik} f_k, \sum_l A_{jl} f_l \right) \\ &= \sum_{k,l} A_{ik} A_{jl} \delta_{kl} \\ &= \sum_k A_{ik} A_{jk} = \left(A A^\top \right)_{ij} , \end{aligned}$$

so A must be orthogonal. Thus P^g is an $\mathrm{O}(n)$-structure. This gives a 1-1 correspondence between Riemannian metrics and $\mathrm{O}(n)$-structures. Note that the metric g only determines what subfibres of the frame bundle F are 'selected' when constructing P^g, but the $O(n)$-action, of course, does not depend on g.

Example 2.10. It is easy to see that an orientation on M is equivalent to a $\mathrm{GL}^+(n, \mathbb{R})$-structure.

From any principal bundle we can mould a host of vector bundles by using representations of the fibre G.

Definition 2.11. Let M be a smooth manifold and $P \to M$ a principal bundle over M with fibre a Lie group G. Let ρ be a representation of G on a vector space V (a homomorphism $\mathrm{G} \to \mathrm{GL}(V)$). Then G acts on the product space $P \times V$ by the principal bundle action on the first factor and ρ on the second. We define $\rho(P) := (P \times V)/\mathrm{G}$, the quotient of the product by this G-action. Of course, $P/\mathrm{G} = M$, so the obvious projection from $(P \times V)/\mathrm{G}$ to $P/\mathrm{G} = M$ gives a projection $\rho(P) \to M$. Since G acts freely on P, this projection has fibre V, and thus $\rho(P)$ is a vector bundle over M.

2.2 Connections on Principal Bundles

Suppose P is a principal bundle over a manifold M with fibre G and projection $\pi : P \to M$. Let $p \in P$, and set $m = \pi(p)$. Then the derivative of π gives a linear map $\mathrm{d}\pi_p : T_p P \to T_m M$. Define $C_p = \ker(\mathrm{d}\pi_p)$. Clearly C_p is a subspace of $T_p P$, and the collection of subspaces C_p as p ranges over P form a vector subbundle C of tangent bundle TP, called the *vertical subbundle*. Clearly $C_p = T_p(\pi^{-1}(m))$, as the kernel of $\mathrm{d}\pi_p$ are exactly the tangents to P at p which have no tangential component to the manifold M. Since the fibres of π are the orbits of the free G-action on P, it follows that there is a natural isomorphism between C_p and the Lie algebra $\mathfrak{g}$ of G.

Definition 2.12. Let M be a manifold and P a principal bundle over M with fibre a Lie group G. A *connection* on P is a vector subbundle D of TP, called the *horizontal subbundle*, that is invariant under the G-action on P and which satisfies

$$T_p P = C_p \oplus D_p$$

for all $p \in P$.

The derivative $d\pi_p$ induces an isomorphism between D_p and T_mM. Indeed, if $\pi(p) = m$, then $d\pi_p$ maps $T_pP \cong C_p \oplus D_p$ onto T_mM, and $C_p = \ker(d\pi_p)$. Thus the horizontal subbundle D is naturally isomorphic to $\pi^*(TM)$, where $\pi^*(TM)$ denotes the pullback bundle defined by

$$\pi^*(TM) := \{(u, p) \in TM \times P : \pi(p) = \hat{\pi}(u)\},$$

where $\hat{\pi} : TM \to M$ is the projection of the tangent bundle of M. Thus if $X \in \Gamma(TM)$ is a vector field on M, there is a unique section $\lambda(X) \in \Gamma(D)$ of the bundle $D \subset TP$ over P such that $d\pi_p(\lambda(X)|_p) = X|_m$ for each $p \in P$. We call $\lambda(X)$ the *horizontal lift* of the vector field X. It is a vector field on P, and is invariant under the action of G on P.

It will be useful to relate connections on principal bundles and the vector bundles that arise from the representations of the principal bundle. As before, let M be a smooth manifold, $\pi : P \to M$ a principal bundle with fibre a Lie group G, and let $\rho : \mathrm{G} \to \mathrm{GL}(V)$ be a representation of G on a vector space V. Let E be the vector bundle $\rho(P)$ (as defined in Definition 2.11) over M. Suppose we are given a connection D on the principal bundle P. We construct the unique connection ∇^E on E as follows. Suppose $\sigma \in \Gamma(E)$ is a smooth section of $E = \rho(P) := (P \times V)/\mathrm{G}$. Then since the pullback bundle of $E = (P \times V)/\mathrm{G}$ along $\pi : P \to M = P/\mathrm{G}$ is clearly $P \times V$, $\pi^*(\sigma)$ is a section of $P \times V$ over P, or a function $\pi^*(\sigma) : P \to V$. So its differential is the linear map $d\pi^*(\sigma)|_p : T_pP \to V$ for each $p \in P$, and so $d\pi^*(\sigma)$ is a smooth section of the vector bundle $V \otimes T^*P$ over P.

Now for each $p \in P$ we have the isomorphisms

$$T_pP \cong C_p \oplus D_p, \quad C_p \cong \mathfrak{g}, \quad \text{and} \quad D_p \cong \pi^*(T_mM),$$

where $m = \pi(p)$. Since here pullback commutes with taking duals, these give the natural decomposition

$$V \otimes T^*P \cong (V \otimes \mathfrak{g}^*) \oplus (V \otimes \pi^*(T^*M)).$$

Let us denote by $\pi_D(d\pi^*(\sigma))$ the $\Gamma(V \otimes \pi^*(T^*M))$-component of $d\pi^*(\sigma)$ in this splitting. Now both $\pi^*(\sigma)$ and the vector bundle splitting are G-invariant, so $\pi_D(d\pi^*(\sigma))$ must be G-invariant. But there is a 1-1 correspondence between G-invariant sections of $V \otimes \pi^*(T^*M)$ over P and sections of the corresponding vector bundle $E \otimes T^*M$ over M. Hence $\pi_D(d\pi^*(\sigma))$ is the pullback of a unique element of $\Gamma(E \otimes T^*M)$, which we write as $\nabla^E\sigma \in \Gamma(E \otimes T^*M)$. The section $\nabla^E\sigma$ is the unique section of $E \otimes T^*M$ with pullback $\pi_D(d\pi^*(\sigma))$ under the natural projection $V \otimes \pi^*(T^*M) \to E$, so this construction defines a connection ∇^E on the vector bundle E over M.

So for each connection D on a principal bundle P there exists a unique corresponding connection ∇^E on the vector bundle $E = \rho(P)$. In general, the map $D \mapsto \nabla^E$ may be neither injective nor surjective. However, if $\mathrm{G} = \mathrm{GL}(n, \mathbb{R})$ and ρ is the standard representation of G on $\mathbb{R}^n$, then P is the frame bundle F of M, and this gives a 1-1 correspondence between connections ∇ on TM and D on F. We will thus usually identify connections ∇ on TM with the corresponding connections D on F and refer to both as connections on M.

Now let M be a manifold of dimension n with frame bundle F, let G be a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$, and P be a G-structure on M. Suppose D is a connection on P. It turns out that then there is a unique connection D' on F that reduces to D on P, as explained by Joyce in [10], §2.3 – §2.6. Conversely, a connection D' on F reduces to a connection D on P if and only if for each $p \in P$, the subspace D'_p of T_pF lies in T_pP. We call a connection ∇ on TM *compatible with the* G*-structure* P if the corresponding connection on F reduces to P. Thus every connection D on P induces a unique connection ∇ on TM, and conversely a connection ∇ on TM arises from a connection D on P if and only if ∇ is compatible with P. For our purposes it will be sufficient to think of this compatibility condition as the requirement for the tensor defining the G-structure to be parallel with respect to ∇.

Definition 2.13. Let P be a subbundle of the frame bundle F of M. We call a connection D on P *torsion-free* if there exists a connection ∇ on TM which is compatible with P and is torsion-free in the sense of Definition 2.4.

Definition 2.14. A *torsion-free* G*-structure* is a G-structure $\mathrm{G}(M)$ on which there exists a torsion-free connection on $\mathrm{G}(M)$.

Note that if $\mathrm{G} \subset \mathrm{O}(n)$, then any torsion-free connection on $\mathrm{G}(M)$ must be the Levi–Civita connection of the induced metric, so in particular is unique.

2.3 Holomorphic Functions

The essential feature that distinguishes the complex plane $\mathbb{C}$ from $\mathbb{R}^2$ is the operation of multiplying by i. The interesting objects on $\mathbb{C}$ then become complex differentiable, or *holomorphic*, functions. Recall that a smooth function $f = u + iv : \mathbb{C} \to \mathbb{C}$ is called holomorphic if it satisfies the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

A different way of phrasing this condition is to require that the $\bar{\partial}$-derivative of f vanishes, that is

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f = 0.$$

We will see that the natural generalizations of these concepts to manifolds yield the notions of an (almost) complex structure on the manifold and a (pseudo-)holomorphic structure on a vector bundle respectively.

We will follow the general approach of regarding complex manifolds as even dimensional real manifolds with extra structure, so to this end let j_1 denote the endomorphism of $\mathbb{R}^2$ corresponding to multiplication by i on $\mathbb{C}$ under the identification $z = x + iy \mapsto (x, y)$. In the standard basis it is given by

$$j_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We can view any complex function $f = u + iv$ as a real function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $(x, y) \mapsto (u(x, y), v(x, y))$, whose differential at a point $p \in \mathbb{R}^2$ is the linear map

$$\mathrm{d}f|_p = \begin{pmatrix} \frac{\partial u}{\partial x}\big|_p & \frac{\partial u}{\partial y}\Big|_p \\ \frac{\partial v}{\partial x}\big|_p & \frac{\partial v}{\partial y}\Big|_p \end{pmatrix}.$$

The Cauchy–Riemann equations at $p \in \mathbb{R}^2$ can then be written as the commutation relation

$$j_1 \circ \mathrm{d}f|_p = \mathrm{d}f|_p \circ j_1$$

This algebraic, rather than analytic, characterization allows us to extend the notion of holomorphicity to higher complex dimensions. Consider the identification of $\mathbb{C}^m$ with $\mathbb{R}^{2m}$ by

$$(z_1, \ldots, z_m) = (x_1 + iy_1, \ldots, x_m + iy_m) \longmapsto (x_1, \ldots, x_m, y_1, \ldots y_m),$$

and denote by j_m the endomorphism of $\mathbb{R}^{2m}$ corresponding to multiplication by i on $\mathbb{C}^m$. In the standard basis

$$j_m = \begin{pmatrix} 0 & -\mathbb{1}_m \\ \mathbb{1}_m & 0 \end{pmatrix}.$$

We then say a function $f : \mathbb{C}^l \to \mathbb{C}^m$ is holomorphic at $p \in \mathbb{C}^l$ if and only if, considered as a real function $f : \mathbb{R}^{2l} \to \mathbb{R}^{2m}$, its differential satisfies the commutation relation

$$j_m \circ \mathrm{d}f|_p = \mathrm{d}f|_p \circ j_l. \tag{1}$$

3 Complex and Almost Complex Structures

3.1 (Almost) Complex Structures

Definition 3.1 (Complex Manifolds #1)**.** Let M be a real topological manifold of dimension $2m$. A *complex chart* on M is a pair (U, φ_U), where U is an open subset of M and $\varphi_U : U \to \mathbb{C}^m$ is a homeomorphism between U and some open subset of $\mathbb{C}^m$. A *complex manifold* of complex dimension m is a real topological $2m$-dimensional manifold M with an atlas of charts (U, φ_U), $U \in \mathcal{U}$, whose transition functions $\varphi_{UV} : \varphi_V(U \cap V) \to \varphi_U(U \cap V)$, given by $\varphi_{UV} = \varphi_U \circ \varphi_V^{-1}$, are required to be holomorphic as maps between open subsets of $\mathbb{C}^m$. The atlas of holomorphic charts is called a *holomorphic structure*.

A holomorphic structure consists of holomorphic maps between open sets of $\mathbb{C}^m$, so in particular defines a set of smooth maps between open sets of $\mathbb{R}^{2m}$. So every complex manifold M of complex dimension m defines a smooth real manifold $M_{\mathbb{R}}$ of real dimension $2m$, where $M_{\mathbb{R}}$ is the same as M as a topological space. The converse is not true, of course, since not every smooth map is holomorphic. However, the holomorphic structure of M is encoded in a single tensor J on the real manifold $M_{\mathbb{R}}$. To see how, for every vector $X \in T_pM_{\mathbb{R}}$ choose any $U \in \mathcal{U}$ containing p and define

$$J_U(X) = (\mathrm{d}\varphi_U)^{-1} \circ j_m \circ (\mathrm{d}\varphi_U)(X).$$

In fact J_U is independent of U, since for any other $V \in \mathcal{U}$ containing p, $\varphi_{VU} = \varphi_V \circ \varphi_U^{-1}$ is holomorphic (satisfies relation (1)), and $\varphi_V = \varphi_{VU} \circ \varphi_U$, so

$$\begin{aligned} J_V(X) &= (\mathrm{d}\varphi_V)^{-1} \circ j_m \circ (\mathrm{d}\varphi_V)(X) \\ &= (\mathrm{d}\varphi_U)^{-1} \circ (\mathrm{d}\varphi_{VU})^{-1} \circ j_m \circ (\mathrm{d}\varphi_{VU}) \circ (\mathrm{d}\varphi_U)(X) \\ &= (\mathrm{d}\varphi_U)^{-1} \circ j_m \circ (\mathrm{d}\varphi_U)(X) = J_U(X). \end{aligned}$$

Thus the collection of maps J_U, $U \in \mathcal{U}$, which we refer to as just J, defines a tensorial field of endomorphisms on the tangent bundle of $M_{\mathbb{R}}$ satisfying $J^2 = -\mathbb{1}$. We capture the essential properties of J in the following definition.

Definition 3.2. Let M be a smooth *real* manifold of dimension $2m$. An *almost complex structure* on M is a tensor $J^a_b : TM \to TM$ such that $J^2 = -\mathbb{1}$. The pair (M, J) is then called an *almost complex manifold.*

The discussion above shows that a complex manifold is in a canonical way an almost complex manifold, and we will always identify a complex manifold M with its underlying real manifold $M_{\mathbb{R}}$ equipped with a tensor J. The converse is not true in general, however; the condition for the converse to hold is provided by the Newlander–Nirenberg Theorem below.

Definition 3.3. To every almost complex structure J we associate a tensor $N^J = {}^J N^c{}_{ab}$, called the *Nijenhuis tensor*, defined by

$$N^J(X,Y) := [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]$$

for all vector fields X, Y.

We can give another definition of complex manifolds as real manifolds with extra structure, one we will use throughout the rest of this essay.

Definition 3.4 (Complex Manifolds #2)**.** A *complex manifold* is an almost complex manifold (M, J) whose Nijenhuis tensor vanishes, $N^J = 0$.

The equivalence of the two definitions of complex manifolds is a consequence of the Newlander–Nirenberg Theorem.

Theorem 3.5 (Newlander–Nirenberg)**.** *Let (M, J) be an almost complex manifold. The almost complex structure J comes from a holomorphic structure if and only if the Nijenhuis tensor of J vanishes, $N^J = 0$.*

Proof. The proof can be found in [8]. □

Finally, we can give a third, elegant definition of a complex manifold by using the language of G-structures introduced in Section 2.1.

Definition 3.6 (Complex Manifolds #3)**.** Let M be a smooth real manifold of dimension $2m$. Then a *complex structure* on M is a torsion-free $\mathrm{GL}(m, \mathbb{C})$-structure on M.

Indeed, let $F \to M$ be the frame bundle of M and J an almost complex structure on M. Define $P \subseteq F$ to be the subbundle of frames in F in which the components of J assume the standard form

$$J = \begin{pmatrix} 0 & -\mathbb{1}_m \\ \mathbb{1}_m & 0 \end{pmatrix} = j_m.$$

Then P is a principal subbundle of F with fibre $\mathrm{GL}(m, \mathbb{C})$. We have the homomorphic embedding $\mathrm{GL}(m, \mathbb{C}) \hookrightarrow \mathrm{GL}(2m, \mathbb{R})$,

$$A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}, \tag{2}$$

so P is a $\mathrm{GL}(m, \mathbb{C})$-structure on M. This defines a 1-1 correspondence between almost complex structures J and $\mathrm{GL}(m, \mathbb{C})$-structures on P on M. It remains to show that J is a complex structure if and only if P is torsion-free. Suppose ∇ is a torsion-free connection on TM. Consider the connection defined by

$$\nabla^{\#}_X Y = \nabla_X Y - A_X Y,$$

where

$$A_X Y = \frac{1}{4}\left(2J(\nabla_X J)Y + (\nabla_{JY} J)X + J(\nabla_Y J)X\right).$$

The idea behind this definition is that the new connection $\nabla^{\#}$ preserves J, and thus is compatible with the $\mathrm{GL}(m, \mathbb{C})$-structure. Furthermore, it will turn out that $\nabla^{\#}$ is torsion-free.

Lemma 3.7. *J is a complex structure if and only if $(\nabla_{JX} J)Y = J(\nabla_X J)Y$.*

Proof. The proof is a routine calculation and can be found on pp. 82 – 83 of Moroianu, [13]. □

Proposition 3.8. *A is symmetric if and only if J is a complex structure.*

Proof. Suppose J is a complex structure. Then by Lemma 3.7,

$$4A_X Y = 2J(\nabla_X J)Y + 2J(\nabla_Y J)X = 4A_Y X.$$

Conversely, suppose A is symmetric, so that

$$J(\nabla_X J)Y - J(\nabla_Y J)X = (\nabla_{JX} J)Y - (\nabla_{JY} J)X. \tag{3}$$

We use the fact that ∇ satisfies the Leibniz rule in the second variable to rewrite equation (3) as

$$J\nabla_X(JY) + \nabla_X Y - J\nabla_Y(JX) - \nabla_Y X = \nabla_{JX}(JY) - J(\nabla_{JX} Y) - \nabla_{JY}(JX) + J(\nabla_{JY} X),$$

and use the fact that ∇ is torsion-free to notice that this is precisely the statement that the Nijenhuis tensor of J vanishes, $N^J(X,Y) = 0$. □

Thus J is a complex structure if and only if

$$\nabla^{\#}_X Y - \nabla^{\#}_Y X = \nabla_X Y - \nabla_Y X = [X,Y],$$

that is, $\nabla^{\#}$ is torsion-free. The new connection $\nabla^{\#}$ is compatible with the $\mathrm{GL}(m,\mathbb{C})$-structure defining J, so by the discussion in Section 2.2, we see that a complex structure J is exactly equivalent to a torsion-free $\mathrm{GL}(m,\mathbb{C})$-structure.

Remark 3.9. Interestingly, we can also use the embedding (2) to show that an almost complex structure defines an orientation. We compute the product

$$\begin{pmatrix} \mathbb{1}_m & 0 \\ -i\mathbb{1}_m & \mathbb{1}_m \end{pmatrix} \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} \mathbb{1}_m & 0 \\ i\mathbb{1}_m & \mathbb{1}_m \end{pmatrix} = \begin{pmatrix} A - iB & -B \\ 0 & A + iB \end{pmatrix},$$

which shows that

$$\det \begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \det \begin{pmatrix} A - iB & -B \\ 0 & A + iB \end{pmatrix} = |\det(A + iB)|^2 > 0.$$

Thus $\mathrm{GL}(m,\mathbb{C})$ in fact embeds into $\mathrm{GL}^+(2m,\mathbb{R})$, which defines an orientation by Example 2.10.

3.2 The Complexified Tangent Bundle

Intuitively, J should have eigenvalues $\pm i$, but as TM is a real vector bundle, J does not have eigenspaces in TM. Thus to diagonalize J we have to complexify the tangent bundle,

$$TM^{\mathbb{C}} := TM \otimes_{\mathbb{R}} \mathbb{C},$$

and extend all real endomorphisms and differential operators from TM to $TM^{\mathbb{C}}$ by $\mathbb{C}$-linearity. We then denote by $T^{1,0}M$ the eigenbundle of J in $TM^{\mathbb{C}}$ corresponding to the eigenvalue i, and by $T^{0,1}M$ the eigenbundle corresponding to $-i$. That is,

$$T^{1,0}M = \{Z \in TM^{\mathbb{C}} \,:\, JZ = iZ\},$$

and similarly for $T^{0,1}M$. For calculation purposes it is useful to characterise these eigenbundles more explicitly.

Lemma 3.10. *We have*

$$T^{1,0}M = \{X - iJX \,:\, X \in TM\}, \qquad T^{0,1}M = \{X + iJX \,:\, X \in TM\},$$

and $TM^{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M$.

The proof is an easy calculation. The intuitive way to think about these subbundles of $TM^{\mathbb{C}}$ is that $T^{1,0}M$ captures the holomorphic information about $TM^{\mathbb{C}}$, while $T^{0,1}M$ captures the anti-holomorphic information.

3.3 The Complexified Exterior Bundles

Let (M, J) be an almost complex manifold. For any p we define the complexified exterior bundle $\Lambda^p_{\mathbb{C}}M := \Lambda^p M \otimes_{\mathbb{R}} \mathbb{C}$. We denote by $\Omega^p(M)$ the smooth sections of $\Lambda^p M$; sections of $\Lambda^p_{\mathbb{C}}M$ can then be viewed as formal sums $\omega + i\tau$, where $\omega, \tau \in \Omega^p(M)$.

Definition 3.11. In a way similar to the decomposition of the complexified tangent bundle, we define

$$\Lambda^{1,0}M := \{\xi \in \Lambda^1_{\mathbb{C}}M \,:\, \xi(Z) = 0 \;\; \forall Z \in T^{0,1}M\}$$

and

$$\Lambda^{0,1}M := \{\xi \in \Lambda^1_{\mathbb{C}}M \,:\, \xi(Z) = 0 \;\; \forall Z \in T^{1,0}M\},$$

called bundles of *forms of type* $(1,0)$ and *forms of type* $(0,1)$ respectively.

By Lemma 3.10, we immediately get the following.

Lemma 3.12. *We have*

$$\Lambda^{1,0}M = \{\omega - i\omega \circ J \,:\, \omega \in \Lambda^1 M\}, \qquad \Lambda^{0,1}M = \{\omega + i\omega \circ J \,:\, \omega \in \Lambda^1 M\},$$

and $\Lambda^1_{\mathbb{C}}M = \Lambda^{1,0}M \oplus \Lambda^{0,1}M$.

We denote by $\Lambda^{k,0}$ the k-th exterior power of $\Lambda^{1,0}$, and similarly by $\Lambda^{0,k}$ the k-th exterior power of $\Lambda^{0,1}$. We also write $\Lambda^{p,q}$ for the tensor product $\Lambda^{p,0} \otimes \Lambda^{0,q}$. Using the formula

$$\Lambda^k(E \oplus F) \cong \bigoplus_{i=0}^{k} \Lambda^i E \otimes \Lambda^{k-i} F$$

for the k-th exterior power of a direct sum of vector spaces E and F, we get

$$\Lambda^k_{\mathbb{C}}M \cong \bigoplus_{p+q=k} \Lambda^{p,q}M. \tag{4}$$

We call elements of $\Lambda^{p,q}M$ *forms of type* (p,q) and denote the space of smooth sections of $\Lambda^{p,q}M$ by $\Omega^{p,q}(M)$. It is not difficult to see that a complex k-form belongs to $\Lambda^{p,q}M$, where $p + q = k$, if and only if it vanishes when applied to $p+1$ vectors in $T^{1,0}M$, or to $q+1$ vectors in $T^{0,1}M$. Having defined the above decomposition of forms, we can give yet another characterisation of an almost complex structure J being a complex structure.

Proposition 3.13. *Let (M, J) be an almost complex manifold of real dimension $2m$. Then J is a complex structure if and only if*

$$\mathrm{d}(\Omega^{1,0}(M)) \subset \Omega^{2,0}(M) \oplus \Omega^{1,1}(M), \tag{5}$$

or equivalently

$$\mathrm{d}(\Omega^{p,q}(M)) \subset \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M) \ \ \forall\, 0 \leqslant p, q \leqslant m. \tag{6}$$

Proof. The proof is a short calculation and can be found in [13]. □

3.4 Holomorphicity

Having introduced complex structures, we can now write down the natural generalisation of condition (1) to manifolds.

Definition 3.14. Let (M_1, J_1) and (M_2, J_2) be almost complex manifolds. A smooth map $f : (M_1, J_1) \to (M_2, J_2)$ is called *pseudo-holomorphic* if its differential commutes with the two complex structures at every point, $\mathrm{d}f \circ J_1 = J_2 \circ \mathrm{d}f$. If (M_1, J_1) and (M_2, J_2) are both complex manifolds, then $f : (M_1, J_1) \to (M_2, J_2)$ is called *holomorphic* if its differential commutes with the two complex structures at every point.

It should be noted that a single complex manifold may admit more than one complex structure; indeed, a Riemannian manifold (M, g) is called *hyperkähler* exactly when it admits three complex I, J, and K such that $IJ = K$, and the metric g is Kähler with respect to each of the three complex structures. We do not study hyperkähler manifolds in this essay. For more details the reader should consult §10 of Joyce [10], or Gross, Huybrechts and Joyce, [7].

Let us now consider a complex manifold (M, J) of complex dimension m. The following is a characterisation of holomorphic functions on M.

Proposition 3.15. *Let $f : M \to \mathbb{C}$ be a smooth function on M. The following are equivalent.*

1. *f is holomorphic,*
2. *$\partial_Z f = 0$ for all $Z \in T^{0,1}M$,*
3. *$\mathrm{d}f$ is a form of type $(1, 0)$.*

Proof. *2.* ⇔ *3.* The form $\mathrm{d}f$ is of type $(1, 0)$ if and only if $\mathrm{d}f(Z) = 0$ for all $Z \in T^{0,1}M$, which is by definition equivalent to $\partial_Z f = 0$ for all $Z \in T^{0,1}M$.

1. ⇔ *3.* The function f is holomorphic if and only if $f \circ \varphi_U^{-1}$ is holomorphic for every holomorphic chart (U, φ_U), which is equivalent to $f_* \circ (\varphi_U)_*^{-1} \circ j_m = i f_* \circ (\varphi_U)_*^{-1}$, or $f_* \circ J = i f_*$. Said differently, for every real vector X we have $\mathrm{d}f(JX) = i\mathrm{d}f(X)$, or $\mathrm{d}f(X + iJX) = 0$ for all $X \in TM$.

□

Definition 3.16. A complex vector field Z of type $(1, 0)$ is called *holomorphic* if $\partial_Z f$ is holomorphic for every locally defined holomorphic function f.

Definition 3.17. A real vector field X is called *real holomorphic* if its $(1, 0)$ component $X - iJX$ is a holomorphic vector field.

Now on any complex manifold (M, J) we know that, by condition (6), the exterior derivative d is the sum of two operators, $\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$ and $\bar\partial : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$, i.e. $\mathrm{d} = \partial + \bar\partial$. Since $0 = \mathrm{d}^2 = (\partial + \bar\partial)^2 = \partial^2 + \bar\partial^2 + (\partial\bar\partial + \bar\partial\partial)$, and the three operators in the last term take values in different subbundles of $\Lambda^2_{\mathbb{C}} M$, we have the following identities.

Proposition 3.18. *The operators ∂ and $\bar\partial$ satisfy*

(i) $\partial^2 = 0$,

(ii) $\bar\partial^2 = 0$, *and*

(iii) $\partial\bar\partial + \bar\partial\partial = 0$.

Definition 3.19. A p-form ω of type $(p, 0)$ is called *holomorphic* if $\bar\partial\omega = 0$.

We will see the importance of this final characterisation of holomorphic forms when we try to generalise the concept of holomorphic structures on vector bundles in the next section.

Some of the structure and naturality of the exterior derivative d carries over to the operators ∂ and $\bar\partial$. In particular, the following counterpart of the Poincaré Lemma, called the Dolbeault Lemma, holds.

Lemma 3.20 (Dolbeault Lemma)**.** *A $\bar\partial$-closed $(0, 1)$-form is locally $\bar\partial$-exact.*

Proposition 3.21 (The Local $\partial\bar\partial$-Lemma)**.** *Let $\varphi \in \Omega^{1,1}(M)$ be a real 2-form of type $(1, 1)$ on a complex manifold M. Then φ is closed if and only if every point $p \in M$ has an open neighbourhood U such that the restriction of φ to U equals $i\partial\bar\partial u$ for some smooth real function u on U.*

Proof. The proofs of this and Lemma 3.20 may be found in Griffiths and Harris [6], p. 25. $\square$

Stronger $\partial\bar\partial$ results will turn out to hold on Kähler manifolds, as we will see in Section 7.

4 Complex and Holomorphic Vector Bundles

Definition 4.1 (Complex Vector Bundle #1)**.** A smooth real vector bundle $E \to M$ over a smooth real manifold M is called a *complex vector bundle* over M if each fibre $E_p = \pi^{-1}(p)$ has a complex vector space structure.

Note that the definition of a complex vector bundle is the same as that of a real vector bundle (see Section 2.1), except here the fibres are required to be complex vector spaces. It is easy to see that equivalently, we may define a complex vector bundle as follows.

Definition 4.2 (Complex Vector Bundle #2)**.** A *complex vector bundle* is the pair (E, K), where $E \to M$ is a smooth real vector bundle and $K : E \to E$ is a linear bundle map satisfying $K^2 = \mathbb{1}_E$.

Remark 4.3. Recall that a smooth map $K : E \to F$ between fibre bundles $\pi_E : E \to M$ and $\pi_F : F \to M$ is called a *bundle map* if for every $p \in M$ it maps the fibre $E_p = \pi_E^{-1}(p)$ to the fibre $F_p = \pi_F^{-1}(p)$.

Definition 4.4 (Holomorphic Vector Bundle #1)**.** Let M be a complex manifold. A *holomorphic rank k vector bundle* E over M, written $E \to M$, is a complex manifold E equipped with a holomorphic projection $\pi : E \to M$ such that for each $p \in M$ the *fibre* $E_p := \pi^{-1}(p)$ has the structure of a vector space, there is an open neighbourhood $U(p)$ of p such that there exists a biholomorphism $\psi : \pi^{-1}(U(p)) \to U(p) \times \mathbb{C}^k$, the vector space structures on $E_q = \pi^{-1}(q)$ and $\{q\} \times \mathbb{C}^k$ agree for all $q \in U(p)$, and the following diagram commutes,

$$\begin{tikzcd} \pi^{-1}(U(p)) \arrow[rr, "\psi", "\cong"'] \arrow[dr, "\pi"'] & & U(p) \times \mathbb{C}^k \arrow[dl, "\mathrm{pr}_{U(p)}"] \\ & U(p) & \end{tikzcd}$$

We will see shortly a different characterisation of holomorphic vector bundles, one which will allow us to write down a second definition describing them as real bundles with extra structure.

Proposition 4.5. *The tangent bundle TM of a complex manifold M has the structure of a holomorphic vector bundle.*

Proof. Holomorphicity of the projection is obvious. Take a holomorphic atlas (U, φ_U), $U \in \mathcal{U}$ on M and define $\psi_U : TM|_U \to U \times \mathbb{C}^m$ by $\psi_U(X_p) = (p, \mathrm{d}\varphi_U(X))$. The transition functions are then $g_{UV} = \mathrm{d}\varphi_U \circ (\mathrm{d}\varphi_V)^{-1}$, which are clearly holomorphic. □

Similar arguments show that the cotangent bundle $T^*M \simeq \Lambda^{1,0}M$ and the bundles $\Lambda^{p,0}M$ are holomorphic. The complex bundles $\Lambda^{p,q}M$, however, are not holomorphic for $q \neq 0$.

Definition 4.6. Let $E \to M$ be a holomorphic vector bundle over M. The bundle $\Lambda^{p,q}E$ of E-valued (p,q)-forms is defined as the tensor product of complex vector bundles $\Lambda^{p,q}M \otimes E$. The space $\Gamma(\Lambda^{p,q}M \otimes E)$ of smooth sections of $\Lambda^{p,q}(E)$ is denoted by $\Omega^{p,q}(E)$.

As before, the bundles $\Lambda^{p,0}E$ are holomorphic, but $\Lambda^{p,q}E$ are not for $q \neq 0$.

In the previous section we defined the ∂ and $\bar{\partial}$ operators acting on $\Omega^{p,q}(M)$, which make up the p-part and the q-part of the exterior derivative d respectively. It can be easily checked that in any local coordinate system $\{z_\alpha\}$, ∂ and $\bar{\partial}$ are given by

$$\partial f = \sum_{\alpha=1}^{m} \frac{\partial f}{\partial z_\alpha} \mathrm{d}z_\alpha \quad \text{and} \quad \bar{\partial} f = \sum_{\alpha=1}^{m} \frac{\partial f}{\partial \bar{z}_\alpha} \mathrm{d}\bar{z}_\alpha \tag{7}$$

when acting on functions $f : M \to \mathbb{C}$. In other words, as the notation suggests, $\bar{\partial}$ should be thought of as the complex conjugate of the operator ∂.

Recall that a function $f : \mathbb{C} \to \mathbb{C}$ is called holomorphic if $\bar{\partial} f = 0$ (see section 2.3). Let $E \to M$ be a holomorphic vector bundle. We extend the definition of $\bar{\partial}$ to act on $\Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ as follows. Suppose a section σ is given by $(\sigma_1, \ldots, \sigma_k)$, where σ_i are local (p,q) forms, in some local *holomorphic* trivialisation. We then simply define $\bar{\partial}\sigma = (\bar{\partial}\sigma_1, \ldots, \bar{\partial}\sigma_k)$. If in a different trivialisation we have $\sigma = (\tau_1, \ldots, \tau_k)$, then $\tau_j = g_{jl}\sigma_l$ for some transition functions g_{jl}. These are holomorphic as E is a holomorphic vector bundle, so $\bar\partial\tau_j = g_{jl}\bar\partial\sigma_l$, showing that $\bar\partial\sigma$ does not depend on the choice of trivialisation. Of course, $\bar\partial^2 = 0$ as before, and moreover, $\bar\partial$ satisfies the Leibniz rule:

$$\bar\partial(\omega \wedge \sigma) = (\bar\partial\omega) \wedge \sigma + (-1)^{p+q}\omega \wedge (\bar\partial\sigma) \tag{8}$$

for all $\omega \in \Omega^{p,q}(M)$, $\sigma \in \Omega^{r,s}(E)$. Notice that this construction required the vector bundle E to be holomorphic.

4.1 Holomorphic Structures

Definition 4.7. A *pseudo-holomorphic structure* on a complex vector bundle E is a collection of operators $\bar\partial : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ for all (p, q), all denoted by $\bar\partial$, satisfying the Leibniz rule (8). If, moreover, $\bar\partial^2 = 0$, then $\bar\partial$ is called a *holomorphic structure*.

Theorem 4.8. *A complex vector bundle E is holomorphic if and only if it has a holomorphic structure $\bar\partial$.*

Proof. We do not go through the proof here, but it can be found in Kobayashi [11], and is repeated in §9 of Moroianu, [13]. □

We can thus write down the following definition.

Definition 4.9 (Holomorphic Vector Bundles #2)**.** A *holomorphic vector bundles* is a triplet $(E, K, \bar\partial)$, where $E \to M$ is a real vector bundle, $K : E \to E$ is a linear bundle map satisfying $K^2 = -\mathbb{1}_E$, and $\bar\partial : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ is an operator satisfying the Leibniz rule (8), and $\bar\partial^2 = 0$.

5 Hermitian and Kähler Manifolds

5.1 Hermitian structures and the Chern connection

Let $E \to M$ be a complex rank-k vector bundle over a smooth manifold M.

Definition 5.1. A *Hermitian structure* H on E is a smooth field of Hermitian products on the fibres of E. More precisely, we say E possesses a Hermitian structure if there exists a family of maps $H : E_p \times E_p \to \mathbb{C}$ for all $p \in M$ satisfying the following.

1. $H(u, v)$ is $\mathbb{C}$-linear in u for every $v \in E_p$,
2. $H(u, v) = \overline{H(v, u)}$ for all $u, v \in E_p$,
3. $H(u, u) > 0$ for all $u \neq 0$,
4. $H(u, v)$ is a smooth function on M for every pair of smooth sections u and v of E.

A complex vector bundle endowed with a Hermitian structure, (E, H), is called a *Hermitian vector bundle*.

By definition, H is non-degenerate and $\mathbb{C}$-anti-linear in the second variable, so we may view H as a $\mathbb{C}$-anti-linear isomorphism $H : E \to E^*$, where E^* is the dual bundle of E.

Example 5.2. The standard Hermitian structure on $\mathbb{C}^k$ is of course given by $H(z,w) = w^\dagger z$. More generally, any Hermitian matrix $\mathcal{H}$ induces a Hermitian structure via $H(z,w) = w^\dagger \mathcal{H} z$. Indeed, any rank-$k$ complex vector bundle E admits multiple Hermitian structures by pullback of a Hermitian matrix $\mathcal{H}$. More precisely, take a trivialisation (U_i, ψ_i) of E and a partition of unity (f_i) subordinate to the open cover $\{U_i\}$. For each $p \in U_i$, let $(H_i)_p = \psi_i^* \circ \mathcal{H}$ be the pullback of a Hermitian matrix $\mathcal{H}$ by the $\mathbb{C}$-linear map $\psi_i|_{E_p}$. Then $H := \sum f_i H_i$ is a well-defined Hermitian structure on E.

Definition 5.3. Suppose $E \to M$ is a complex vector bundle over a complex manifold M. Suppose E admits a connection ∇, and consider the projections $\pi^{1,0} : \Lambda^1(E) \to \Lambda^{1,0}(E)$ and $\pi^{0,1} : \Lambda^1(E) \to \Lambda^{0,1}(E)$. We define the $(1,0)$ and $(0,1)$ components of ∇ to be

$$\nabla^{1,0} := \pi^{1,0} \circ \nabla \quad \text{and} \quad \nabla^{0,1} := \pi^{0,1} \circ \nabla.$$

We have seen that these operators extend to $\nabla^{1,0} : \Omega^{p,q}(E) \to \Omega^{p+1,q}(E)$ and $\nabla^{0,1} : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$, and satisfy the Leibniz rule with respect to their corresponding components of the exterior derivative,

$$\nabla^{1,0}(\omega \otimes \sigma) = \partial\omega \otimes \sigma + (-1)^{p+q} \omega \wedge \nabla^{1,0}\sigma,$$

and

$$\nabla^{0,1}(\omega \otimes \sigma) = \bar{\partial}\omega \otimes \sigma + (-1)^{p+q} \omega \wedge \nabla^{0,1}\sigma$$

for all $\omega \in \Omega^{p,q}(M)$, $\sigma \in \Gamma(E)$. Of course, $\nabla^{0,1}$ is a pseudo-holomorphic structure on E for every connection ∇. The curvature operator of ∇ can be written as

$$R^\nabla = \nabla^2 = (\nabla^{1,0} + \nabla^{0,1})^2 = (\nabla^{1,0})^2 + (\nabla^{0,1})^2 + (\nabla^{1,0}\nabla^{0,1} + \nabla^{0,1}\nabla^{1,0}),$$

so the $(0,2)$ component of the curvature is given by

$$(R^\nabla)^{0,2} = (\nabla^{0,1})^2.$$

Since $\nabla^{0,1}$ is a pseudo-holomorphic structure, if $(R^\nabla)^{0,2}$ vanishes for some connection ∇, then E is a holomorphic bundle with holomorphic structure $\bar{\partial} := \nabla^{0,1}$, by Theorem 4.8. The converse is also true and is provided by the following theorem.

Theorem 5.4. *Let E be a holomorphic vector bundle with holomorphic structure $\bar{\partial}$. For every Hermitian structure H on E there exists a unique connection $\nabla = \nabla^{Ch}$, called the* Chern connection*, such that*

(i) $\nabla H = 0$, *and*

(ii) $\nabla^{0,1} = \bar{\partial}$. *This clearly means that the $(0,2)$ component of the curvature of the Chern connection vanishes,*

$$(R^\nabla)^{0,2} = (\nabla^{0,1})^2 = \bar{\partial}^2 = 0.$$

Proof. Firstly, notice that the dual vector bundle E^* of a holomorphic vector bundle E is holomorphic too, with its naturally inherited holomorphic structure still denoted by $\bar{\partial}$. Moreover, any connection ∇ on E canonically induces a connection on E^*, still denoted by ∇, by the formula

$$(\nabla_X \sigma^*)\sigma := \partial_X(\sigma^*(\sigma)) - \sigma^*(\nabla_X \sigma) \tag{9}$$

for all vector fields X, $\sigma \in \Gamma(E)$, $\sigma^* \in \Gamma(E^*)$. Note that the condition $\nabla^{0,1} = \bar{\partial}$ on E just means that $\nabla\sigma \in \Omega^{1,0}(E)$ for every holomorphic section σ of E. By (9), the same characterisation of this condition holds on E^* as well.

To prove the theorem, suppose that ∇ is a connection satisfying $\nabla H = 0$ and $\nabla^{0,1} = \bar{\partial}$, where H is viewed as a $\mathbb{C}$-antilinear isomorphism $E \to E^*$. For every section $\sigma \in \Gamma(E)$ and every real vector X on M we get

$$\nabla_X(H(\sigma)) = \nabla_X(H)(\sigma) + H(\nabla_X\sigma) = H(\nabla_X\sigma).$$

Now by the $\mathbb{C}$-antilinearity of H, for every complex vector $Z \in TM^{\mathbb{C}}$ we then have $\nabla_Z(H(\sigma)) = H(\nabla_{\bar{Z}}\sigma)$. For $Z \in T^{1,0}M$ this then shows that

$$\nabla^{1,0}\sigma = H^{-1} \circ \nabla(H(\sigma)) = H^{-1}(\bar{\partial}(H(\sigma))),$$

whence $\nabla = \bar{\partial} + H^{-1} \circ \bar{\partial} \circ H$. We thus have found a formula for ∇, in particular proving existence and uniqueness. $\square$

In particular, the above formula for the Chern connection also shows that the $(2,0)$-component of the curvature also vanishes, since

$$(R^{\nabla})^{2,0} = (\nabla^{1,0})^2 = (H^{-1} \circ \bar{\partial} \circ H)^2 = H^{-1} \circ \bar{\partial}^2 \circ H = 0.$$

Thus the curvature of the Chern connection is in fact a $(1,1)$-form.

It turns out that the Chern connection is natural with respect to direct sums and tensor products of vector bundles. The precise meaning of this statement is the following. Suppose $E, F \to M$ are holomorphic vector bundles with corresponding Hermitian structures H_E and H_F. Let ∇_E and ∇_F be the Chern connections on E and F respectively. There exist natural definitions of two new connections, $\nabla_{E\oplus F}$ and $\nabla_{E\otimes F}$, from ∇_E and ∇_F, which are characterised by

$$\nabla_{E\oplus F}(e, f) = (\nabla_E e, \nabla_F f), \quad \text{and}$$
$$\nabla_{E\otimes F}(e \otimes f) = (\nabla_E e) \otimes f + e \otimes (\nabla_F f)$$

for all $e \in \Gamma(E)$, $f \in \Gamma(F)$. Moreover, there exist natural definitions of two new Hermitian structures, $H_E \oplus H_F$ on $E \oplus F$ and $H_E \otimes H_F$ on $E \otimes F$. Theorem 5.4 then gives us Chern connections $\nabla'_{E\oplus F}$ and $\nabla'_{E\otimes F}$ on the holomorphic Hermitian bundles $(E \oplus F, H_E \oplus H_F, \bar{\partial})$ and $(E \oplus F, H_E \otimes H_F, \bar{\partial})$ respectively (direct sums and tensor products certainly preserve holomorphicity of vector bundles; this is obvious from our Definition #1 of holomorphic vector bundles). Naturality of the Chern connection is then the statement that

$$\nabla_{E\oplus F} = \nabla'_{E\oplus F} \quad \text{and} \quad \nabla_{E\otimes F} = \nabla'_{E\otimes F}.$$

5.2 Hermitian metrics

Definition 5.5. A *Hermitian metric* on an almost complex manifold (M, J) is a Riemannian metric h such that $h(X, Y) = h(JX, JY)$ for all $X, Y \in TM$.

We also denote by h the extension of h to $TM^{\mathbb{C}}$ by $\mathbb{C}$-linearity. It is easy to check that on $TM^{\mathbb{C}}$ this extension satisfies

1. $h(\overline{Z}, \overline{W}) = \overline{h(Z, W)}$ for all $Z, W \in TM^{\mathbb{C}}$,

2. $h(Z, \overline{Z}) > 0$ for all $Z \in TM^{\mathbb{C}} \setminus \{0\}$, and
3. $h(Z, W) = 0$ whenever both Z and W belong to either $T^{1,0}M$ or $T^{0,1}M$.

Conversely, every symmetric tensor on $TM^{\mathbb{C}}$ satisfying these properties defines a Hermitian metric by restriction to TM.

Definition 5.6. Every Hermitian metric h on an almost complex manifold (M, J) has an associated 2-form, called the *fundamental form* (sometimes *Kähler form*), defined by

$$\omega(X, Y) := h(JX, Y)$$

for all vector fields $X, Y \in TM$.

5.3 Kähler Metrics

Definition 5.7. A Hermitian metric h on an almost complex manifold (M, J) is called a *Kähler metric* if J is a complex structure and the fundamental form ω is closed,

$$h \text{ is Kähler} \iff N^J = 0 \text{ and } \mathrm{d}\omega = 0.$$

A local *real* function u satisfying $\omega = i\partial\bar{\partial}u$ is called a *local Kähler potential* of the metric h.

It turns out that Kähler metrics may be characterised in terms of the Levi–Civita connection ∇.

Theorem 5.8. *A Hermitian metric h on an almost complex manifold (M, J) is Kähler if and only if J is parallel with respect to the Levi–Civita connection of h, $\nabla J = 0$.*

Sketch proof. Heuristically speaking, it turns out that

$$\nabla J = N^J \oplus h^{-1} \cdot \mathrm{d}\omega,$$

so that J being parallel corresponds precisely to the condition that h is Kähler, $N^J = 0$ and $\mathrm{d}\omega = 0$. □

Furthermore, as mentioned in the introduction, we may characterise Kähler metrics in a manner similar to the existence of normal coordinates for Riemannian metrics.

Theorem 5.9. *A Hermitian metric h on a complex manifold (M, J) is Kähler if and only if around each point in M there exist* holomorphic *coordinates in which h osculates to the standard Hermitian metric to second order.*

Proof. See pp. 83 – 84 of Moroianu, [13]. □

The tangent bundle of a Hermitian manifold (M, J, h) has two, generally distinct, natural linear connections: the Levi–Civita connection ∇^{LC} and the Chern connection ∇^{Ch}. Another special feature of Kähler manifolds is the following.

Proposition 5.10. *Let (M, J, h) be a Hermitian manifold and let ∇^{LC}, ∇^{Ch} be the Levi–Civita and the Chern connections on TM respectively. The manifold (M, J, h) is Kähler if and only if $\nabla^{LC} = \nabla^{Ch}$.*

Proof. This is a routine calculation using Theorem 5.8 and can be found on pp. 85 – 86 of Moroianu, [13]. □

5.4 The Ricci Form

Let (M^{2m}, J, h) be a Kähler manifold and ∇ the Levi–Civita connection of h. Since J is parallel with respect to ∇, $\nabla J = 0$, the Riemann curvature tensor satisfies

$$R(X,Y)JZ = JR(X,Y)Z.$$

Symmetries of the fully covariant Riemann tensor then imply that

$$R(X,Y,JZ,JT) = R(JX,JY,Z,T) = R(X,Y,Z,T),$$

so after taking the trace we see that

$$\mathrm{Ric}(JX,JY) = \mathrm{Ric}(X,Y).$$

Thus $\mathrm{Ric}(J\cdot,\cdot)$ is skew-symmetric in its arguments, i.e. is a 2-form.

Definition 5.11. The *Ricci form* ρ of a Kähler manifold is defined by

$$\rho(X,Y) := \mathrm{Ric}(JX,Y)$$

for all $X, Y \in TM$.

Let (z_α) be a system of local holomorphic coordinates around some point $p \in M$. We write

$$Z_\alpha := \frac{\partial}{\partial z_\alpha} \quad \text{and} \quad Z_{\bar\alpha} := \frac{\partial}{\partial \bar z_\alpha}, \qquad 1 \leqslant \alpha \leqslant m.$$

Let the Roman indices $a, b, c, \ldots$ run over the set $\{1, \ldots, m, \bar 1, \ldots, \overline{m}\}$ and the Greek indices $\alpha, \beta, \gamma, \ldots$ run over the set $\{1, \ldots, m\}$ (so that $\bar\alpha, \bar\beta, \bar\gamma, \ldots$ run over $\{\bar 1, \ldots, \overline{m}\}$). The components of the Kähler metric in the above local coordinates are given by

$$h_{ab} := h(Z_a, Z_b).$$

By Hermiticity we immediately get

$$h_{\alpha\beta} = 0 = h_{\bar\alpha\bar\beta}, \qquad h_{\alpha\bar\beta} = \overline{h_{\beta\bar\alpha}} = h_{\bar\beta\alpha}.$$

Let us denote by

$$d = \det(h_{\alpha\bar\beta}).$$

Proposition 5.12. *The Ricci tensor of the Levi–Civita connection on a Kähler manifold is locally given by*

$$\mathrm{Ric}_{\alpha\bar\beta} = -\frac{\partial^2 \log d}{\partial z_\alpha \partial \bar z_\beta}.$$

Proof. See §12.2 of Moroianu, [13], for a proof. □

By definition, we then have the expression for the Ricci form

$$\rho = -i\partial\bar\partial \log d. \tag{10}$$

Proposition 5.13. *The Ricci form is closed,* $\mathrm{d}\rho = 0$.

Proof. Using (10) and Proposition 3.18, we immediately have

$$\mathrm{d}\rho = -i(\partial + \bar{\partial})\partial\bar{\partial}\log d = 0.$$

□

The Ricci form turns out to be one of the most important objects on a Kähler manifold; its cohomology class is essentially equal to the Chern class of the canonical bundle of M, as we will explain in Section 9.

6 Natural Operators

Let (M^n, g) be an oriented Riemannian manifold with volume form $\mathrm{d}v$, and let E and F be Hermitian vector bundles over M with Hermitian structures $\langle \cdot, \cdot \rangle_E$ and $\langle \cdot, \cdot \rangle_F$ respectively.

Definition 6.1. Let $P : \Gamma(E) \to \Gamma(F)$ and $Q : \Gamma(F) \to \Gamma(E)$ be smooth linear differential operators. We say Q is a *formal adjoint* of P if

$$\int_M \langle P\alpha, \beta \rangle_F \, \mathrm{d}v = \int_M \langle \alpha, Q\beta \rangle_E \, \mathrm{d}v$$

for all compactly supported smooth sections $\alpha \in C_c^\infty(E)$, $\beta \in C_c^\infty(F)$.

A formal adjoint of a smooth linear differential operator P always exists, is unique, and is denoted by P^*. Standard arguments showing existence and uniqueness can be found in §14 of Moroianu, [13]. Clearly $(P^*)^* = P$, and $(P \circ Q)^* = Q^* \circ P^*$.

In the particular case of the exterior bundle, the metric g induces a natural scalar product on $\Lambda^k M$, which we denote by $\langle \cdot, \cdot \rangle$ (see Appendix B). With respect to this scalar product the interior product $\lrcorner$ and the exterior product $\wedge$ are adjoint operators in the following sense. Let X be a vector field, $\omega \in \Lambda^k M$, and $\tau \in \Lambda^{k-1} M$. In index notation,

$$\begin{aligned}
X^a \omega_{ab\ldots c} \tau^{b\ldots c} &= \omega_{ab\ldots c} X^a \tau^{b\ldots c} \\
&= \frac{1}{2}\left(\omega_{ab\ldots c} X^a \tau^{b\ldots c} - \omega_{ba\ldots c} X^a \tau^{b\ldots c}\right) \\
&= \frac{1}{2}\left(\omega_{ab\ldots c} X^a \tau^{b\ldots c} - \omega_{ab\ldots c} X^b \tau^{a\ldots c}\right) \\
&= \cdots = \omega_{ab\ldots c} X^{[a} \tau^{b\ldots c]},
\end{aligned}$$

which in coordinate-free notation reads

$$\langle X \lrcorner \omega, \tau \rangle = \langle \omega, X \wedge \tau \rangle.$$

So $X\wedge$ is the adjoint of $X\lrcorner$.

6.1 The Codifferential and the Laplacian

Let $\{e_1, \ldots, e_n\}$ be a local orthonormal frame on M which is parallel at some point $p \in M$. Recall (Proposition A.5) that the exterior derivative $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ is given by

$$\mathrm{d} = e_i \wedge \nabla_{e_i}.$$

The formal adjoint of d is given by $\delta : \Omega^{k+1}(M) \to \Omega^k(M)$,

$$\delta = -(-1)^{nk} * \mathrm{d} *,$$

where $*$ denotes the Hodge star operator (see Appendix B). Indeed, suppose $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{k+1}(M)$ are smooth compactly supported forms. Then

$$\begin{aligned}
-(-1)^{nk} \int_M \langle \alpha, *\mathrm{d} * \beta \rangle \, \mathrm{d}v &= -(-1)^{nk} \int_M \alpha \wedge *^2 \mathrm{d} * \beta \\
&= -(-1)^{nk} (-1)^{(n-k)k} \int_M \alpha \wedge \mathrm{d} * \beta \\
&= -(-1)^{k^2} \left((-1)^k \int_M \mathrm{d}(\alpha \wedge *\beta) - (-1)^k \int_M \mathrm{d}\alpha \wedge *\beta \right) \\
&= \int_M \mathrm{d}\alpha \wedge *\beta = \int_M \langle \mathrm{d}\alpha, \beta \rangle \, \mathrm{d}v,
\end{aligned}$$

where we used Stokes' Theorem in the penultimate line.

Definition 6.2. The formal adjoint of the exterior derivative d is called the *codifferential* and is denoted by $\delta := \mathrm{d}^*$. We say a form α is *coclosed* if $\delta\alpha = 0$.

In local coordinates

$$\begin{aligned}
\langle \alpha, \delta\beta \rangle &= \langle \mathrm{d}\alpha, \beta \rangle \\
&= \langle e_i \wedge \nabla_{e_i} \alpha, \beta \rangle \\
&= \langle \nabla_{e_i} \alpha, e_i \lrcorner \beta \rangle \\
&= \langle \alpha, -\nabla_{e_i} (e_i \lrcorner \beta) \rangle + \mathrm{d}(\dots) \\
&= \langle \alpha, -e_i \lrcorner \nabla_{e_i} \beta \rangle + \mathrm{d}(\dots),
\end{aligned}$$

so in local coordinates the codifferential is given by

$$\delta = -e_i \lrcorner \nabla_{e_i}.$$

Definition 6.3. The *Laplacian* is the operator $\Delta : \Omega^k(M) \to \Omega^k(M)$ defined by

$$\Delta := \mathrm{d}\delta + \delta\mathrm{d}.$$

A form $\omega \in \Omega^k(M)$ satisfying $\Delta\omega = 0$ is called *harmonic*. Note that the Laplacian is clearly self-adjoint, $\langle \Delta\alpha, \beta \rangle = \langle \alpha, \Delta\beta \rangle$.

Let us now consider a Hermitian manifold (M^{2m}, J, h). We can consider the Hodge star as acting on complex-valued forms $\Omega^{p,q}(M)$ by extending it to be $\mathbb{C}$-linear. The Hodge star operator then maps (p, q) forms to $(m - q, m - p)$ forms[1]. This can be seen by assuming the contrary, wedging the Hodge star of a (p, q) form with test forms of carefully chosen type, and using the orthonormality of local normal frames.

Recall that on complex manifolds we have the decomposition

$$\mathrm{d} = \partial + \bar{\partial}.$$

[1]Many authors define the extension of the Hodge star to complex forms to be $\mathbb{C}$-antilinear. In that case it maps (p, q) forms to $(m - p, m - q)$ forms.

Corresponding to this decomposition we have the Dolbeault splitting of the codifferential,

$$\delta = \partial^* + \bar{\partial}^*,$$

where

$$\partial^* : \Omega^{p,q}(M) \to \Omega^{p-1,q}(M), \quad \partial^* := - * \bar{\partial} *,$$

and

$$\bar{\partial}^* : \Omega^{p,q}(M) \to \Omega^{p,q-1}(M), \quad \bar{\partial}^* := - * \partial * .$$

Note that ∂^* and $\bar{\partial}^*$ are formal adjoints of ∂ and $\bar{\partial}$ with respect to the Hermitian product H on complex forms given by

$$H(\omega, \tau) = \langle \omega, \overline{\tau} \rangle.$$

Definition 6.4. The ∂-Laplace and the $\bar{\partial}$-Laplace operators are defined to be

$$\Delta^{\partial} := \partial\partial^* + \partial^*\partial \quad \text{and} \quad \Delta^{\bar{\partial}} := \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}.$$

While on general Hermitian manifolds these operators are genuinely different, another remarkable feature of Kähler manifolds is that on them all three Laplacians are essentially the same.

Theorem 6.5. *On any Kähler manifold one has*

$$\Delta^{\partial} = \frac{1}{2}\Delta = \Delta^{\bar{\partial}}.$$

Proof. The proof is an exercise in computing commutation relations between the operators ∂, $\bar{\partial}$, ∂^*, and $\bar{\partial}^*$ and can be found in §14 of Moroianu, [13], or in §5 of Ballmann, [2]. In particular, analogously to Proposition 3.18, one finds

$$\begin{aligned} \partial\bar{\partial}^* + \bar{\partial}^*\partial = 0, \\ \bar{\partial}\partial^* + \partial^*\bar{\partial} = 0. \end{aligned} \tag{11}$$

□

6.2 Twisted Differentials

Definition 6.6. The *twisted differential* on M is the operator $\mathrm{d}^c : \Omega^k(M) \to \Omega^{k+1}(M)$ defined by

$$\mathrm{d}^c = i(\bar{\partial} - \partial).$$

Definition 6.7. The formal adjoint of the twisted differential is given by $\delta^c : \Omega^{k+1}(M) \to \Omega^k(M)$,

$$\delta^c := - * \mathrm{d}^c * = i(\partial^* - \bar{\partial}^*).$$

Now it is easy to check that the identification of TM with T^*M, extended by $\mathbb{C}$-linearity, sends vectors of type $(1,0)$ to forms of type $(0,1)$, and vice versa, so in a local orthonormal frame parallel at a point we immediately get the local expressions

$$\partial = \frac{1}{2}(e_j + iJe_j) \wedge \nabla_{e_j} \quad \text{and} \quad \bar{\partial} = \frac{1}{2}(e_j - iJe_j) \wedge \nabla_{e_j}.$$

Subtracting these we get an expression for the twisted differential,

$$\mathrm{d}^c := Je_i \wedge \nabla_{e_i},$$

and calculations similar to the ones we did above for d and δ show

$$\delta^c = -Je_i \lrcorner \nabla_{e_i}.$$

Proposition 6.8. *On every Kähler manifold there hold the commutation relations*

$$0 = \mathrm{dd}^c + \mathrm{d}^c\mathrm{d} = \mathrm{d}\delta^c + \delta^c\mathrm{d} = \delta\delta^c + \delta^c\delta = \delta\mathrm{d}^c + \mathrm{d}^c\delta.$$

Proof. The identities $\mathrm{dd}^c + \mathrm{d}^c\mathrm{d} = 0$ and $\delta\delta^c + \delta^c\delta = 0$ are a consequence of Proposition 3.18, while the other two are a consequence of (11). $\square$

7 Cohomology

7.1 Hodge Theory

There are several different ways to define the *cohomology* of topological spaces, singular, Alexander–Spanier and Čech being a few. If the topological space is sufficiently nice, e.g. a smooth manifold, all the corresponding cohomology groups are isomorphic; see Bott and Tu [3], or Warner [14].

Let (M^n, g) be a smooth oriented compact Riemannian manifold. Denote by $\Omega^k_{\mathbb{C}}(M) := \Gamma(\Lambda^k M \otimes \mathbb{C})$ the space of smooth $\mathbb{C}$-valued k-forms. As the exterior derivative satisfies $\mathrm{d}^2 = 0$, it makes sense to define the cohomology groups of the complex

$$0 \longrightarrow \Omega^0_{\mathbb{C}}(M) \xrightarrow{\ \mathrm{d}\ } \Omega^1_{\mathbb{C}}(M) \xrightarrow{\ \mathrm{d}\ } \cdots \xrightarrow{\ \mathrm{d}\ } \Omega^n_{\mathbb{C}}(M) \longrightarrow 0$$

as follows.

Definition 7.1. The *k-th de Rham cohomology group* is the quotient of the vector space of closed k-forms on M by the vector space of exact k-forms on M,

$$H^k_{dR}(M,\mathbb{C}) := \frac{\ker\left(\mathrm{d} : \Omega^k_{\mathbb{C}}(M) \to \Omega^{k+1}_{\mathbb{C}}(M)\right)}{\operatorname{im}\left(\mathrm{d} : \Omega^{k-1}_{\mathbb{C}}(M) \to \Omega^k_{\mathbb{C}}(M)\right)}.$$

Definition 7.2. We denote by $\mathcal{H}^k(M,\mathbb{C})$ the space of *complex harmonic k-forms* on M,

$$\mathcal{H}^k(M,\mathbb{C}) := \{\alpha \in \Omega^k_{\mathbb{C}}(M) \,:\, \Delta\alpha = 0\}.$$

Lemma 7.3. *A form is harmonic if and only if it is closed and coclosed.*

Proof. One direction is clear. Conversely, suppose that α is harmonic. Since M is compact, the operators d and δ are formally adjoint with respect to the Hermitian structure $H(\alpha,\beta) := \langle\alpha,\bar{\beta}\rangle$. Thus

$$0 = \int_M H(\Delta\alpha,\alpha)\,\mathrm{d}v = \int_M H(\mathrm{d}\delta\alpha + \delta\mathrm{d}\alpha,\alpha)\,\mathrm{d}v = \int_M |\delta\alpha|^2 + |\mathrm{d}\alpha|^2\,\mathrm{d}v,$$

showing that $\mathrm{d}\alpha = 0 = \delta\alpha$. $\square$

Theorem 7.4 (Hodge Decomposition Theorem)**.** *The space of k-forms $\Omega^k_{\mathbb{C}}(M)$ decomposes as*

$$\Omega^k_{\mathbb{C}}(M) = \mathcal{H}^k(M,\mathbb{C}) \oplus \delta\Omega^{k+1}_{\mathbb{C}}(M) \oplus \mathrm{d}\Omega^{k-1}_{\mathbb{C}}(M).$$

Proof. The orthogonality with respect to H of the three summands may be quickly checked using Lemma 7.3, but the difficulty lies in proving that these three summands span the whole space. The full proof may be found in Griffiths and Harris [6], pp. 84–100. $\square$

Theorem 7.5 (Hodge's Theorem)**.** *Every de Rham cohomology class on M contains a unique harmonic representative, and $\mathcal{H}^k(M,\mathbb{C}) \cong H^k_{dR}(M,\mathbb{C})$.*

Proof. This is a routine application of Theorem 7.4. □

Definition 7.6. The complex dimension of $H^k_{dR}(M,\mathbb{C})$,

$$b_k(M) := \dim_{\mathbb{C}}(H^k_{dR}(M,\mathbb{C})),$$

is called the *k-th Betti number of M.* The Betti numbers are topological invariants.

Theorem 7.7 (Poincaré Duality)**.** *The vector spaces $\mathcal{H}^k(M,\mathbb{C})$ and $\mathcal{H}^{n-k}(M,\mathbb{C})$ are isomorphic. In particular, $b_k(M) = b_{n-k}(M)$ for every compact n-dimensional manifold M.*

Proof. The isomorphism is simply given by the Hodge star operator, which maps harmonic k-forms to harmonic $(n-k)$-forms. Indeed, if $\eta \in \mathcal{H}^k(M,\mathbb{C})$, then $\mathrm{d}\eta = 0 = \delta\eta$ by Lemma 7.3, and so

$$\mathrm{d} * \eta = \pm * * \mathrm{d} * \eta = \mp * \delta\eta = 0,$$

and

$$\delta * \eta = \pm * * \delta * \eta = \mp * \mathrm{d}\eta = 0.$$

□

There is an interesting application of the Hodge Decomposition Theorem to Kähler manifolds which is of particular importance to the study of symmetries on Kähler manifolds. Let (M, J, h) be a Kähler manifold.

Lemma 7.8. *A real vector field X on a complex manifold (M, J) is real holomorphic if and only if $\mathcal{L}_X J = 0$.*

Proof. First notice that a complex vector field Z is of type $(0,1)$ if and only if $\partial_Z f = 0$ for every locally defined holomorphic function f. Suppose that X is real holomorphic, and let Y be an arbitrary vector field and f a local holomorphic function. As $\partial_{X+iJX} f = 0$, we have $\partial_{X-iJX} f = 2\partial_X f$, so that $\partial_X f$ is holomorphic. Thus $\partial_{Y+iJY}\partial_X f = 0$, and $\partial_{Y+iJY} f = 0$, which in particular implies that $\partial_{[Y+iJY,X]} f = 0$. As f was an arbitrary holomorphic function, this shows that $[Y + iJY, X]$ is of type $(0,1)$, that is $[JY, X] = J[Y, X]$. Hence

$$(\mathcal{L}_X J)Y = \mathcal{L}_X(JY) - J(\mathcal{L}_X Y) = [X, JY] - J[X, Y] = 0$$

for all vector fields Y, implying that $\mathcal{L}_X J = 0$.

Conversely, suppose $\mathcal{L}_X J = 0$. Now we want to show that $\partial_{X-iJX} f = 2\partial_X f$ is holomorphic for every locally defined holomorphic function f. As before, $\mathcal{L}_X J = 0$ is equivalent to $J[Y, X] = [JY, X]$ for every vector field Y. By Proposition 3.15, it is enough to show that $\partial_Z \partial_X f = 0$ for every vector field Z of type $(0,1)$ and every locally defined holomorphic function f. Let $Z = W + iJW$ for some real vector field W. Then

$$\begin{aligned}
\partial_Z \partial_X f = \partial_{W+iJW}\partial_X f &= \mathcal{L}_W \mathcal{L}_X f + i\mathcal{L}_{JW}\mathcal{L}_X f \\
&= \mathcal{L}_W \mathcal{L}_X + i\left(\mathcal{L}_X \mathcal{L}_{JW} + \mathcal{L}_{[JW,X]} f\right) \\
&= \mathcal{L}_X \mathcal{L}_W f + \mathcal{L}_{[W,X]} f + \mathcal{L}_X \mathcal{L}_{iJW} f + \mathcal{L}_{iJ[W,X]} f \\
&= \mathcal{L}_X \mathcal{L}_{W+iJW} f + \mathcal{L}_{[W,X]+iJ[W,X]} f = 0,
\end{aligned}$$

as both $W + iJW$ and $[W, X] + iJ[W, X]$ are of type $(0,1)$. □

Lemma 7.9. *The Kähler form ω of M is harmonic.*

Proof. The closedness of ω is immediate from M being Kähler, so the statement amounts to showing that $\delta\omega = 0$. By Theorem 5.8, J is parallel with respect to the Levi–Civita (equivalently, Chern) connection, $\nabla J = 0$, and since ∇ preserves the metric h, it follows that ω is parallel with respect to ∇. Now since the Hodge star is defined in terms of the metric, $\nabla\omega = 0$ implies that $\nabla * \omega = 0$. This, of course, means that $\mathrm{d} * \omega = 0$, since $\mathrm{d}\alpha$ is the antisymmetrization of $\nabla\alpha$. Thus $\delta\omega = 0$, after applying the Hodge star. ☐

Proposition 7.10. *Every Killing vector field on a* compact *Kähler manifold (M, J, h) is real holomorphic.*

Proof. Let K be a Killing vector field, $\mathcal{L}_K h = 0$. By the Cartan formula (Theorem A.4), the Lie derivative of the Kähler form ω of M is

$$\mathcal{L}_K\omega = \mathrm{d}(K \lrcorner\, \omega) + K \lrcorner\, \mathrm{d}\omega = \mathrm{d}(K \lrcorner\, \omega),$$

showing that $\mathcal{L}_K\omega$ is exact. Since, by definition, the flow of K is isometric, it commutes with the Hodge star operator; thus $\mathcal{L}_K \circ * = * \circ \mathcal{L}_K$. Since the Lie derivative commutes with the exterior derivative, we also see that $\mathcal{L}_K \circ \delta = \delta \circ \mathcal{L}_K$, whence

$$\delta(\mathcal{L}_K\omega) = \mathcal{L}(\delta\omega) = 0,$$

because ω is coclosed by the above lemma. Thus $\mathcal{L}_K\omega$ is coclosed and exact, meaning it is harmonic and exact, so it has to vanish by Theorem 7.4. We have shown that the flow of K preserves the Kähler form ω, which means that $\mathcal{L}_K J = 0$, showing that K is real holomorphic by Lemma 7.8. ☐

7.2 Dolbeault Theory

Let (M^{2m}, J, h) be a compact Hermitian manifold. Analogously to the exterior derivative acting on spaces of k-forms, we may consider the Dolbeault operator $\bar\partial$ acting on the spaces of (p,q)-forms, $\Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$.

Definition 7.11. We define the *Dolbeault cohomology groups* by

$$H^{p,q}(M) := \frac{\ker\left(\bar\partial : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)\right)}{\mathrm{im}\left(\bar\partial : \Omega^{p,q-1}(M) \to \Omega^{p,q}(M)\right)}.$$

In contrast to the de Rham cohomology, the Dolbeault cohomology is no longer a topological invariant of the manifold, since it strongly depends on the complex structure J.

Analogously to harmonic k-forms, we define the following.

Definition 7.12. The space of $\bar\partial$-harmonic (p,q)-forms on M is defined to be

$$\mathcal{H}^{p,q}(M) := \{\alpha \in \Omega^{p,q}(M) \,:\, \Delta^{\bar\partial}\alpha = 0\}.$$

In exactly the same fashion as in the previous subsection, we may prove (by referring, as before, to Griffiths [6], pp. 84–100) the following lemma and theorem.

Lemma 7.13. *A (p,q)-form $\alpha \in \Omega^{p,q}(M)$ is $\bar\partial$-harmonic if and only if $\bar\partial\alpha = 0$ and $\bar\partial^*\alpha = 0$.*

Theorem 7.14 (Dolbeault Decomposition Theorem)**.** *The space of (p, q)-forms decomposes as*

$$\Omega^{p,q}(M) = \mathcal{H}^{p,q}(M) \oplus \bar{\partial}^* \Omega^{p,q+1}(M) \oplus \bar{\partial}\Omega^{p,q-1}(M).$$

Thus we can write any (p, q)-form as $\alpha = \bar{\partial} = \bar{\partial}\alpha_1 + \bar{\partial}^*\alpha_2 + \alpha_h$, where $\alpha_1 \in \Omega^{p,q-1}(M)$, $\alpha_2 \in \Omega^{p,q+1}(M)$, and $\alpha_h \in \mathcal{H}^{p,q}(M)$. Applying $\bar{\partial}$ to both sides and integrating, we see that if $\bar{\partial}\alpha = 0$, then

$$\int_M \langle \bar{\partial}\alpha, \alpha_2 \rangle \, \mathrm{d}v = \int_M \langle \bar{\partial}\bar{\partial}^* \alpha_2, \alpha_2 \rangle \, \mathrm{d}v = \int_M |\bar{\partial}^* \alpha_2|^2 \, \mathrm{d}v,$$

showing that $\bar{\partial}^*\alpha_2 = 0$ if and only if $\bar{\partial}\alpha = 0$. Setting $q = 0$, we then have the following.

Proposition 7.15. *A $(p, 0)$-form on a compact Hermitian manifold is holomorphic if and only if it is $\bar{\partial}$-harmonic.*

Corollary 7.16. *Every Dolbeault cohomology class on M contains a unique $\bar{\partial}$-harmonic representative, and $\mathcal{H}^{p,q}(M) \cong H^{p,q}(M)$*

Definition 7.17. We define the *Hodge number* $h^{p,q}$ to be the complex dimension of $H^{p,q}(M)$,

$$h^{p,q}(M, J) = \dim_{\mathbb{C}}(H^{p,q}(M)).$$

It should be noted that they are associated to the given complex structure J on M, and are not topological invariants of M.

Theorem 7.18 (Serre Duality)**.** *The vector spaces $\mathcal{H}^{p,q}(M)$ and $\mathcal{H}^{m-p,m-q}(M)$ are isomorphic. In particular, $h^{p,q} = h^{m-p,m-q}$ for every compact Hermitian manifold (M^{2m}, J, h).*

Proof. By analogy with Poincaré duality, we expect that the isomorphism should be given by the Hodge star operator. Recall that our extension of the Hodge star operator to complex-valued forms maps (p, q)-forms to $(m - q, m - p)$-forms. Consider the composition of the Hodge star operator with complex conjugation, $\bar{*} : \Omega^{p,q}(M) \to \Omega^{m-p,m-q}(M)$, $\bar{*}\alpha := *\bar{\alpha}$. We calculate

$$\begin{aligned}
\bar{*}\Delta^{\bar{\partial}}\alpha &= *\overline{(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})\alpha} \\
&= *(\partial\partial^* + \partial^*\partial)\bar{\alpha} \\
&= - * (\partial * \bar{\partial} * + * \bar{\partial} * \partial)\bar{\alpha} \\
&= \bar{\partial}^*\bar{\partial}(\bar{*}\alpha) - *^2\bar{\partial} * \partial\bar{\alpha} \\
&= \bar{\partial}^*\bar{\partial}(\bar{*}\alpha) - \bar{\partial} * \partial *^2 \bar{\alpha} \\
&= \bar{\partial}^*\bar{\partial}(\bar{*}\alpha) + \bar{\partial}\bar{\partial}^*(\bar{*}\alpha) = \Delta^{\bar{\partial}}(\bar{*}\alpha),
\end{aligned}$$

which shows that $\bar{*}$ is a $\mathbb{C}$-antilinear isomorphism from $\mathcal{H}^{p,q}(M)$ to $\mathcal{H}^{m-p,m-q}(M)$. $\square$

7.3 Global Results on Compact Kähler Manifolds

If (M^{2m}, J, h) is a compact Kähler manifold, more can be said about the Hodge and Betti numbers by virtue of Theorem 6.5. Firstly, the fact that $\Delta = 2\Delta^{\bar{\partial}}$ shows that $\mathcal{H}^{p,q}(M) \subset \mathcal{H}^{p+q}(M)$. Secondly, since $\Delta^{\bar{\partial}}$ leaves the spaces $\Omega^{p,q}(M)$ invariant, so does Δ, showing that the components of a harmonic form in its decomposition by type (4) are all $\bar{\partial}$-harmonic. Thus the space of harmonic k-forms decomposes as

$$\mathcal{H}^k(M) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(M).$$

Furthermore, $\Delta = 2\Delta^{\bar{\partial}}$ shows that $\Delta^{\bar{\partial}}$ is a real operator on Kähler manifolds, so it commutes with complex conjugation[2]. Thus on Kähler manifolds complex conjugation defines an isomorphism between $\mathcal{H}^{p,q}(M)$ and $\mathcal{H}^{q,p}(M)$.

Now consider the Kähler form $\omega \in \Omega^{1,1}(M)$ of M^{2m}. Since ω^m is a non-zero multiple of the volume form, we deduce that all exterior powers $\omega^p \in \Omega^{p,q}(M)$ are non-zero. Moreover, they are all harmonic by Lemma 7.9. In summary, we have the following.

Proposition 7.19. *On any compact Kähler manifold (M^{2m}, J, h) the Hodge and Betti numbers are related by*

$$b_k(M) = \sum_{p+q=k} h^{p,q}(M,J), \quad h^{p,q}(M,J) = h^{q,p}(M,J), \quad \textit{and} \quad h^{p,p}(M,J) \geqslant 1 \ \ \forall\ 0 \leqslant p \leqslant m.$$

In particular, all Betti numbers of odd order are even, and all Betti numbers of even order are non-zero.

Furthermore, on compact Kähler manifolds a global analogue of Proposition 3.21 holds.

Proposition 7.20 (The Global $\partial\bar{\partial}$-Lemma)**.** *Let φ be an exact real $(1,1)$-form on a a compact Kähler manifold M. Then φ is globally $i\partial\bar{\partial}$-exact, that is there exists a globally defined smooth real function u satisfying $\varphi = i\partial\bar{\partial}u$.*

Proof. For the proof we again refer the reader to Griffiths and Harris, [6]. ☐

Due to the identity $\mathrm{dd}^c = 2i\partial\bar{\partial}$, both the global and the local $\partial\bar{\partial}$-lemmas are sometimes referred to as the dd^c-lemmas.

Proposition 7.21. *There can exist no global Kähler potential on a compact Kähler manifold (M^{2m}, J, h).*

Proof. Assume the contrary. Then we may write the Kähler form ω of M globally as

$$\omega = i\partial\bar{\partial}u = \mathrm{d}(i\bar{\partial}u) = \mathrm{d}v$$

for some globally defined 1-form v. Then

$$\begin{aligned}\mathrm{Vol}(M) &= \frac{1}{m!}\int_M \omega^m = \frac{1}{m!}\int_M \omega \wedge \omega^{m-1} \\ &= \frac{1}{m!}\int_M \mathrm{d}v \wedge \omega^{m-1} = \frac{1}{m!}\int_M \mathrm{d}(v \wedge \omega^{m-1}) = 0\end{aligned}$$

by Stokes' Theorem. This is absurd, since $\mathrm{Vol}(M) > 0$. ☐

[2]In general we only have $\overline{\Delta^{\bar{\partial}}\alpha} = \Delta^{\partial}\bar{\alpha}$.

8 The First Chern Class

Characteristic classes are a way of associating a cohomology class to each principal bundle over a topological space. They measure the extent to which the bundle is twisted; they are topological invariants measuring the deviation of a local product structure from a global product structure. Chern classes are the characteristic classes of complex vector bundles over smooth manifolds. For our purposes it will suffice to understand the first Chern class, but it should be noted that the theory of characteristic classes is very rich and has wide applications. A comprehensive account of the theory of Chern classes can be found in §12 of Kobayashi and Nomizu, [12].

Let $E \to N$ be a complex vector bundle over a smooth manifold N. We take the following complete characterisation of the first Chern class $c_1(E)$ of E as a definition.

Definition 8.1. The first Chern class of E, $c_1(E)$, is an element of $H^2(N, \mathbb{Z})$ satisfying the following axioms.

1. (Naturality) For every smooth map $f : M \to N$ and complex vector bundle E over N one has
$$f^*(c_1(E)) = c_1(f^*E). \tag{12}$$

2. (Whitney sum formula) For all vector bundles E, F over M one has
$$c_1(E \oplus F) = c_1(E) + c_1(F), \tag{13}$$
where $E \oplus F$ is the direct sum of the vector bundles E and F.

3. (Normalisation) The first Chern class of the tautological bundle $L \to \mathbb{CP}^1$ of $\mathbb{CP}^1$ is equal to -1,
$$c_1(L) = -1 \in \mathbb{Z} \simeq H^2(\mathbb{CP}^1, \mathbb{Z}). \tag{14}$$

Comments.

1. The right-hand side of eq. (12) is the first Chern class of the pullback bundle f^*E of $\pi : E \to N$. Recall that the pullback bundle is defined by $(f^*E)_p = E_{f(p)}$ for all $p \in M$, or more precisely
$$f^*E := \{(u, p) \in E \times M \ : \ \pi(u) = f(p)\}.$$
The left-hand side of eq. (12) is the pullback of $c_1(E)$ *in cohomology*. On smooth manifolds all cohomology theories are isomorphic, so pullback in cohomology (at least over $\mathbb{R}$ or $\mathbb{C}$) may be understood in terms of differential forms and the de Rham cohomology. Recall that the smooth map $f : M \to N$ induces a pullback on differential forms $\omega \in \Omega^k(N)$, $f^* : \Omega^k(N) \to \Omega^k(M)$, defined by
$$(f^*\omega)_p (X_1, \dots, X_k) = \omega_{f(p)} (\mathrm{d}f_p(X_1), \dots, \mathrm{d}f_p(X_k)),$$
where $\mathrm{d}f_p : T_pM \to T_{f(p)}N$ is the differential of f at $p \in M$, so induces a pullback of cohomology,
$$f^* : H^l_{dR}(N, \mathbb{R}) \to H^l_{dR}(M, \mathbb{R}).$$

2. A pedantic definition: the direct sum of two vector bundles $E \to M$ and $F \to M$ is the pullback bundle $\iota^*(E \times F)$, where $\iota : M \to M \times M$ is the diagonal inclusion $p \mapsto (p, p)$, and $E \times F$ the natural product bundle $\pi_{E\times F} = (\pi_E, \pi_F) : E \times F \to M \times M$. This is sometimes called the *Whitney sum*.

3. The *tautological bundle* of $\mathbb{CP}^1$ is the complex line bundle $L \to \mathbb{CP}^1$ whose fibre $L_{[z]}$ over point $[z] \in \mathbb{CP}^1$ is the complex line $\langle z \rangle$ in $\mathbb{C}^2$, $\langle z \rangle = \{\lambda z : \lambda \in \mathbb{R}\}$.

It turns out that one is able to express the images in real cohomology of the first Chern class of a complex vector bundle $E \to M$ using the curvature of an *arbitrary* connection ∇ on E. This is known as the *Chern–Weil theory.*

Theorem 8.2. *Let ∇ be any connection on a complex vector bundle E over M. The real cohomology class*

$$c_1(\nabla) := \left[\frac{i}{2\pi} \mathrm{Tr}(R^\nabla) \right] \tag{15}$$

is equal to the image of $c_1(E)$ in $H^2(M, \mathbb{R})$.

Proof. The proof consists of checking that $c_1(\nabla)$, defined by (15), satisfies the three hypotheses of Definition 8.1. We do not reproduce it here, but refer the interested reader to pp. 113 – 116 of Moroianu, [13]. □

Definition 8.3. Let (M, J) be an almost complex manifold. We define the *first Chern class of M*, denoted by $c_1(M)$, to be the first Chern class of the tangent bundle TM,

$$c_1(M) := c_1(TM).$$

9 The Ricci Form Again

9.1 Kähler Metrics as Torsion-Free $\mathrm{U}(m)$-structures

Let M be a $2m$-dimensional smooth manifold. Recall that an almost complex structure J on M is equivalent to a $\mathrm{GL}(m, \mathbb{C})$-structure on M, as explained in Section 3.1. Furthermore, if M possesses a Hermitian metric h, it has an $\mathrm{O}(2m)$-structure. Now $\mathrm{GL}(m, \mathbb{C}) \cap \mathrm{O}(2m) = \mathrm{U}(m)$, so the pair (J, h) defines a U(m)-structure.

Proposition 9.1. *The $\mathrm{U}(m)$-structure defined by an almost complex structure J together with a Hermitian metric h on a manifold M is torsion-free if and only if the metric h is Kähler.*

Proof. Since $\mathrm{U}(m) \subset \mathrm{O}(2m)$, there exists at most one torsion-free connection on the $\mathrm{U}(m)$-structure by the uniqueness of the Levi–Civita connection. The Levi–Civita connection is compatible with the $\mathrm{U}(m)$-structure if and only if $\nabla^{\mathrm{LC}} J = 0$ (see the discussion in Section 2.2), which means that M is Kähler by Theorem 5.8. □

9.2 The Curvature of the Canonical Bundle

Let us now return to our main objects of study and let (M^{2m}, J, h) be a Kähler manifold with Ricci form ρ. We call

$$K := \Lambda^{m,0} M$$

the *canonical bundle* of M. Recall that the tangent bundle TM has the structure of a holomorphic Hermitian vector bundle over M (Proposition 4.5), with multiplication by i corresponding to the tensor J and the Hermitian structure given by $H = h - i\omega$. Recall also that by Proposition 5.10

the Levi–Civita connection on M coincides with the Chern connection on TM. Since the curvature operator R^∇ of the Chern connection agrees with the curvature tensor R of the Levi–Civita connection, we will not distinguish between them and write simply $R^\nabla = R$.

Now for any connection ∇ on a complex bundle E, we let ∇^* be the induced connection on the dual bundle E^*, defined by

$$(\nabla^*_X \sigma^*)\sigma := \partial_X(\sigma^*(\sigma)) - \sigma^*(\nabla_X \sigma)$$

for all vector fields X and sections $\sigma \in \Gamma(E)$, $\sigma^* \in \Gamma(E^*)$. Let ∇ be the Levi–Civita connection on M (equivalently, the Chern connection on TM), and let R^∇ be the curvature of ∇.

Lemma 9.2. *The curvature of the dual connection ∇^* is related to the curvature of ∇ by the formula*

$$R^{\nabla^*}(X,Y) = -\left(R^\nabla(X,Y)\right)^*,$$

where $A^* \in \mathrm{End}(E^*)$ denotes the adjoint of the operator $A \in \mathrm{End}(E)$, defined by $A^*(\sigma^*)(\sigma) := \sigma^*(A(\sigma))$.

Proof. The proof is a simple (though tedious) calculation:

$$\begin{aligned}
R^{\nabla^*}(X,Y)(Z^*)\sigma &= (\nabla^*_X\nabla^*_Y Z^* - \nabla^*_Y\nabla^*_X Z^* - \nabla^*_{[X,Y]}Z^*)\sigma \\
&= \partial_X(\partial_Y(Z^*(\sigma)) - Z^*(\nabla_Y\sigma)) - \partial_Y(Z^*(\nabla_X(\sigma))) \\
&\quad - \partial_Y(\partial_X(Z^*(\sigma)) - Z^*(\nabla_X\sigma)) + Z^*(\nabla_Y\nabla_X\sigma) \\
&\quad + \partial_X(Z^*(\nabla_Y\sigma)) - Z^*(\nabla_X\nabla_Y\sigma) - \partial_{[X,Y]}(Z^*(\sigma)) + Z^*(\nabla_{[X,Y]}\sigma) \\
&= -Z^*(\nabla_{[X,Y]} - \nabla_Y\nabla_X - \nabla_{[X,Y]})\sigma \\
&= -Z^* R^\nabla(X,Y)\sigma \\
&= -\left(R^\nabla(X,Y)\right)^*(Z^*)\sigma.
\end{aligned}$$

□

Proposition 9.3. *The curvature of the Chern connection on the canonical bundle is equal to $i\rho$ acting by scalar multiplication.*

Sketch Proof. Let r be the curvature of the Chern connection of $K = \Lambda^{m,0}M$ and r^* the curvature of the Chern connection of $K^* = \Lambda^{0,m}M$. By Lemma 9.2, these are related by $r = -r^*$, since a Hermitian structure H induces an isomorphism between the complex vector bundles $\bar{E}$ and E^* (indeed, this is easy to see from the fact that $H : E \to E^*$ is a $\mathbb{C}$-antilinear isomorphism). The Hermitian structure H on TM induces a Hermitian structure, which we also denote by H, on $\Lambda^m(TM)$, and the connection on $\Lambda^m(TM)$ induced by the Chern connection on (TM, H) is clearly the Chern connection of $(\Lambda^m(TM), H)$. Since $\Lambda^m(TM)$ is isomorphic to K^*, we find

$$r^*(X,Y) = \mathrm{Tr}(R^\nabla(X,Y)) = \mathrm{Tr}(R(X,Y)).$$

It is easily checked that on a Kähler manifold

$$\mathrm{Ric}(X,Y) = \frac{1}{2}\mathrm{Tr}(R(X,JY)\circ J),$$

so we obtain

$$\begin{aligned}
i\rho(X,Y) &= i\mathrm{Ric}(JX,Y) \\
&= \frac{i}{2}\mathrm{Tr}^{\mathbb{R}}(R(X,Y)\circ J) \\
&= \frac{i}{2}(2i\mathrm{Tr}^{\mathbb{C}}(R(X,Y)) \\
&= -\mathrm{Tr}^{\mathbb{C}}(R(X,Y)) \\
&= -r^*(X,Y) = r(X,Y),
\end{aligned}$$

where we used the fact that

$$\mathrm{Tr}^{\mathbb{R}}(A^{\mathbb{R}}\circ J) = 2i\mathrm{Tr}^{\mathbb{C}}(A)$$

for every skew-Hermitian endomorphism A of $\mathbb{C}^m$ with corresponding real endomorphism $A^{\mathbb{R}}$ of $\mathbb{R}^{2m}$. □

9.3 Ricci Flat Kähler Manifolds

Having proven Proposition 9.3, we thus can state the following.

Theorem 9.4. *Suppose (M^{2m}, J, h) is a simply connected Kähler manifold with canonical bundle K and Ricci form ρ. The following are equivalent.*

1. *M is Ricci flat.*
2. *The Chern connection of the canonical bundle K is flat.*
3. *There exists a ∇-parallel complex volume form, that is a ∇-parallel smooth section of $\Lambda^{m,0}M$.*
4. *M possesses a torsion-free $\mathrm{SU}(m)$-structure with induced Kähler structure (J,h).*

Proof. As M is simply connected, the only non-obvious equivalence is between 1 and 2, which is provided by Proposition 9.3. The pedantic reader may like to consult Joyce [10] for more information. □

10 The Calabi Conjecture

Theorem 10.1 (Calabi, Aubin, Yau)**.** *Let (M^{2m}, J, h) be a compact Kähler manifold with Kähler form ω and Ricci form ρ. Then for every closed real $(1,1)$ form ρ_1 in the cohomology class of $2\pi c_1(M)$ there exists a unique Kähler metric h_1 with Kähler form ω_1 in the same cohomology class as ω, whose Ricci form is exactly ρ_1. In particular, if the first Chern class of a compact Kähler manifold vanishes, then M carries a Ricci flat Kähler metric.*

Discussion. The Calabi Conjecture was originally posed by Calabi in 1954, [4, 5], who also showed that if h_1 exists, it must be unique. It was eventually proved by Yau in 1976, [15, 16]. Before that, Aubin had made significant progress towards a proof, [1]. A simplified version of the proof is given by Joyce in [9] and [10].

The proof proceeds by rephrasing the statement of the conjecture in terms of a nonlinear second order elliptic partial differential equation. In precise terms, it turns out that the following is a reformulation of Theorem 10.1. For details the reader should consult §6 of Joyce, [10].

Theorem 10.2 (Reformulation of the Calabi Conjecture)**.** *Let* (M^{2m}, J, h) *be a compact Kähler manifold with Kähler form* ω*. Let* f *be a smooth real function on* M *and define* $A > 0$ *by*

$$A \int_M \mathrm{e}^f \, \mathrm{d}v_h = \mathrm{Vol}_h(M).$$

Then there exists a unique real function φ *such that*

(i) $\omega + \mathrm{d}\mathrm{d}^c\varphi$ *is a positive* $(1,1)$ *form,*

(ii) $\int_M \varphi \, \mathrm{d}v_h = 0$*, and*

(iii) $(\omega + \mathrm{d}\mathrm{d}^c\varphi)^m = A\mathrm{e}^f \omega^m$ *on* M*.*

In local coordinates condition *(iii)* takes on the form of the equation

$$\det\left(h_{\alpha\bar{\beta}} + \frac{\partial^2 \varphi}{\partial z_\alpha \partial \bar{z}_\beta}\right) = A\mathrm{e}^f \det(h_{\alpha\bar{\beta}}), \tag{16}$$

and the crux of the Calabi Conjecture then becomes proving the existence of a smooth solution to equation (16). It is a severely nonlinear second order elliptic PDE of a type known as a *Monge–Ampére equation.* The proof proceeds by a *continuity method.* One considers the equation

$$\det\left(h_{\alpha\bar{\beta}} + \frac{\partial^2 \varphi_t}{\partial z_\alpha \partial \bar{z}_\beta}\right) = \left((1-t) + tA\mathrm{e}^f\right) \det\left(h_{\alpha\bar{\beta}}\right) \tag{17}$$

for $t \in [0,1]$ with the associated set $S = \{t \in [0,1] \,:\, \text{a solution } \varphi_t \text{ to (17) exists}\}$. If one can show that S is non-empty, open, and closed, then $t = 1 \in S$, and one has existence of a solution to (16). Clearly $t = 0 \in S$, whilst openness also turns out to be straightforward, and was proven by Calabi in [4, 5]. The hard part turns out to be showing that S is closed, which amounts to showing that S contains its limit points, or that if $t_i \to t$ for a sequence $\{\varphi_{t_i}\}$ of solutions to (17), then $\varphi_{t_i} \to \varphi_t$. This amounts to finding uniform *a priori* estimates for the solution, and was done by Yau in [16].

A Exterior Derivative and Lie Derivative

Let M be a smooth manifold of dimension n. The following theorem may be taken as the definition of the exterior derivative d on M. The proof may be found in §3 of Moroianu, [13].

Theorem A.1. *There exists a unique $\mathbb{R}$-linear endomorphism* d *on smooth exterior forms on M, called the* exterior derivative*, satisfying the following axioms.*

1. $\mathrm{d} : \Omega^p(M) \to \Omega^{p+1}(M)$ *for all* $0 \leqslant p \leqslant n$,
2. $\mathrm{d}f(X) = \partial_X(f)$ *for all functions* $f \in \Gamma(M)$ *and all vector fields* $X \in \Gamma(TM)$,
3. *For all* $\alpha \in \Omega^p(M)$, $\beta \in \Omega^q(M)$ *there holds the Leibniz rule*
$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^p \alpha \wedge \mathrm{d}\beta,$$
4. $\mathrm{d}^2 = 0$,
5. *For all smooth maps of manifolds* $\varphi : M \to N$ *the exterior derivative commutes with the pullback,*
$$\mathrm{d}(\varphi^* \alpha) = \varphi^*(\mathrm{d}\alpha)$$
for every smooth form α.

A related concept, and another way to differentiate tensors on a manifold, is the Lie derivative.

Definition A.2. Let X be a vector field on M and let φ_t be the local flow of X. The *Lie derivative* of a tensor field K along X is
$$\mathcal{L}_X K := \lim_{t \to 0} \frac{K - (\varphi_t)_*(K)}{t}.$$

Theorem A.3. *Let X be a vector field on M. The Lie derivative $\mathcal{L}_X$ satisfies the following properties.*

1. $\mathcal{L}_X(K \otimes T) = (\mathcal{L}_X K) \otimes T + K \otimes (\mathcal{L}_X T)$ *for all tensor fields K and T,*
2. $\mathcal{L}_X$ *commutes with contractions, for example* $\mathcal{L}_X(\mathrm{Tr}(\eta \otimes X)) = \mathrm{Tr}(\mathcal{L}_X(\eta \otimes X))$ *for all* 1*-forms* η *and vector fields* X,
3. $\mathcal{L}_X f = \partial_X f$ *for all smooth functions f on M,*
4. $\mathcal{L}_X Y = [X, Y]$ *for all vector fields* $Y \in \Gamma(TM)$, *and*
5. *If* $\varphi : M \to N$ *is a diffeomorphism, then* $\varphi_*(\mathcal{L}_X K) = \mathcal{L}_{\varphi_*(X)} \varphi_*(K)$.
6. *The Lie derivative commutes with the exterior derivative,* $[\mathcal{L}_X, \mathrm{d}] = 0$.

Proof. The proof may be found in §2 of Moroianu, [13]. □

Theorem A.4 (Cartan's Formula)**.** *For every vector field X and every exterior form α on M one has*
$$\mathcal{L}_X \alpha = \mathrm{d}(X \lrcorner \alpha) + X \lrcorner \mathrm{d}\alpha.$$

Proof. The proof can be found in §3 of Moroianu, [13]. □

Proposition A.5. *Let us now suppose that M possesses a Riemannian metric g and let ∇ denote the Levi–Civita connection of g. Let $\{e_1, \dots, e_n\}$ be a local orthonormal frame parallel with respect to ∇ at some point $p \in M$. The exterior derivative is locally given by*

$$\mathrm{d} = e_i \wedge \nabla_{e_i}.$$

Lemma A.6. *For every 1-form $\alpha \in \Omega^1(M)$ we have*

$$\mathrm{d}\alpha(X, Y) = \partial_X(\alpha(Y)) - \partial_Y(\alpha(X)) - \alpha([X, Y])$$

for all vector fields $X, Y \in \Gamma(TM)$.

Proof. By Cartan's Formula,

$$\begin{aligned}
\mathrm{d}\alpha(X, Y) = (X \lrcorner \mathrm{d}\alpha)(Y) &= (\mathcal{L}_X \alpha)(Y) - \mathrm{d}(X \lrcorner \alpha)(Y) \\
&= \mathcal{L}_X(\alpha(Y)) - \alpha(\mathcal{L}_X Y) - \mathrm{d}(\alpha(X))(Y) \\
&= \partial_X(\alpha(Y)) - \partial_Y(\alpha(X)) - \alpha([X, Y]).
\end{aligned}$$

□

Sketch proof of Proposition A.5. The claim is trivial to check for d acting on functions, $\mathrm{d} : \Gamma(M) \to \Omega^1(M)$. Consider now $\mathrm{d} : \Omega^1(M) \to \Omega^2(M)$. Since ∇ is torsion-free, $\nabla_X Y - \nabla_Y X = [X, Y]$, so by the above lemma we have

$$\begin{aligned}
\mathrm{d}\alpha(X, Y) &= (\nabla_X \alpha)(Y) - (\nabla_Y \alpha)(X) - \alpha(\nabla_X Y) + \alpha(\nabla_Y X) \\
&= (\nabla_X \alpha)(Y) - (\nabla_Y \alpha)(X) \\
&= (e_i \wedge \nabla_{e_i} \alpha)(X, Y).
\end{aligned}$$

More generally, an inductive argument along the same lines shows that for $\alpha \in \Omega^p(M)$ one has

$$\mathrm{d}\alpha(X_0, \dots, X_p) = \sum_{k=0}^{p} (-1)^k (\nabla_{X_k} \alpha)(X_0, \dots, \hat{X}_k, \dots, X_p),$$

where $\hat{X}_k$ indicates that the k-th argument has been omitted, thus showing the result. □

B The Hodge Star

Let (M^n, g) be an oriented Riemannian manifold with volume form $\mathrm{d}v$. Let $\{e_1, \dots, e_n\}$ be a local orthonormal frame on M which is parallel at a point. As usual, we use the metric g to identify vectors and 1-forms so that, for example, we may write $\mathrm{d}v = \mathrm{d}v_g = e_1 \wedge \cdots \wedge e_n$. Furthermore, we naturally identify $\Lambda^k M$ with a subset of $(T^*M)^{\otimes k}$ by writing

$$e_1 \wedge \cdots \wedge e_k = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \varepsilon(\sigma) e_{\sigma_1} \otimes \cdots \otimes e_{\sigma_k}.$$

Contraction by the metric then induces a scalar product on $\Lambda^k M$, denoted $\langle \cdot, \cdot \rangle$, which is given by

$$\langle \omega, \tau \rangle := g(\omega, \tau).$$

We then define the *Hodge star operator* $* : \Lambda^k M \to \Lambda^{n-k} M$ by

$$\omega \wedge *\tau := \langle \omega, \tau \rangle \, \mathrm{d}v \tag{18}$$

for all $\omega, \tau \in \Lambda^k M$. It is easy to check that on $\Lambda^k M$

$$*^2 = (-1)^{k(n-k)},$$

$$*1 = \mathrm{d}v, \quad *\mathrm{d}v = 1,$$

and

$$\langle *\omega, *\tau \rangle = \langle \omega, \tau \rangle.$$

C The Riemann Curvature Tensor

Let (M, g) be a Riemannian manifold of dimension n. We briefly recall results about the Levi–Civita connection and the Riemann curvature tensor below.

Theorem C.1 (Fundamental Theorem of Riemannian Geometry)**.** *On any (pseudo-)Riemannian manifold there exists a unique torsion-free linear connection ∇, called the* Levi–Civita connection*, which preserves the metric g.*

Theorem C.2. *Around every point p on M there exists a local orthonormal frame $\{e_1, \dots e_n\}$ parallel at p with respect to ∇, meaning that $\nabla_{e_i} e_j = 0$ for all i, j.*

Definition C.3. The tensor $R^a{}_{bcd}$ defined by

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for all vector fields X, Y, Z, is called the *Riemann curvature tensor.*

Proposition C.4. *The metric g allows one to identify the tensor $R^a{}_{bcd}$ with a fully covariant tensor R_{abcd}, which then possesses the following symmetries.*

1. $R_{abcd} = -R_{abdc}$,
2. $R_{abcd} = R_{cdab}$,
3. $R_{a[bcd]} = 0$ *(first Bianchi identity),*
4. $R_{ab[cd;e]} = 0$ *(second Bianchi identity).*

Definition C.5. The *Ricci tensor* is the trace of the Riemann curvature tensor defined by

$$R_{ab} = R^c{}_{acb}.$$

Definition C.6. The *scalar curvature* is the trace of the Ricci curvature tensor,

$$S = g^{ab} R_{ab}.$$

D The Curvature Operator of a Connection

Let M be a smooth manifold of dimension n and let $E \to M$ be a real or complex vector bundle over M.

Definition D.1. A connection on E is a $\mathbb{C}$-linear differential operator $\nabla : \Gamma(E) \to \Omega^1(E)$ satisfying the Leibniz rule

$$\nabla(f\sigma) = \mathrm{d}f \otimes \sigma + f\nabla\sigma$$

for all $f \in C^\infty(M)$ and all $\sigma \in \Gamma(E)$, where $\Omega^1(E)$ denotes the space of smooth sections of $\Lambda^1 M \otimes E$, that is the space of E-valued 1-forms.

We extend this definition to act on $\Omega^p(E) = \Gamma(\Lambda^p(M) \otimes E)$ by $\mathbb{C}$-linearity and the formula

$$\nabla(\omega \otimes \sigma) = \mathrm{d}\omega \otimes \sigma + (-1)^p \omega \wedge \nabla\sigma$$

for all $\omega \in \Omega^p(M)$ and all $\sigma \in \Gamma(E)$.

Definition D.2. The *curvature operator* of ∇ is the End(E)-valued 2-form $R^\nabla : \Gamma(E) \to \Omega^2(E)$ defined by

$$R^\nabla(\sigma) := \nabla(\nabla\sigma)$$

for all $\sigma \in \Gamma(E)$.

It is an easy application of the Leibniz rule for ∇ to check that R^∇ is linear over $C^\infty(M)$, so is a tensor. One can also characterise R^∇ more explicitly as follows. Let $\{\sigma_1, \dots, \sigma_k\}$ be local sections of E which form a basis of each fibre of E over some open set $U \subset M$. We define the *local connection forms* $\omega_{ij} \in \Omega^1(U)$ (relative to this basis) by

$$\nabla\sigma_i = \omega_{ij} \otimes \sigma_j,$$

and the *local curvature* 2*-forms* R^∇_{ij} by

$$R^\nabla(\sigma_i) = R^\nabla_{ij} \otimes \sigma_j.$$

A calculation then shows that

$$R^\nabla_{ij} \otimes \sigma_j = R^\nabla(\sigma_i) = \nabla(\omega_{ij} \otimes \sigma_j) = (\mathrm{d}\omega_{ij}) \otimes \sigma_j - \omega_{ik} \wedge \omega_{kj} \otimes \sigma_j,$$

that is

$$R^\nabla_{ij} = \mathrm{d}\omega_{ij} - \omega_{ik} \wedge \omega_{kj}. \tag{19}$$

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